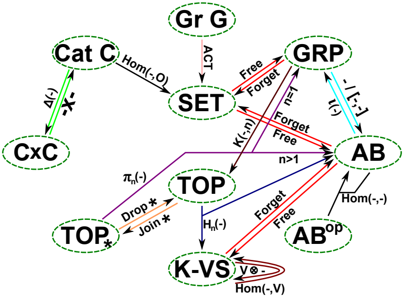
Why category theory?

Daniel Tubbenhauer - 02.04.2012

Georg-August-Universität Göttingen



A generalisation of well-known notions

	Closed	Total	Associative	Unit	Inverses
Group	Yes	Yes	Yes	Yes	Yes
Monoid	Yes	Yes	Yes	Yes	No
Semigroup	Yes	Yes	Yes	No	No
Magma	Yes	Yes	No	No	No
Groupoid	Yes	No	Yes	Yes	Yes
Category	No	No	Yes	Yes	No
Semicategory	No	No	Yes	No	No
Precategory	No	No	No	No	No

Euler's polyhedron theorem

Leonard Euler
(15.04.1707-18.09.1783)

Polyhedron theorem (1736)

Let $P \subset \mathbb{R}^3$ be a convex polyhedron with V vertices, E edges and F faces. Then:

$$\chi = V - E + F = 2.$$

Here χ denotes the Euler characteristic.

Euler's polyhedron theorem

Polyhedron	**E**	**K**	**F**	χ
Tetrahedron	4	6	4	2
Cube	8	12	6	2
Oktahedron	6	12	8	2
Dodekahedron	20	30	12	2
Isokahedron	12	30	20	2

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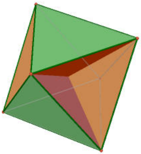

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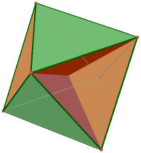
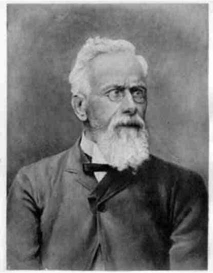


- A more general version would be nice!

Two important mathematicians

Georg Friedrich Bernhard Riemann
(17.09.1826-20.07.1866)



Enrico Betti
(21.10.1823-11.08.1892)

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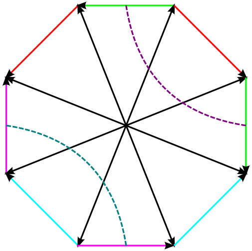

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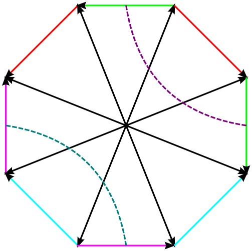

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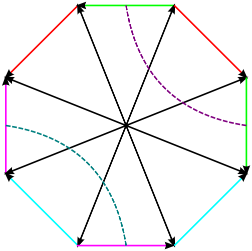

- From a modern viewpoint $\mathcal{Z} = 2\dim \mathrm{H}_1(S, \mathbb{Z}/2)$ and the interaction between the cuts and the curves is a first hint for the Poincaré duality.

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- His construction dependent on the choice for the basis of $\mathbb{R}^3$.
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Two fundamental concepts of topology

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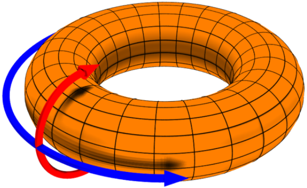

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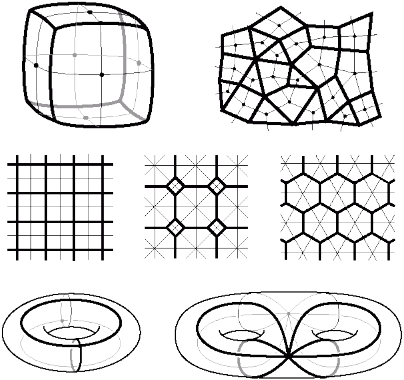

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- But it takes quite long and lots of theorems have a complicated proof.

The Göttingen connection

Amalie Emmy Noether
(23.03.1882-14.05.1935)

Heinz Hopf
(19.11.1894-03.06.1971)

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- From today's perspective one would say that Emmy Noether and Heinz Hopf categorified the notion „Bettinumber“ .

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Here $\delta_i \circ \delta_{i+1} = 0$. Therefore he was allowed to define $\mathrm{H}_i(\cdot) = \ker(\delta_i)/\mathrm{im}(\delta_{i+1})$. The Euler characteristic becomes the alternating sum $\sum_k (-1)^k \mathrm{rk}(\mathrm{H}_k(\cdot))$.

