Categorification and (virtual) knots

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If you really want to understand something - (try to) categorify it!

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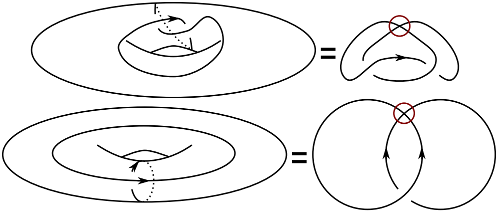


What is categorification?

Categorification is a scary word, but it refers to a very simple idea and is a huge business nowadays. If I had to explain the idea in one sentence, then I would choose

Some facts can be best explained using a categorical language.

Do you need more details?
Categorification can be easily explained by two basic examples - the categorification of the natural numbers through the category of finite sets **FinSet** and the categorification of the Betti numbers through homology groups.
Let us take a look on these two examples in more detail.

Let us consider the category **FinSet** - objects are finite sets and morphisms are maps between these sets. The set of isomorphism classes of its objects are the natural numbers $\mathbb{N}$ with 0.
This process is the inverse of categorification, called decategorification - the spirit should always be that decategorification should be simple while categorification could be hard.
We note the following observations.

Finite Combinatorics and counting

- Much information is lost, i.e. we can only say that two objects are isomorphic instead of how they are isomorphic.
- The extra structure of the natural numbers (they form a so-called commutative ordered rig) is decoded in the category **FinSet**, e.g:
  - The product and coproduct in **FinSet** are the Cartesian product and the disjoint union and we have $|X \times Y| = |X| \cdot |Y|$ and $|X \amalg Y| = |X| + |Y|$, i.e. they categorify multiplication and addition.
  - The category has $\emptyset$ and $\{*\}$ as initial and terminal objects and we have $X \amalg \emptyset \simeq X$ and $X \times \{*\} \simeq X$, i.e. we can even categorify the identities.
  - We have $X \hookrightarrow Y$ iff $|X| \leq |Y|$ and $X \twoheadrightarrow Y$ iff $|X| \geq |Y|$, i.e. injections and surjections categorify the order relation.

Finite Combinatorics and counting

One can write down the categorified statements of each of following properties. If you are really up for a challenge, show that all the isomorphisms are natural.

- Addition and multiplication are associative.
- Addition and multiplication are commutative.
- Multiplication distributes over addition.
- Addition and multiplication preserve order.

Hence, we can say the following.

Theorem(Folklore)

Finite combinatorics, i.e. the category **FinSet** is a categorification of finite arithmetic, i.e. the commutative, ordered rig $\mathbb{N}$.

Another well-known example

Theorem(Noether, Hopf, Mayer)

Let X be a reasonable finite-dimensional spaces. Then the homology groups $H_k(X)$ are a categorification of the Betti numbers of X and the singular chain complex (C, d_i) is categorification of the Euler characteristic of X.

To be a little bit more precise, we give the category $\mathcal{C}$ such that the isomorphism classes of objects $\mathrm{DECAT}(\mathcal{C})$ gives a functor $\mathrm{decat}\colon \mathrm{DECAT}(\mathcal{C}) \to \mathcal{D}$.

- In the first case take $\mathcal{C} =$**FinVec**$_K$, i.e. the category of finite dimensional vector spaces over a field K, and $\mathcal{D} = \mathbb{N}$ and $\mathrm{decat}(V) = \dim V$.
- In the second case take $\mathcal{C} =$**FinChain**, i.e. the category of finite chain complexes, and $\mathcal{D} = \mathbb{Z}$ and $\mathrm{decat}(C) = \chi(C)$.

Summary of the examples

Note the following common features of the two examples above.

