Web bases and categorification

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Joint work with Marco Mackaay and Weiwei Pan

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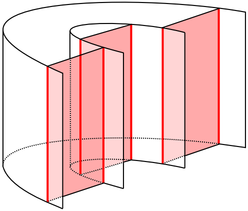


What is categorification?

Forced to reduce this presentation to one sentence, the author would choose:

Interesting integers are shadows of richer structures in categories.

The basic idea can be seen as follows. Take a "set-based" structure S and try to find a "category-based" structure $\mathcal{C}$ such that S is just a shadow of $\mathcal{C}$.

Categorification, which can be seen as "remembering" or "inventing" information, comes with an "inverse" process called decategorification, which is more like "forgetting" or "identifying".

Note that decategorification should be easy.

Exempli gratia

Examples of the pair categorification/decategorification are:

$$\text{Bettinumbers of a manifold } M \underset{\text{decat}=\mathrm{rank}(\cdot)}{\overset{\text{categorify}}{\rightleftarrows}} \text{Homology groups}$$

$$\text{Polynomials in } \mathbb{Z}[q,q^{-1}] \underset{\text{decat}=\chi_{\mathrm{gr}}(\cdot)}{\overset{\text{categorify}}{\rightleftarrows}} \text{complexes of gr.VS}$$

$$\text{The integers } \mathbb{Z} \underset{\text{decat}=K_0(\cdot)}{\overset{\text{categorify}}{\rightleftarrows}} K-\text{vector spaces}$$

$$\text{An } A-\text{module} \underset{\text{decat}=K_0^{\oplus}(\cdot)\otimes_{\mathbb{Z}}A}{\overset{\text{categorify}}{\rightleftarrows}} \text{additive category}$$

Usually the categorified world is much more interesting.

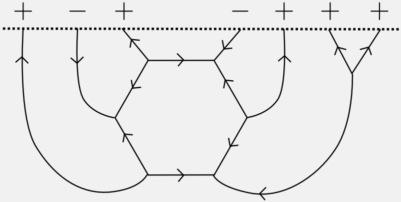
Today categorification = Grothendieck group!

Kuperberg's $\mathfrak{sl}_3$-webs

Definition(Kuperberg)

A $\mathfrak{sl}_3$-web w is an oriented trivalent graph, such that all vertices are either sinks or sources. The boundary ∂w of w is a sign string $S = (s_1, \ldots, s_n)$ under the convention $s_i = +$ iff the orientation is pointing in and $s_i = -$ iff the orientation is pointing out.

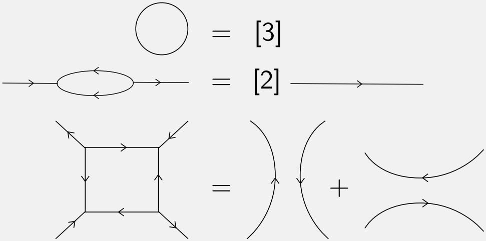
Example



Kuperberg's $\mathfrak{sl}_3$-webs

Definition(Kuperberg)

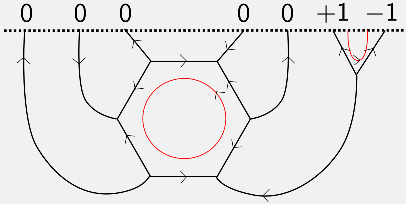
The $\mathbb{C}(q)$-web space W_S for a given sign string $S = (\pm, \ldots, \pm)$ is generated by $\{w \mid \partial w = S\}$, where w is a web, subject to the relations



Here $[a] = \frac{q^a - q^{-a}}{q - q^{-1}} = q^{a-1} + q^{a-3} + \cdots + q^{-(a-1)}$ is the quantum integer.

Kuperberg's $\mathfrak{sl}_3$-webs

Example



Webs can be coloured with flow lines. At the boundary, the flow lines can be represented by a state string J. By convention, at the i-th boundary edge, we set $j_i = \pm 1$ if the flow line is oriented downward/upward and $j_i = 0$, if there is no flow line. So $J = (0, 0, 0, 0, 0, +1, -1)$ in the example.

Given a web with a flow w_f, attribute a weight to each trivalent vertex and each arc in w_f and take the sum. The weight of the example is -4.

