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Web calculi in representation theory

Or: Why playing with diagrams is fun

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Abstract

Fix a Lie algebra $\mathfrak{g}$. The goal of my research is to give a presentation, via diagrammatic generators and relations, of the category of finite-dimensional $\mathfrak{g}$-modules or of some well-behaved subcategories. These presentations, having a topological flavor, can then be applied in low-dimensional topology, combinatorics and combinatorial algebraic geometry. Moreover, these presentations are amenable to categorification and provide inside on higher levels as well.

Introduction

The symmetric group S_d can be for example described either as the set of all automorphisms of $\{1,\dots,d\}$ or, alternatively, via generators and relations:

$$S_d = \left\langle \sigma_1,\dots,\sigma_{d-1} \middle| \begin{cases} \sigma_i^2 = 1, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & \text{if } |i-j| = 1, \\ \sigma_i\sigma_j = \sigma_j\sigma_i, & \text{if } |i-j| > 1. \end{cases} \right\rangle \tag{1}$$

The former description shows why S_d *is interesting* to study, while the one from (1) is usually *very handy* for showing theorems about S_d.

Thus, it is a natural question to ask if we can give a generators and relations presentation of $\mathfrak{g}$-$\mathbf{Mod}_{\mathrm{fd}}$, i.e. the category of finite-dimensional $\mathfrak{g}$-modules. Since $\mathfrak{g}$-$\mathbf{Mod}_{\mathrm{fd}}$ is ubiquitous in modern mathematics and physics, one could expect that an analogue of (1) for $\mathfrak{g}$-$\mathbf{Mod}_{\mathrm{fd}}$ would be very useful. One could even aim for a *diagrammatic* presentation, since it is a known, but non-trivial, fact that $\mathfrak{g}$-intertwiners (linear maps preserving the $\mathfrak{g}$-action; our morphisms in $\mathfrak{g}$-$\mathbf{Mod}_{\mathrm{fd}}$) have a *topological behaviour*.

Main Objectives

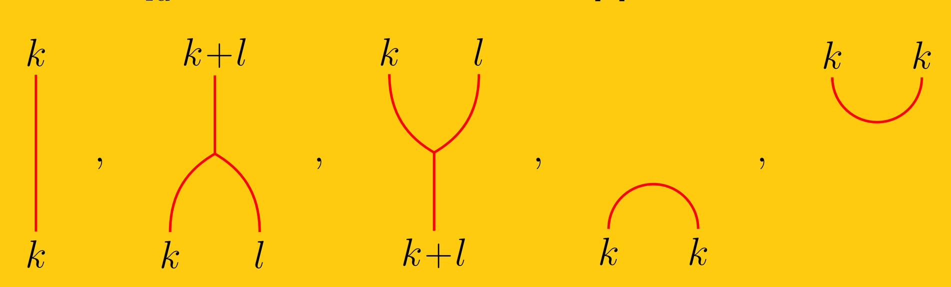
1. Find a (diagrammatic) generators and relations presentation of $\mathfrak{g}$-$\mathbf{Mod}_{\mathrm{fd}}$ or appropriate subcategories.
2. Use this presentation to show hidden symmetries within link polynomials and Witten-Reshetikhin-Turaev invariants of 3-manifolds (yes, everything can be quantized).
3. Try to get a better understanding of related concepts like crystal bases and associated buildings.
4. Categorify everything in sight!

Materials and Methods

The methods we use are a mixture between representation theory, (low-dimensional) topology, combinatorics, Lie theory, category theory and, the most powerful one, *naively playing* with diagrams.

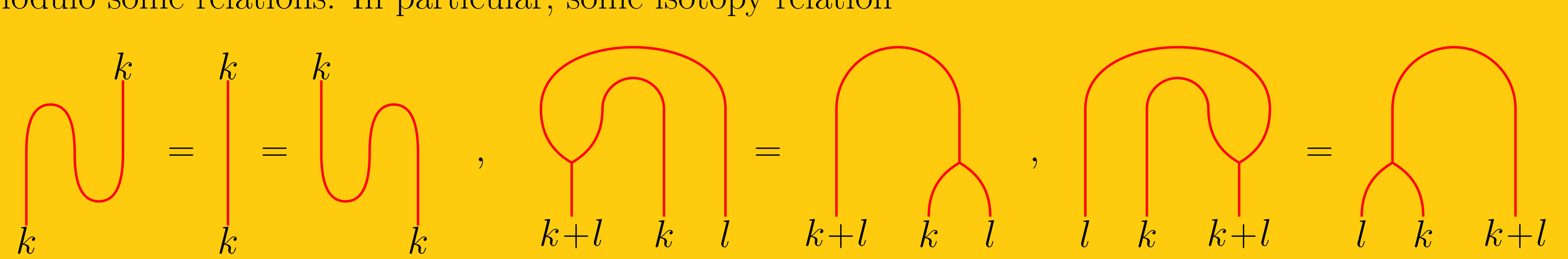
Mathematical Section

It turns out that *literally nothing is known* about a generators and relations presentation of $\mathfrak{g}$-$\mathbf{Mod}_{\mathrm{fd}}$. In fact, the description of $\mathfrak{sl}_2$-$\mathbf{Mod}_{\mathrm{fd}}$ was only accomplished in [1]: we show that $\mathfrak{sl}_2$-$\mathbf{Mod}_{\mathrm{fd}}$ is generated by