- The natural numbers and the Betti numbers/Euler characteristic can be seen as parts of "bigger, richer" structures.
- In both categorifications it is very easy to decategorify.
- Both notions are not obvious, e.g. the first notion of "Betti numbers" was in the year 1857 (B. Riemann) and the first notion of "homology groups" was in the year 1925.
- Note that the two categories $\mathcal{C} =$**FinSet** and $\mathcal{C} =$**FinVec**$_K$ can be seen as a categorification of the natural numbers, i.e. categorification is not unique. We will use the second today since it can be naturally extended to $\mathcal{C} =$**FinChain** and can be seen as a categorification of the integers $\mathbb{Z}$.

Of course, there exit more "fancier" examples of categorification.

The framework

The idea of categorification can be summarised in the following table.

Set based mathematics	**Categorification**
Elements	Objects
Equations between elements	Isomorphisms between objects
Sets	Categories
Functions	Functors
Equations between functions	Natural isomorphisms between functors

The ladder of categories

$$
\begin{tikzcd}[row sep=large]
\vdots \arrow[d, bend right, "\text{forget 3-arrows}"'] & \vdots & \vdots \\
{} \arrow[d, bend right, "\text{forget 2-arrows}"'] & 2-\text{categories} = \text{“arrows between arrows”} & {} \arrow[u, bend right, "\text{add 3-arrows}"'] \\
{} \arrow[d, bend right, "\text{forget arrows}"'] & 1-\text{categories} = \text{usual categories} & {} \arrow[u, bend right, "\text{add 2-arrows}"'] \\
{} \arrow[d, bend right, "\text{size}"'] & 0-\text{categories} = \text{sets, vector spaces} & {} \arrow[u, bend right, "\text{add arrows}"'] \\
{} & -1-\text{categories} = \text{cardinals, truth values} & {} \arrow[u, bend right, "\text{internal structure}"']
\end{tikzcd}
$$

A pun about categorification - “flatland”

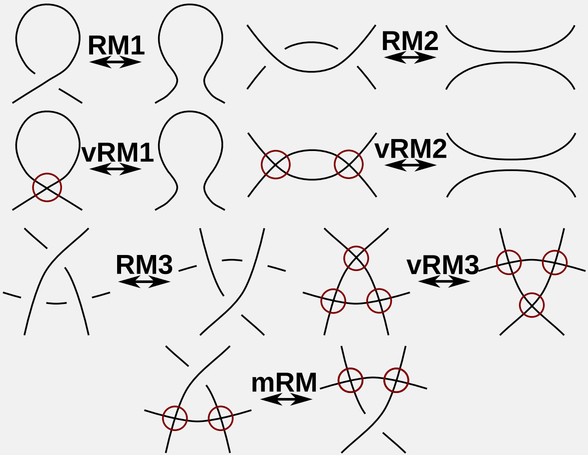
If you live in a two-dimensional world, then it is easy to imagine a one-dimensional world, but hard to imagine a three-dimensional world!

Definition

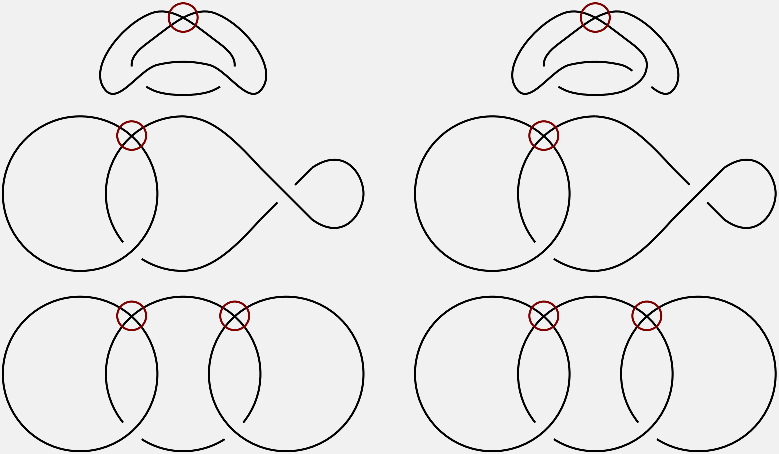
A virtual knot or link diagram L_D is a four-valent graph embedded in the plane. Moreover, every vertex is marked with an overcrossing, an undercrossing or a virtual crossing.
We call such a diagram without over- and undercrossings classical. An oriented virtual knot or link diagram is defined in the obvious way.
A virtual knot or link L is an equivalence class of virtual knot or link diagrams modulo the so-called generalised Reidemeister moves. An oriented virtual knot or link is defined in the obvious way.