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$$\Lambda_f = \sum_{k \geq 0} (-1)^k \mathrm{Tr}(\mathrm{H}_k(f, \mathbb{Q}) \colon \mathrm{H}_k(X, \mathbb{Q}) \to \mathrm{H}_k(X, \mathbb{Q}))$$

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The proof is of course impossible without the maps (morphisms) between the groups.

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Fundamental theorem of algebra

Fundamental theorem of algebra (folklore)

Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with $n > 0$ and $a_k \in \mathbb{C}$. Then p has a root in $\mathbb{C}$.

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Beweis.

We have $\mathrm{H}_1(S^1) = \mathbb{Z}$ and the only group homomorphisms $\mathbb{Z} \to \mathbb{Z}$ are multiplication with $\pm n$.
Moreover $\mathrm{H}_1(z \to z^n) = \cdot n$ is the multiplication with n for all $n \in \mathbb{N}$. So we assume p has no root.

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Beweis.

We define $H, H' \colon S^1 \times [0,1] \to S^1$ by

$$H_t(z) = \frac{p(tz)}{|p(tz)|} \text{ und } H'_t(z) = \frac{(1-t)H_t(z) + tz^n}{|(1-t)H_t(z) + tz^n|}$$

(it is easy to show that both denominators never become zero if p has no roots!) two homotopies from the constant map to p and from p to $z \to z^n$.
This is a contradiction because we get
$\cdot 0 = \mathrm{H}_1(\mathrm{const}) = \mathrm{H}_1(z \to z^n) = \cdot n.$

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- The two points are even more important for higher categories.

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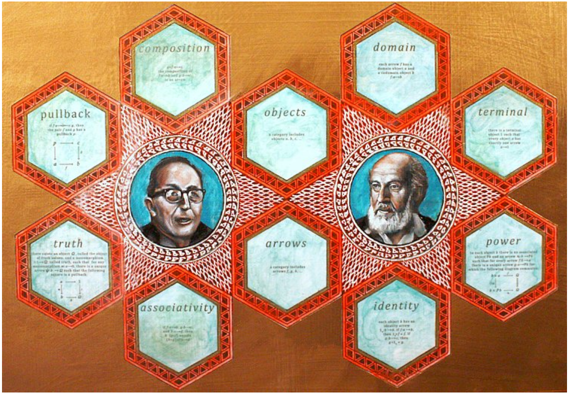
- But much more...

Two historical figures



Left: Saunders Mac Lane (04.08.1909-14.04.2005)
Right: Samuel Eilenberg (30.09.1913-30.01.1998)

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The title of their paper already suggests that they were more interested in natural transformations then in categories. But they invented the natural transformation "just" to study effects in homological algebra (e.g. effects involving homology groups $\mathrm{H}_n(\cdot)$).

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The first appearance of the notion "category" in Samuel Eilenbergs and Saunders Mac Lanes paper „General Theory of Natural Equivalences" (1945) came almost out of nowhere. There was only one and restricted to groups notation in the year 1942 in one of their papers.
The title of their paper already suggests that they were more interested in natural transformations then in categories. But they invented the natural transformation "just" to study effects in homological algebra (e.g. effects involving homology groups $\mathrm{H}_n(\cdot)$).
The notions "functor", "limes" and "colimes" also appeared in the paper for the first time.

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They already denoted that the objects could be almost omitted and that the morphisms are much more important.
In the year 1945 it was not clear that category theory is more then just a good syntax to describe effects in homological algebra, e.g. the notation groupoid for $\pi_n(\cdot)$ (without a base point).

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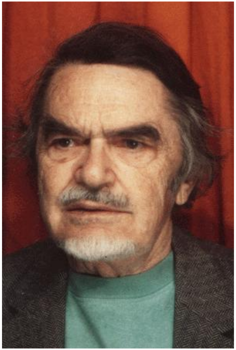
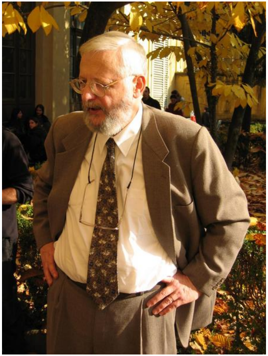
- Grothendieck used the notations for the first time outside of homological algebra, i.e. in algebraic geometry (1957);
- very influential was the deductive definition by Lambek and Lawvere. Their notions got widespread around 1960 because of their universal elegance.