The quantum algebra $\mathbf{U}_q(\mathfrak{sl}_n)$

Definition

For $n \in \mathbb{N}_{>1}$ the quantum special linear algebra $\mathbf{U}_q(\mathfrak{sl}_n)$ is the associative, unital $\mathbb{C}(q)$-algebra generated by $K_i^{\pm 1}$ and E_i and F_i, for $i = 1, \ldots, n-1$, subject to the relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$K_iE_j = q^{(\epsilon_i,\alpha_j)}E_jK_i,$$

$$K_iF_j = q^{-(\epsilon_i,\alpha_j)}F_jK_i,$$

$$E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0, \qquad \text{if} \quad |i-j| = 1,$$

$$F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0, \qquad \text{if} \quad |i-j| = 1.$$

Representation theory of $\mathbf{U}_q(\mathfrak{sl}_3)$

A sign string $S = (s_1, \dots, s_n)$ corresponds to tensors

$$V_S = V_{s_1} \otimes \cdots \otimes V_{s_n},$$

where V_+ is the fundamental $\mathbf{U}_q(\mathfrak{sl}_3)$-representation and V_- is its dual, and webs correspond to intertwiners.

Theorem(Kuperberg)

$$W_S \cong \mathrm{Hom}_{\mathbf{U}_q(\mathfrak{sl}_3)}(\mathbb{C}(q), V_S) \cong \mathrm{Inv}_{\mathbf{U}_q(\mathfrak{sl}_3)}(V_S)$$

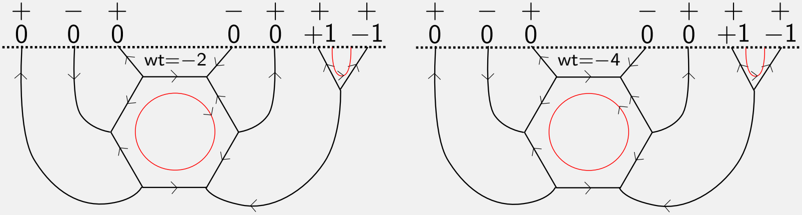
In fact, the so-called spider category of all webs modulo the Kuperberg relations is equivalent to the representation category of $\mathbf{U}_q(\mathfrak{sl}_3)$.
As a matter of fact, the $\mathfrak{sl}_3$-webs without internal circles, digons and squares form a basis B_S, called web-basis, of W_S!

Representation theory of $\mathbf{U}_q(\mathfrak{sl}_3)$

Theorem(Khovanov, Kuperberg)

Pairs of sign S and a state strings J correspond to the coefficients of the web basis relative to tensors of the standard basis $\{e^{\pm}_{-1}, e^{\pm}_{0}, e^{\pm}_{+1}\}$ of $V_{\pm}$.

Example



$$w_S = \cdots + (q^{-2} + q^{-4})(e^+_0 \otimes e^-_0 \otimes e^+_0 \otimes e^-_0 \otimes e^+_0 \otimes e^+_{-1} \otimes e^+_{+1}) \pm \cdots .$$

Skew Howe-duality

The natural actions of GL_k and GL_n on

$$\bigwedge^{p}(\mathbb{C}^k \otimes \mathbb{C}^n)$$

are Howe dual (skew Howe duality).
This implies that

$$\mathrm{Inv}_{\mathrm{SL_k}}(\Lambda^{p_1}(\mathbb{C}^k) \otimes \cdots \otimes \Lambda^{p_n}(\mathbb{C}^k)) \cong W(p_1, \ldots, p_n),$$

where $W(p_1, \ldots, p_n)$ denotes the $(p_1, \ldots, p_n)$-weight space of the irreducible GL_n-module $W_{(k^\ell)}$, if $n = k\ell$.

The idempotent version

For each $\lambda \in \mathbb{Z}^{n-1}$ adjoin an idempotent 1_λ (think: projection to the λ-weight space!) to $\mathbf{U}_q(\mathfrak{sl}_n)$ and add the relations

$$\begin{aligned} 1_\lambda 1_\mu &= \delta_{\lambda,\nu} 1_\lambda, \\ E_i 1_\lambda &= 1_{\lambda+\alpha_i} E_i, \\ F_i 1_\lambda &= 1_{\lambda-\alpha_i} F_i, \\ K_{\pm i} 1_\lambda &= q^{\pm\lambda_i} 1_\lambda \ \ \text{(no } K's \text{ anymore!)}. \end{aligned}$$