Classical and virtual knots and links

Generalised Reidemeister moves



Example(the so-called basic faces)

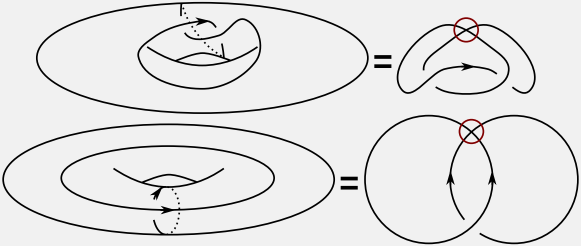


More(?) then combinatorial nonsense

Theorem(Kauffman, Kuperberg)

Let Σ_g denote a surfaces with genus g. Virtual knots and links are a combinatorial description of copies of S^1 embedded in $\Sigma_g \times [0,1]$. Two such links are equivalent iff there projections to Σ_g are stable equivalent, i.e. up to homeomorphisms of surfaces, adding/removing "unimportant" handles and classical Reidemeister moves and isotopies of the projections.

Example(virtual trefoil and virtual Hopf link)



Goal: Find a "good" invariant

The obvious question is, given two virtual link diagrams L_D, L'_D, if they are equivalent or not. Since the combinatorial complexity of virtual links is much higher then for classical links, every invariant is helpful. Not much is known at the moment.

There are much more virtual links then classical links:

	$n \leq 3$	$n = 4$	$n = 5$	$n = 6$
classical	2	3	5	8
virtual	8	109	2448	90235

The number of different knots with n crossings.

The famous (virtual) Jones polynomial

In the mid eighties V. Jones found an amazing invariant of classical knots and links, the so-called Jones polynomial. V. Jones original description came from the study of von-Neumann algebras.

The Jones polynomial is simple, strong and connects to different branches of mathematics and physics, e.g. N. Reshetikhin, V. Turaev (and others) found a connection, using R. Kirby's calculus, to representation theory of the quantum group $U_q(\mathfrak{sl}_2)$ and invariants of 3-manifolds and E. Witten (and others) found a connection to quantum physics and L. Kauffman found a relation to the Tutte-polynomial.

We give a combinatorial exposition of the (virtual) Jones polynomial found by L. Kauffman.

The famous (virtual) Jones polynomial

Let L_D be an oriented link diagram. The bracket polynomial $\langle L_D \rangle \in \mathbb{Q}[q, q^{-1}]$ can be recursively computed by the rules:

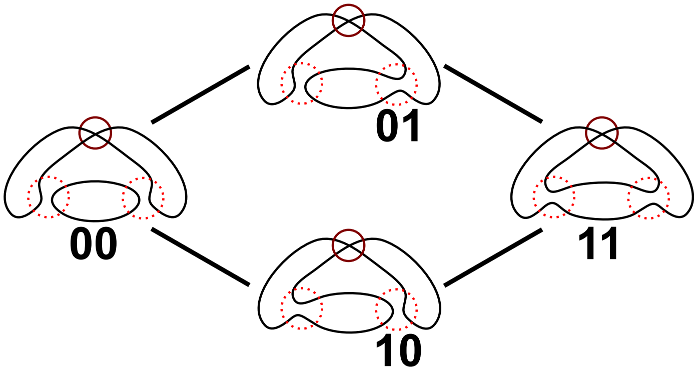
- $\langle \emptyset \rangle = 1$ (normalisation).
- $\langle \; \rangle = \langle \; \rangle - q\langle \; \rangle$ (recursion step 1).
- $\langle \text{Unknot} \amalg L_D \rangle = (q + q^{-1})\langle L_D \rangle$ (recursion step 2).