Vertices and arrows



Joachim Lambek
(05.12.1922-ongoing)



Francis William Lawvere
(09.02.1937-ongoing)

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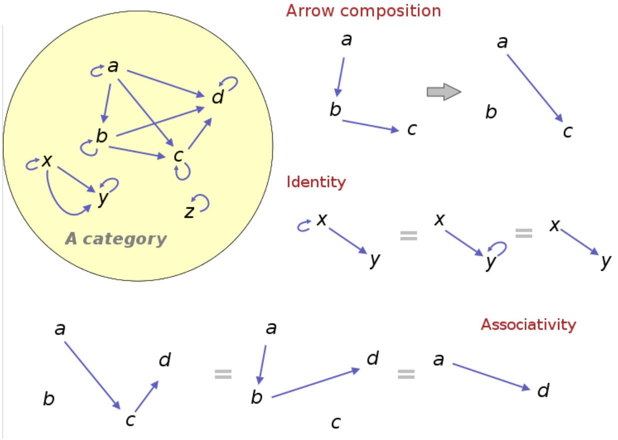
Arrows are for example not necessary maps but logical symbols.
The calculus only gets a concrete interpretation by a model.
This is much more descriptive and shows the idea behind category theory direct: hunt diagrams and find universal vertices/arrows.

Vertices and arrows

A good example for a category from their point of view is:



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Only one little step was missing to establish the category theory as independent research field.
This one last step was probably Dan Kan's observation that so-called adjunctions appear everywhere in mathematics.

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These extrema, i.e. equality and "all is equal", are almost always to fine or to course. A reasonable notions is in between.

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Hence, in a lot of cases there is no such thing like a unique answer, just a "good" one.
What is a "good" notion for category theory?

A "good" equivalence

Daniel Marinus Kan
(??-ongoing)

Dan Kan's answer (1958)

Isomorphic functors almost never appear. Natural equivalence is what we want but adjunctions is what we mostly get.

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Daniel Marinus Kan (called Dan Kan) defined in his paper „Adjoint Functors“ (1958) the notion of adjoint equivalence of functors. This notions becomes central for category theory in the following years. And that although it was overlooked by everyone until then. One could, casually speaking, say that isomorphisms equal isotopies, natural equivalences equals homoemorphisms and adjunctions equals homotopies.

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Indeed: the free group! We denote with F the functor which associates a set to its corresponding free group.

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Or different: F is the best approximation to an inverse of V. This motivated Dan Kan to define adjunctions, i.e. a pair of functors F, G together with an unit and counit and natural isomorphisms between Hom(F-,-) und Hom(-,G-).

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Thus it is a crucial question which functors have adjoints.

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- Kan-extensions and simplicial sets by Dan Kan (1960);

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A very interesting connection is shown in next next section.

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Monoidal categories

Let $\mathcal{C}$ a category together with a functor, called tensor product, $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. He fixed four classes of natural isomorphisms and an object 1. Let x, y, z be objects of $\mathcal{C}$.

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He had proven the following theorem. This is some kind of justification for the carelessness of mathematicians.

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But there is also a problem with his observation:

Monoidal categories

Mac Lane's coherence theorem

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the ring of matrices already shows that commutativity is not as natural as associativity.

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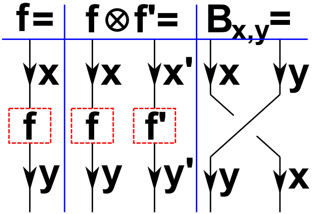

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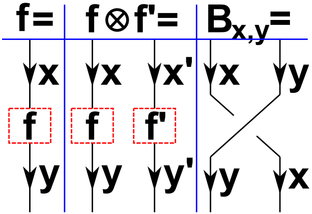

We mention that monoidal categories have, in contrast to "usual" categories, a two dimensional structure, i.e. horizontal (standard) and vertical (tensor) composition.

Braids and category theory

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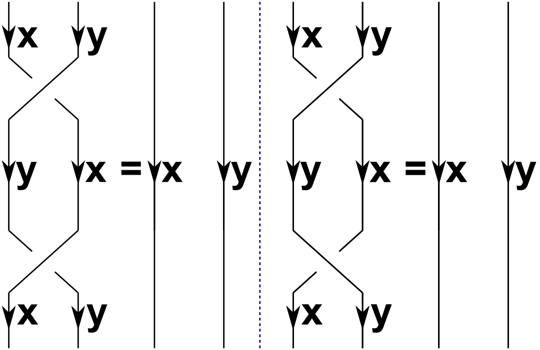

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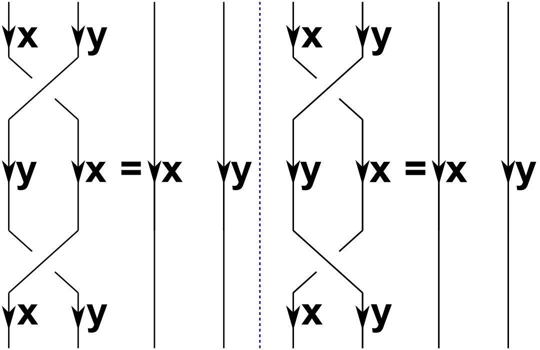

The left equation is in three dimensions false in general because otherwise every knot would be trivial.