Definition

The idempotented quantum special linear algebra is defined by

$$\dot{\mathbf{U}}(\mathfrak{sl}_n) = \bigoplus_{\lambda,\mu \in \mathbb{Z}^n} 1_\lambda \, \mathbf{U}_q(\mathfrak{sl}_n) 1_\mu.$$

En(c)hanced sign strings

Definition

An enhanced sign sequence is a sequence $S=(s_1,\dots,s_n)$ with $s_i\in\{\circ,-,+,\times\}$, for all $i=1,\dots,n$. The corresponding weight $\mu=\mu_S\in\Lambda(n,d)$ is given by the rules

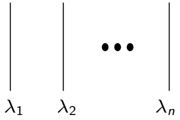
$$\mu_i=\begin{cases}0, & \text{if } s_i=\circ,\\ 1, & \text{if } s_i=+,\\ 2, & \text{if } s_i=-,\\ 3, & \text{if } s_i=\times.\end{cases}$$

Let $\Lambda(n,d)_3\subset\Lambda(n,d)$ be the subset of weights with entries between 0 and 3. Note that 1 corresponds to the $\dot{\mathbf{U}}(\mathfrak{sl}_3)$-representation V_+, 2 to its dual V_- and $0,3$ to the trivial $\dot{\mathbf{U}}(\mathfrak{sl}_3)$-representation.

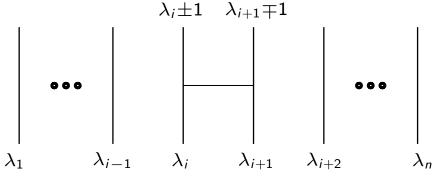
The $\mathfrak{sl}_3$-webs form a $\dot{\mathbf{U}}(\mathfrak{sl}_n)$-module

We defined an action ϕ of $\dot{\mathbf{U}}(\mathfrak{sl}_n)$ on $W_{(3^\ell)} = \bigoplus_{S \in \Lambda(n,n)_3} W_S$ by

$1_\lambda \mapsto$



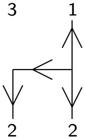
$E_i 1_\lambda,\ F_i 1_\lambda \mapsto$



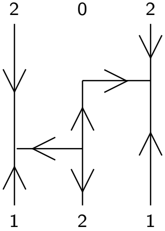
We use the convention that vertical edges labeled 1 are oriented upwards, vertical edges labeled 2 are oriented downwards and edges labeled 0 or 3 are erased. The hard part was to show that this is well-defined.

Exempli gratia

$E_1 1_{(22)} \mapsto$



$F_2 E_1 1_{(121)} \mapsto$



Very nice bases of $W_{(3^\ell)}$

The $\dot{\mathbf{U}}(\mathfrak{sl}_n)$-module $W_{(3^\ell)}$ has different bases. But there are two particular nice ones, called Lusztig-Kashiwara's lower and upper global crystal basis $B_T = \{b_T\}$ and $B^T = \{b^T\}$ (sometimes also called canonical and dual canonical basis), indexed by standard tableaux $T \in \mathrm{Std}((3^\ell))$. One of its nice properties is for example

$$b_T = x_T + \sum_{\tau \prec T} \underbrace{\delta_{\tau T}(q)}_{\in \mathbb{Z}[q]} x_\tau$$

This gives, under q-skew Howe duality, a upper and lower global crystal basis of the invariant $\mathbf{U}_q(\mathfrak{sl}_3)$-tensors.

In contrast to the tensor basis x_T, which is easy to write down, but lacks a good behavior, the lower and upper global crystal bases are hard to write down, but have a good behavior.

An intermediate crystal basis

Leclerc and Toffin gave an intermediate crystal basis of $\dot{\mathbf{U}}(\mathfrak{sl}_n)$-modules, denoted by $LT_T = \{A_T\}$, by the rule

$$A_T = F_{i_s}^{(r_s)} \cdots F_{i_1}^{(r_1)} v_\Lambda, \text{ with } F^{(k)} = \frac{F^k}{[k]!},$$

where the string of $F's$ is obtained by an explicit, combinatorial algorithm from the tableau T. They showed that the crystal bases b_T and the tensor basis x_T are related by a unitriangular matrix

$$A_T = x_T + \sum_{\tau \prec T} \alpha_{\tau T}(q) x_\tau \text{ and } b_T = A_T + \sum_{S \prec T} \beta_{ST}(q) A_S,$$

with certain coefficients $\alpha_{\tau T}(q) \in \mathbb{N}[q, q^{-1}]$ and $\beta_{ST}(q) \in \mathbb{Z}[q, q^{-1}]$.