The Kauffman polynomial is $K(L_D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L_D \rangle$, with $n_+ =$number and $n_- =$number of).

Theorem(Kauffman)

The Kauffman polynomial $K(L)$ is an invariant of virtual links and $K(L) = \hat{J}(K)$, where $\hat{J}(K)$ denotes the unnormalised Jones polynomial.

Exempli gratia



The Jones polynomial for the virtual trefoil T can be computed easily from the cube shape. The 00 component gives $(q+q^{-1})^2$, 01 and 10 give $-q(q+q^{-1})$ and the 11 gives $q^2(q+q^{-1})$. Hence, the normalised Kauffman polynomial yields $K(T) = q + 1 - q^{-2}$.

More complicated representation theory

Indeed, the (virtual) Jones polynomial is related to the quantum group $U_q(\mathfrak{sl}_2)$. The same principle can also be done for the quantum group $U_q(\mathfrak{sl}_n)$, but the relations change to the so-called Skein relations. To be precise, we denote by $P_n(L_D)$ the n-th normalised HOMFLY polynomial.

- $P_n(\diagup\!\!\!\!\diagdown) = q^{n-1}P_n(\uparrow\uparrow) - q^n P_n(\asymp)$ and
  $P_n(\diagdown\!\!\!\!\diagup) = q^{1-n}P_n(\uparrow\uparrow) - q^{-n} P_n(\asymp)$ (local rules).
- Some relations to evaluate crossing-free trivalent graphs (we do not need them today).

Note that only in the case $n = 2$ one can simplify the Skein relations as shown before, i.e. avoiding trivalent vertices.
The relation of knot polynomials and representation of quantum groups is very deep and rich and is not restricted to $\mathfrak{sl}_n$.

Please, fasten your seat belts!

Let's categorify everything!

What we want - reverse engineering

We want to categorify a polynomial in $\mathbb{Z}[q, q^{-1}]$, i.e. what every the decategorification functor $\mathrm{decat}(\cdot)$ turns out to be, it should give us a polynomial in $\mathbb{Z}[q, q^{-1}]$. Note the following.

- For polynomials with coefficients in $\mathbb{N}$ we can use an enriched version of the categorification of the Betti numbers, i.e. we take $\mathcal{C} =$**grFinVec**$_K$, i.e. the category of finite dimensional, graded vector spaces, and

$$\mathrm{decat}(V = \bigoplus_{i\in\mathbb{Z}} V_i) = \mathrm{grdim}\, V = \sum_{i\in\mathbb{Z}} q^i \dim V_i.$$

- If the coefficients are in $\mathbb{Z}$, then we can use an enriched version of the categorification of the Euler characteristic, i.e. we take $\mathcal{C} =$**grFinChain**, i.e. the category of finite, graded chain complexes and $\mathrm{decat}(C) = \chi_q(C)$.

What we want - reverse engineering

Let us take a look at the decategorified skein relations. The first one is just a normalisation, so we start with the last one, i.e.

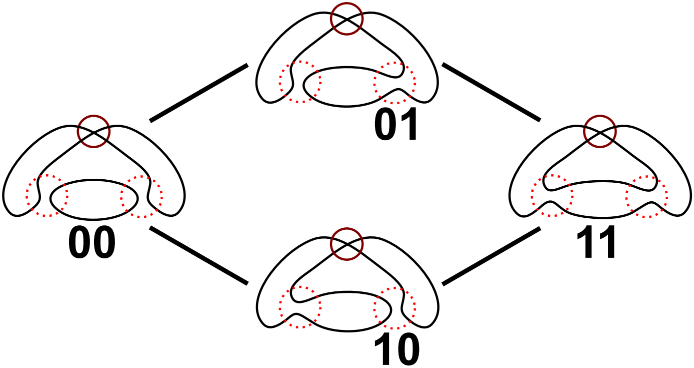
$$\langle \text{Unknot} \amalg L_D \rangle = (q + q^{-1}) \langle L_D \rangle.$$