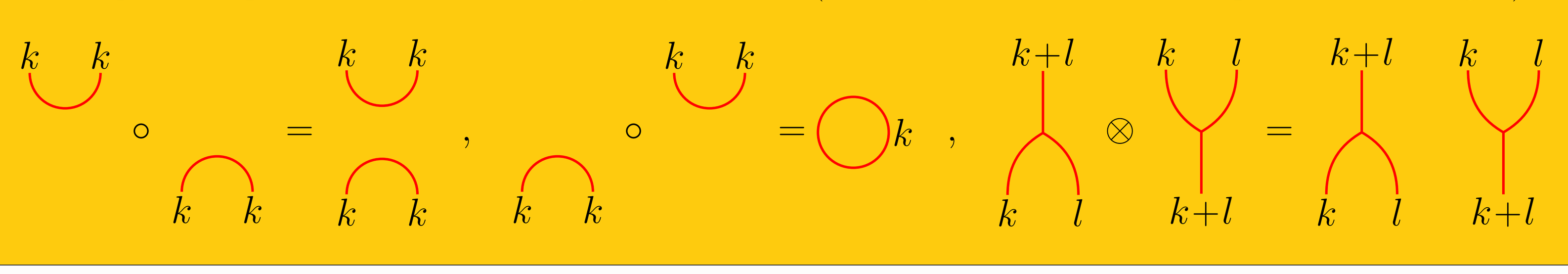
(2)

modulo some relations. In particular, some isotopy relation



(3)

which allows one to see the so-called *webs*, which are generated by the basic pieces from (2), as *topological* objects: embedded, trivalent graph with edge labels. Our description gives $\mathfrak{sl}_2$-$\mathbf{Mod}_{\mathrm{fd}}$ a topological flavor. For example, composition and tensoring are given via (reading from bottom to top and left to right)

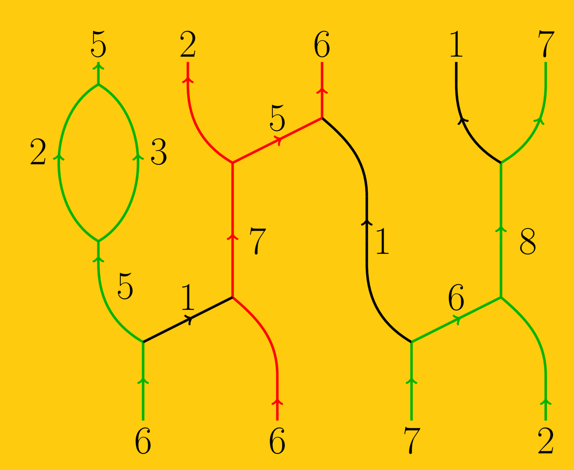


(4)

and a lot of question regarding finite-dimensional $\mathfrak{sl}_2$-modules and their intertwiners can be reduced to *"topologically playing with diagrams"*. And yes: all of these diagrams "are" $\mathfrak{sl}_2$-intertwiners.

Results

We have generalized in [3] the approach followed in [1] and can describe the subcategories of $\mathfrak{gl}_N$-$\mathbf{Mod}_{\mathrm{fd}}$ tensor generated by $\bigwedge^k\mathbb{C}^N$ and $\mathrm{Sym}^k\mathbb{C}^N$ via a rather neat diagrammatic calculus involving webs as



(5)

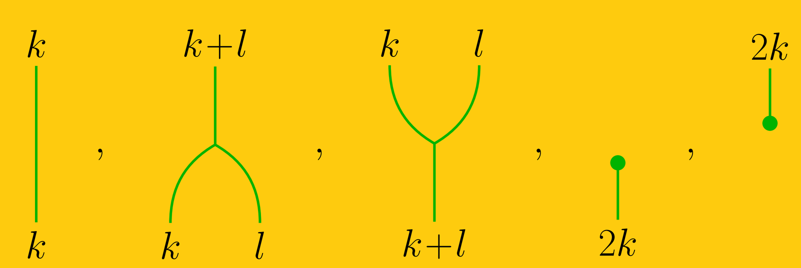
where the colors indicate $\bigwedge^k\mathbb{C}^N$ (green), $\mathrm{Sym}^k\mathbb{C}^N$ (red) or $\mathbb{C}^N$ (black) - as before for $\mathfrak{sl}_2$ where we only needed red. It turns out that this description is very symmetric and powerful: as a direct (almost trivial) application we were able to prove a (hidden) *symmetry within associated link polynomials*. Moreover, our approach easily generalizes to $\mathfrak{gl}_{N|M}$-modules and their $\mathfrak{gl}_{N|M}$-intertwiners as well.

Conclusions

Although relatively new at the moment, the diagrammatic presentations, left aside that they give a fairly neat calculus, of $\mathfrak{gl}_N$-$\mathbf{Mod}_{\mathrm{fd}}$ and associated categories have already led to new insights and there is a good chance for exiting developments in the years to come.

Forthcoming Research

The next step is to extend everything from above to other types. This is ongoing work [2]. For instance, in type $\boldsymbol{C}_n$ webs are generated by



(6)

Indeed, the calculi we have in mind will *generalize* a classical diagrammatic description due to Brauer. But let us see what the future brings.

References

[1] D.E.V. Rose and D. Tubbenhauer, Symmetric webs, Jones-Wenzl recursions and q-Howe duality, 2015, http://arxiv.org/abs/1501.00915.

[2] A. Sartori and D. Tubbenhauer, Webs and skew q-Howe dualities in types $\boldsymbol{B}_n, \boldsymbol{C}_n, \boldsymbol{D}_n$, in preparation.

[3] D. Tubbenhauer, P. Vaz and P. Wedrich, Super q-Howe duality and web categories, 2015, http://arxiv.org/abs/1504.05069.

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