An intermediate crystal basis

Proposition(Mackaay, T)

The Kuperberg web basis B_S is Leclerc-Toffin's intermediate crystal basis under q-skew Howe duality, i.e.

$$LT_T = \{F_{i_s}^{(r_s)} \cdots F_{i_1}^{(r_1)} v_{3^\ell} \mid T \in \mathrm{Std}((3^\ell))\} \stackrel{\mathrm{sHD}}{\longmapsto} LT_S.$$

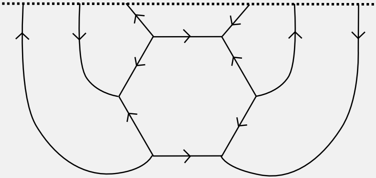
(No K's and E's anymore!)

Thus, the B_S is a good candidate for categorification (can be written down explicitly and has (some) good properties!).

Exempli gratia

Example

$w =$



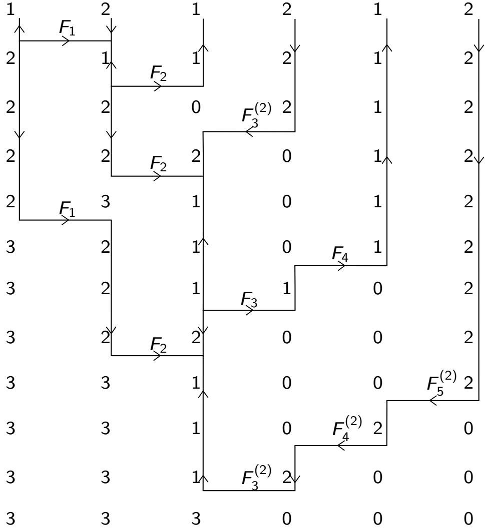
$$\leftrightsquigarrow \quad T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 & 5 \\ \hline 4 & 6 & 6 \\ \hline \end{array}$$

Form T we obtain the string

$$LT(w) = F_1 F_2 F_3^{(2)} F_2 F_1 F_4 F_3 F_2 F_5^{(2)} F_4^{(2)} F_3^{(2)}.$$

Exempli gratia

$$LT(w)v_{(3^3)} = F_1F_2F_3^{(2)}F_2F_1F_4F_3F_2F_5^{(2)}F_4^{(2)}F_3^{(2)}v_{(3^3)}$$



Please, fasten your seat belts!

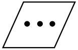
Let's categorify everything!

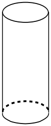
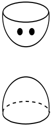
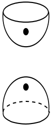
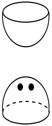
A pre-foam is a cobordism with singular arcs between two webs. Composition consists of placing one pre-foam on top of the other. The following are called the zip and the unzip respectively.

They have dots that can move freely about the facet on which they belong, but we do not allow dot to cross singular arcs.

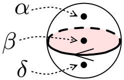
A foam is a formal $\mathbb{C}$-linear combination of isotopy classes of pre-foams modulo the following relations.

The foam relations $\ell = (3D, NC, S, \Theta)$

 $= 0$ (3D)

 $= -$  $-$  $-$  (NC)

 $=$  $= 0$,  $= -1$ (S)



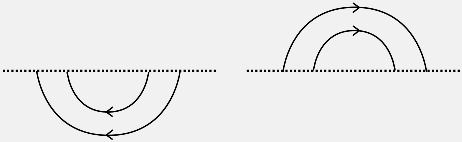
$$= \begin{cases} 1, & (\alpha,\beta,\delta) = (1,2,0) \text{ or a cyclic permutation,} \\ -1, & (\alpha,\beta,\delta) = (2,1,0) \text{ or a cyclic permutation,} \\ 0, & \text{else.} \end{cases} \qquad (\Theta)$$

Adding a closure relation to ℓ suffice to evaluate foams without boundary!

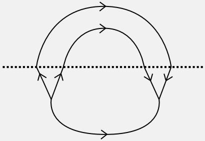
Involution on webs and closed webs

Definition

There is an involution * on the webs.



A closed web is defined by closing of two webs.



A closed foam is a foam from $\emptyset$ to a closed web.