So, for any classical crossing-free diagram of of an unknot $\bigcirc$ we assign the graded vector space $A = \mathbb{Q}[X]/(X^2 = 0)$ with $\deg 1 = 1, \deg X = -1$. In the same vain, we assign to n-copies of $\bigcirc$ the space $\bigotimes_n A$.
The first one, i.e the relation

$$\langle \diagup\!\!\!\!\diagdown \rangle = \langle)\,(\rangle - q\langle \asymp \rangle$$

can be seen as a degree shift $A\{t\}$ that depends on the homology degree t.

Exempli gratia



We read now $A \otimes A$ for the first, $A\{1\} \oplus A\{1\}$ for the second and $A\{2\}$ for the last component.

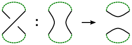
Hence, to turn this into a graded chain complex only the differential are missing.

What we want - reverse engineering

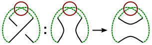
Let us see, what kind of maps we expect. There are three different types.

- For a crossing of the form  we need a multiplication $m\colon A \otimes A \to A$. We set

$$m(1 \otimes 1) = 1, m(1 \otimes X) = X = m(X \otimes 1) \text{ and } m(X \otimes X) = 0.$$

- For a crossing of the form  we need a comultiplication $\Delta\colon A \to A \otimes A$. We set

$$\Delta(1) = 1 \otimes X + X \otimes 1 \text{ and } \Delta(X) = X \otimes X.$$

- For a crossing of the form  we need a map $\theta\colon A \to A$. We set $\theta = 0$.

Note that all the maps shift the degree by -1. Hence, that is exactly what we need to get a graded chain complex at the end.

What we want - reverse engineering

It turns out that, in order to ensure that the differential d satisfies $d^2 = 0$, we need another map $\Phi\colon A \to A$ with $\Phi(1) = 1$ and $\Phi(X) = -X$. Moreover, one has a birth $\iota\colon R \to A$ and death $\varepsilon\colon A \to R$ map. Hence, A together with the sextuple $(m, \Delta, \theta, \Phi, \iota, \varepsilon)$ forms a skew-extended Frobenius algebra, i.e. a Frobenius algebra together with an element $\theta \in A$ and a skew-involution $\Phi\colon A \to A$. Note the name skew-involution, because

$$m \circ \Phi \otimes \Phi = \Phi \circ m, \text{ but } \Phi \otimes \Phi \circ \Delta = -\Delta \circ \Phi.$$

It is well-known, that Frobenius algebras and two dimensional TQFTs are the "same". It turns out that skew-extended Frobenius algebras and two dimensional possible unorientable TQFTs are the "same", i.e.

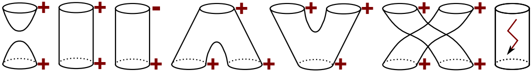
Theorem(T)

The category of (1+1)-dimensional uTQFTs and the category of skew-extended Frobenius algebras are equivalent.

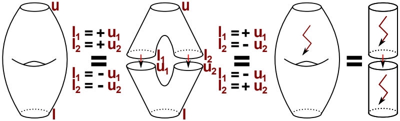
A cobordisms approach

The pre-additive monoidal category $\mathbf{uCob}^2{}_R(\emptyset)$ of possible unorientable, decorated cobordisms has

- Objects are resolutions of virtual link diagrams, i.e. virtual link diagrams without classical crossings.
- Morphisms are decorated cobordisms immersed into $\mathbb{R}^2 \times [-1,1]$ generated by (last is a two times punctured $\mathbb{RP}^2$)



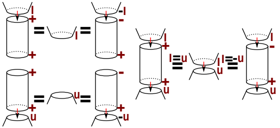
- Some relations like (last one is a two times punctured Klein bottle)



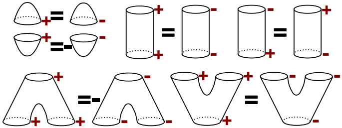
- The monoidal structure is given by the disjoint union.