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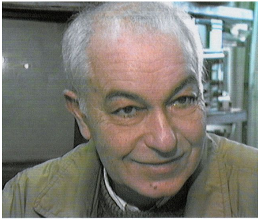
The detailed study of categorical structures has proven useful once again.

2-categories



Jean Bénabou
(03.06.1932-ongoing)

Jean Bénabou (1967)

The monoidal categories are two dimensional but rarely strict. Hence, the two dimensional composition should be defined only up to 2-isomorphisms.

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objects	1-morphisms	2-morphisms
$x\bullet$	$x\bullet \xrightarrow{f} \bullet y$	$\begin{tikzcd} x\bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet y \end{tikzcd}$

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\end{tikzcd}
\qquad
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$$\begin{tikzcd} x \arrow[rr, bend left=60, "f"] \arrow[rr, "g"] \arrow[rr, bend right=60, "h"'] & & y \end{tikzcd} \qquad \alpha: f \Rightarrow g, \quad \beta: g \Rightarrow h$$

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This suggests that one could imagine categories on a pure pictorial scale. Categories have a combinatorial structure and 2-categories have an additional topological structure.

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But the two examples above satisfy unit and associativity direct - a really rare phenomena.
Let us mention a nicer example, i.e. **BiMOD**.

Examples

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Hence, unit and associativity only true up to isomorphisms.

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That's why 2-categories are studied by lots of people until today.

Grothendieck's dream

Alexandre Grothendieck
(28.03.1928-ongoing)

Grothendieck's dream (1983)

Let X be a topological space. Then there is a category $\prod_\omega(X)$, called fundamental ω-groupoid, which is a complete invariant of the homotopy type of X.

n-categories

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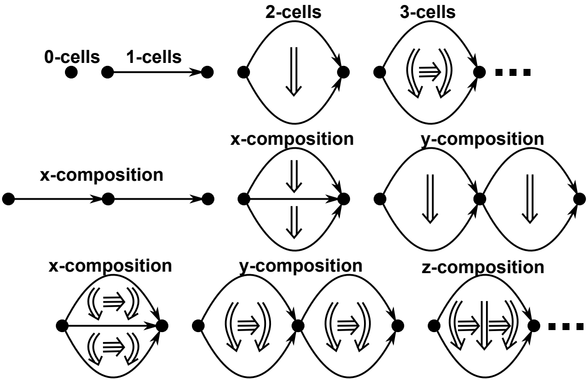

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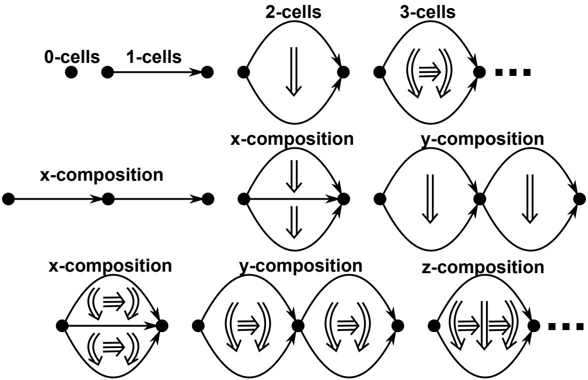

Again everything is only up to some kind of n-isomorphisms defined. But we mention that there is not an unique approach for the definition, i.e. there are more definitions by different authors.

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We note that in this example all is only up to some kind of equivalence (homotopies) defined, e.g. even the composition of paths is only up to homotopies associative.

More interesting examples

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If the world is fair then there should be a weak ω-functor $\prod_\omega$. Because of even more interesting examples *n*-categories were and are intensively studied.

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	n=0	n=1	n=2
m=0	sets	categories	2-categories
m=1	monoids	monoidal cat.	monoidal 2-cat.
m=2	comm. monoids	braided cat.	braided 2-cat.
m=3	"	sym. mon. cat.	sylleptic 2-cat.
m=4	"	"	sym. mon. 2-cat.
m=5	"	"	"

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Beweis.

We set $n = 0$ and $m = k$ in the periodic system. For example $\pi_2(X, x)$ contains one point x ($i = 0$), the constant loop ($i = 1$) and continuous maps $[0, 1]^2 \to X$. □

There is still much to do...

Thanks for your attention!