The $\mathfrak{sl}_3$-foam category

Foam$_3$ is the category of foams, i.e. objects are webs w and morphisms are foams F between webs. The category is graded by the $q-$degree

$$q\mathrm{deg}(F) = \chi(\partial F) - 2\chi(F) + 2d + b,$$

where d is the number of dots and b is the number of vertical boundary components. The foam homology of a closed web w is defined by

$$\mathcal{F}(w) = \mathbf{Foam}_3(\emptyset, w).$$

$\mathcal{F}(w)$ is a graded, complex vector space, whose q-dimension can be computed by the Kuperberg bracket (that is counting all flows on w and their weights).

Definition(MPT)

Let $S = (s_1, \ldots, s_n)$. The $\mathfrak{sl}_3$-web algebra K_S is defined by

$$K_S = \bigoplus_{u,v \in B_S} {}_uK_v,$$

with

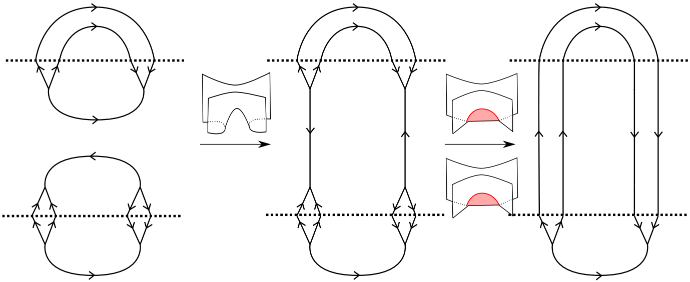
$${}_uK_v = \mathcal{F}(u^*v)\{n\}, \text{ i.e. all foams} \colon \emptyset \to u^*v.$$

Multiplication is defined as follows.

$${}_uK_{v_1} \otimes {}_{v_2}K_w \to {}_uK_w$$

is zero, if $v_1 \neq v_2$. If $v_1 = v_2$, use the multiplication foam m_v, e.g.

The $\mathfrak{sl}_3$-web algebra



Theorem(s)(MPT)

The multiplication is well-defined, associative and unital. The multiplication foam m_v has q-degree n. Hence, K_S is a finite dimensional, unital and graded algebra. Moreover, it is a graded Frobenius algebra.

Moreover, for $n = d = 3^k$ we define

$$W_{(3^k)} = \bigoplus_{\mu_s \in \Lambda(n,n)_3} W_S$$

on the level of webs and on the level of foams we define

$$\mathcal{W}^{(p)}_{(3^k)} = \bigoplus_{\mu_s \in \Lambda(n,n)_3} K_S - (\mathrm{p})\mathbf{Mod}_{gr}\,.$$

With this constructions we obtain our first categorification result.

Theorem(MPT)

$$K_0(\mathcal{W}_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \cong W_{(3^k)} \text{ and } K_0^{\oplus}(\mathcal{W}^p_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \cong W_{(3^k)}.$$

Cellular algebras have a “simple” representation theory

Definition(Graham-Lehrer, Hu-Mathas)

A graded cellular basis c^{λ}_{st} of a graded algebra A is a basis with nice structure coefficients (and other nice properties that we do not need today), i.e.

$$ac^{\lambda}_{st} = \sum_{u \in \mathcal{T}(\lambda)} r_a(s,u) c^{\lambda}_{ut} \pmod{A^{\rhd\lambda}},$$

where the λ's are from a poset $(\mathfrak{P}, \rhd)$ and $\mathcal{T}(\lambda)$ is finite for all $\lambda \in \mathfrak{P}$.

Theorem(Graham-Lehrer, Hu-Mathas)

For $\lambda \in \mathfrak{P}$ one can explicitly (using the structure coefficients) define the graded cell module C^{λ}. Set $D^{\lambda} = C^{\lambda}/\mathrm{rad}(C^{\lambda})$ and $\mathfrak{P}_0 = \{\lambda \in \mathfrak{P} \mid D^{\lambda} \neq 0\}$. Then the set $\{D^{\lambda}\{k\} \mid \lambda \in \mathfrak{P}_0,\ k \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic graded, simple A-modules. The same works for their projective covers!