More important relations

Two other important relations are



and



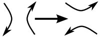
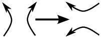
Note that these three relations ensure, that the chain complex will be well-defined.

How to form a chain complex

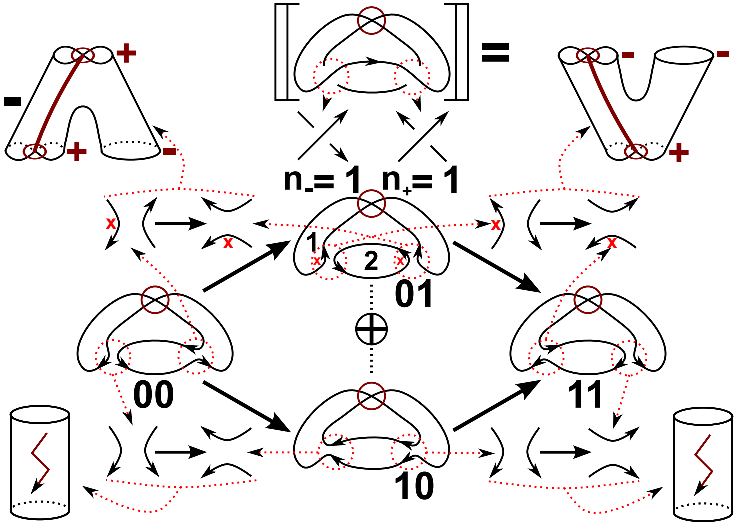
Define the category $\mathrm{Mat}(\mathbf{uCob}^2{}_R(\emptyset))$ to be the category of matrices over the category $\mathbf{uCob}^2{}_R(\emptyset)$, i.e. objects are formal direct sums of the objects of $\mathbf{uCob}^2{}_R(\emptyset)$ and morphisms are matrices whose entries are morphisms from $\mathbf{uCob}^2{}_R(\emptyset)$.
Define the category $\mathbf{uKob}_R(\emptyset)$ to be the category of chain complexes over the category $\mathrm{Mat}(\mathbf{uCob}^2{}_R(\emptyset))$. Note that we assume that the category is pre-additive. Hence, the notion $d^2 = 0$ makes sense.
As a reminder, to every virtual link diagram L_D we want to assign an object in $\mathbf{uKob}_R(\emptyset)$ that is an invariant of virtual links. By construction, this invariant will decategorify to the virtual Jones polynomial.

How to form a chain complex

For a virtual link diagram L_D with $n = n_+ + n_-$ crossings the geometric complex should be:

- For $i = 0, \dots, n$ the $i - n_-$ chain module is the formal direct sum of all resolutions of length i.
- Between resolutions of length i and $i+1$ the morphisms should be saddles between the resolutions.
- The decorations for the saddles can be read of by choosing an orientation for the resolutions. Locally they look like , which is called standard. Now compose with Φ iff the orientations differ or if both are non-alternating  use θ.
- Extra formal signs - placement is rather complicated and skipped today.

The complex for an unknot diagram



Everything is well-defined

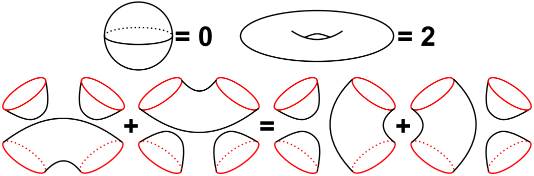
Note that it is not obvious that this definition over a ring of characteristic $\neq 2$ gives a well-defined chain complex. Moreover, a lot of choices are involved. But we get the following.

Proposition(T)

For fixed choices the geometric complex of a virtual link diagram L_D is a well-defined chain complex in the category $\mathbf{uKob}_R(\emptyset)$.
Moreover, different choices give the same object in the skeleton of $\mathbf{uKob}_R(\emptyset)$, i.e. they are the same complexes modulo chain isomorphisms.

It is an invariant!

Denote $\mathbf{uKob}_R(\emptyset)_{hl}$ the category $\mathbf{uKob}_R(\emptyset)$ modulo chain homotopy and the so-called local relations



Theorem(T)

The geometric complex of two equivalent virtual link diagrams are the same in $\mathbf{uKob}_R(\emptyset)_{hl}$, i.e. the complex is an invariant up to chain homotopy.

And it is computable!

It follows from the discussion before that every uTQFT $\mathcal{F}$, i.e. a functor from

$$\mathcal{F}\colon \mathbf{uCob}^2{}_R(\emptyset) \to \text{R-}\mathbf{MOD}$$

that satisfies the local relations can be seen as an invariant of virtual links. Such an uTQFT should be additive. Hence, we can lift it to a functor

$$\mathcal{F}\colon \mathbf{uKob}_R(\emptyset) \to \mathbf{FinChain}\,.$$

We call the $\mathcal{F}(\cdot)$ image of a geometric complex algebraic.

Theorem(T)

Let $\mathcal{F}$ be an uTQFT that satisfies the local relations. Then the homology groups of the algebraic complex are virtual link invariants.

And it is a categorification!

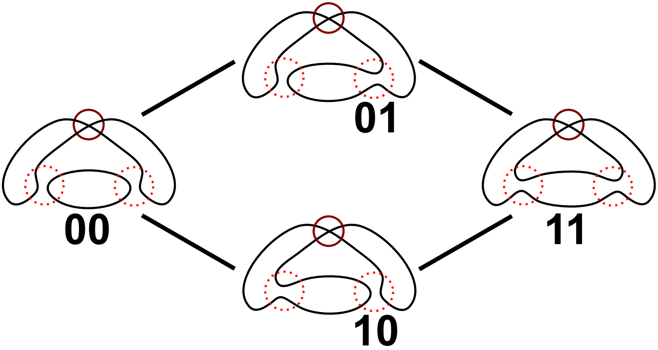
Note that we can give a complete list of skew-extended Frobenius algebras that can be used as virtual link invariants. One of them is the skew-extended Frobenius algebra from before, also called virtual Khovanov homology.

Note that this leads to a categorification of the virtual Jones polynomial.

Theorem(T)

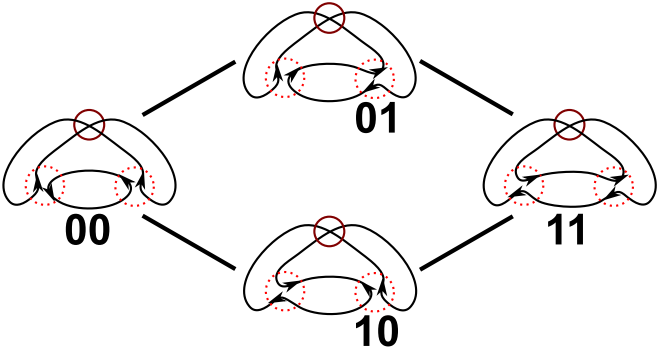
The virtual Khovanov homology of a virtual link is a categorification of the virtual Jones polynomial, i.e. taking the graded Euler characteristic gives the virtual Jones polynomial.