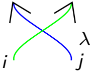
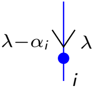
The approach

Recall that the intermediate crystal basis satisfies

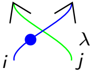
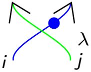
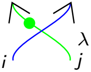
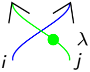
$$A_T = x_T + \sum_{\tau \prec T} \alpha_{\tau T}(q) x_\tau \text{ and } b_T = A_T + \sum_{S \prec T} \beta_{ST}(q) A_S.$$

Idea: If q-skew Howe duality can be used to obtain from the $\dot{\mathbf{U}}(\mathfrak{sl}_n)$-module $W_{(3^\ell)}$ the intermediate crystal basis on the level of webs, then categorified q-skew Howe duality can be used to obtain a cellular basis from a categorified intermediate crystal basis on the level of foams!

Khovanov and Lauda's diagrammatic categorification of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, denoted $\mathcal{U}(\mathfrak{sl}_n)$, is also related to our framework! Roughly, it consist of string diagrams of the form

 $: \mathcal{E}_i\mathcal{E}_j\mathbf{1}_\lambda \Rightarrow \mathcal{E}_j\mathcal{E}_i\mathbf{1}_\lambda\{(\alpha_i,\alpha_j)\}$,  $: \mathcal{F}_i\mathbf{1}_\lambda \Rightarrow \mathcal{F}_i\mathbf{1}_\lambda\{\alpha^{ii}\}$

with a weight $\lambda \in \mathbb{Z}^{n-1}$ and suitable shifts and relations like

 $=$  and  $=$ , if $i \neq j$.

$\mathfrak{sl}_3$-foamation

We define a 2-functor

$$\Psi\colon \mathcal{U}(\mathfrak{sl}_n) \to \mathcal{W}^{(p)}_{(3^k)}$$

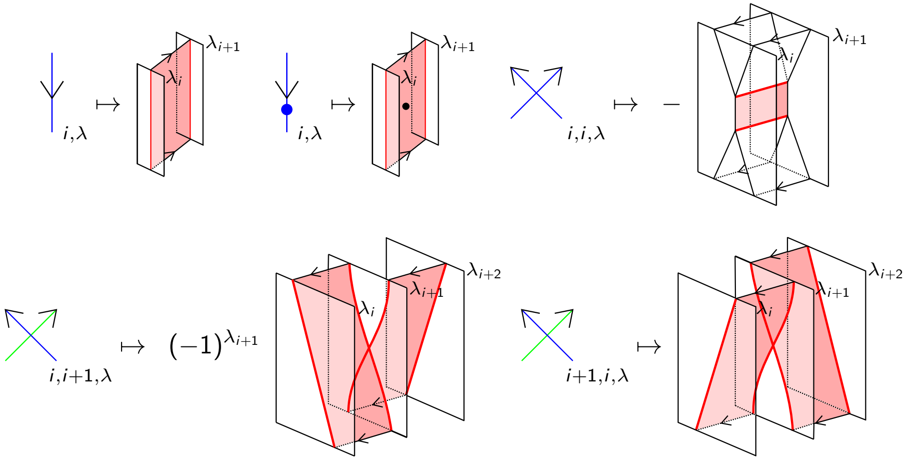
called foamation, in the following way.

On objects: The functor is defined by sending an $\mathfrak{sl}_k$-weight $\lambda = (\lambda_1, \ldots, \lambda_{k-1})$ to an object $\Psi(\lambda)$ of $\mathcal{W}^{(p)}_{(3^k)}$ by

$$\Psi(\lambda) = S,\ S = (a_1, \ldots, a_k),\ a_i \in \{0,1,2,3\},\ \lambda_i = a_{i+1} - a_i,\ \sum_{i=1}^{k} a_i = 3^k.$$

On morphisms: The functor on morphisms is by glueing the ladder webs from before on top of the $\mathfrak{sl}_3$-webs in $W_{(3^k)}$.

On 2-**cells:** We define



And some others that we do not need today.

The idea!

Let $\lambda \in \Lambda(n,n)^+$ be a dominant weight. Define the cyclotomic KL-R algebra R_λ to be the subquotient of $\mathcal{U}(\mathfrak{sl}_n)$ defined by the subalgebra of only downward (only F's!) pointing arrows modulo the so-called cyclotomic relations and set $\mathcal{V}_\lambda = R_\lambda - (\mathrm{p})\mathbf{Mod}_{gr}$. The cyclotomic KL-R algebra R_λ is isomorphic to a certain cyclotomic Hecke algebra H_λ of type A.

Theorem(s)(MPT)

There exists an equivalence of categorical $\mathcal{U}(\mathfrak{sl}_n)$-representations

$$\Phi\colon \mathcal{V}^{(p)}_{(3^k)} \to \mathcal{W}^{(p)}_{(3^k)}.$$

Idea(T)

The combinatorics can be easier worked out in the cyclotomic Hecke algebra H_λ, while the topology is easier in our framework. Use foamation to pull Hu-Mathas graded cellular basis from H_λ to K_S.

A growth algorithm for foams

Definition(T)

Given a pair of a sign string and a state string (S, J), the corresponding 3-multipartition $\vec{\lambda}$ and two Kuperberg webs $u, v \in B_S$ that extend J to f_u and f_v receptively. We define a foam by

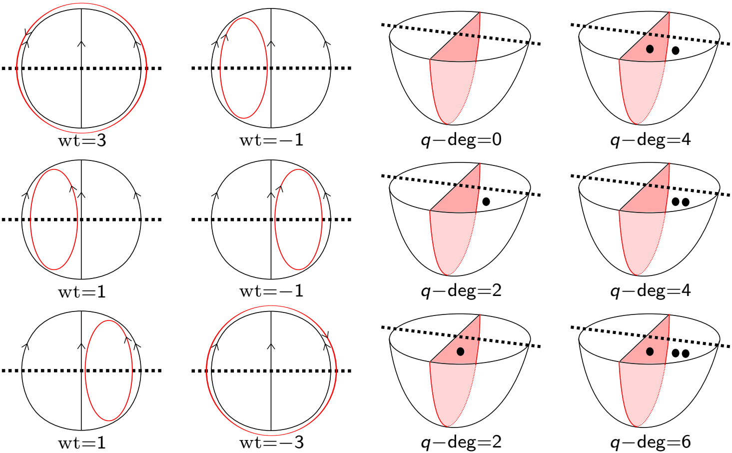
$$\mathcal{F}^{\vec{\lambda}}_{\vec{T}(u_{f_u}), \vec{T}(v_{f_v})} = \underbrace{\mathcal{F}_{\sigma_u}}_{\text{Topology}} \underbrace{e(\vec{\lambda})}_{\text{Idempotent}} \underbrace{d(\vec{\lambda})}_{\text{Dots}} \underbrace{\mathcal{F}^*_{\sigma_v}}_{\text{Topology}} .$$

Theorem(T)

The growth algorithm for foams is well-defined, the only input data are webs and flows on webs, works inductively and gives a graded cellular basis of K_S.

Exempli gratia

Every web has a graded cellular basis parametrised by flow lines.



That these foams are really a graded cellular basis follows from our theorem. Note that the Kuperberg bracket gives $[2][3]=q^{-3}+2q^{-1}+2q+q^{3}$.

Categorification and crystal bases

Recall our first categorification result, i.e.

$$K_0(\mathcal{W}_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong W_{(3^k)} \text{ and } K_0^{\oplus}(\mathcal{W}^p_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong W_{(3^k)}.$$

A natural question is how do the two nice bases, i.e. the lower $\{b_T\}$ and the upper $\{b^T\}$ global crystal basis, of $W_{(3^k)}$ show up in $K_0(\mathcal{W}_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$ or $K_0^{\oplus}(\mathcal{W}^p_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$?

Recall that a graded cellular basis $\{c^{\lambda}_{st}\}$ gives rise to a set of graded cell modules $\{C^{\lambda}\}$, their simple heads $\{D^{\lambda} = C^{\lambda}/\mathrm{rad}(C^{\lambda})\}$ and the corresponding projective covers $\{C^{\lambda}_p\}$ and $\{D^{\lambda}_p\}$.

Categorification and crystal bases

Theorem(T)

We have $\psi([D^T]) = b_T$ and $\psi_p([D_p^T]) = b^T$ under the two isometries

$$\psi\colon K_0(\mathcal{W}_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \to W_{(3^k)} \text{ and } \psi_p\colon K_0^{\oplus}(\mathcal{W}^p_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \to W_{(3^k)},$$

that is the simple heads D^T of the cell modules C^T (who give a complete list of all simple K_S-modules) categorify the lower global crystal basis b_T and their projective covers D_p^T (who give a complete list of all projective, irreducible K_S-modules) categorify the upper global crystal basis b^T.

There is still much to do...

Thanks for your attention!