Exempli gratia



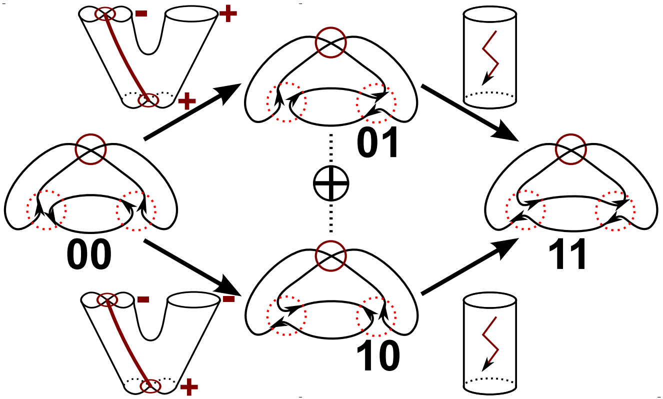
Let us show how the calculation works. We consider the virtual trefoil and suppress grading shifts and signs placement. First let us add some orientations.

Exempli gratia



Let us show how the calculation works. We consider the virtual trefoil and suppress grading shifts and signs placement. First let us add some orientations. Now we can read of the cobordisms.

Exempli gratia



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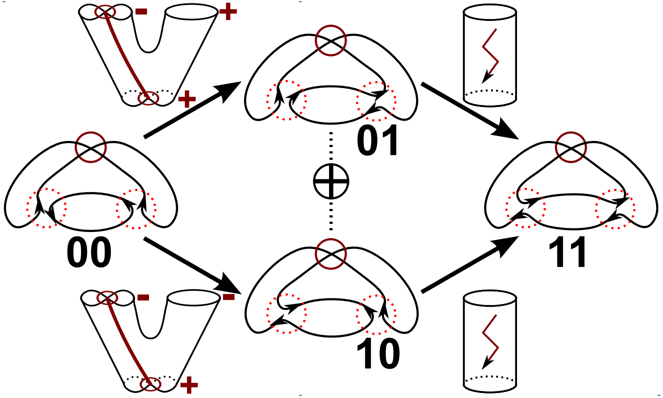
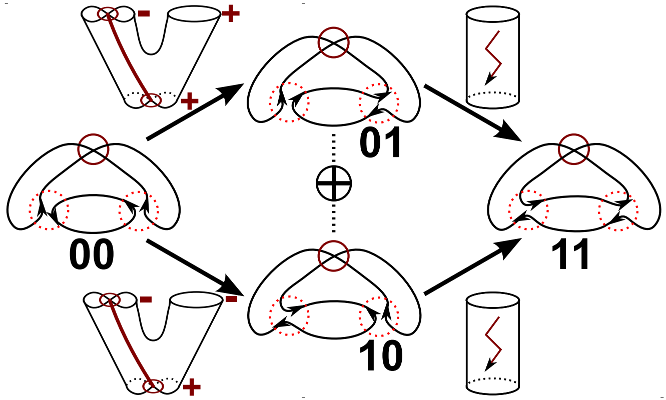


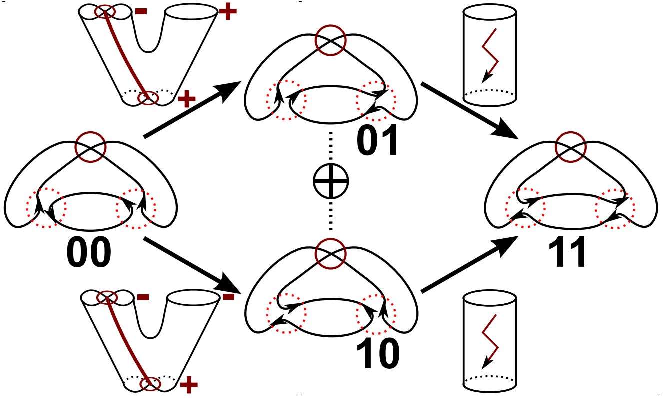
Note that this is the geometric complex.

Exempli gratia



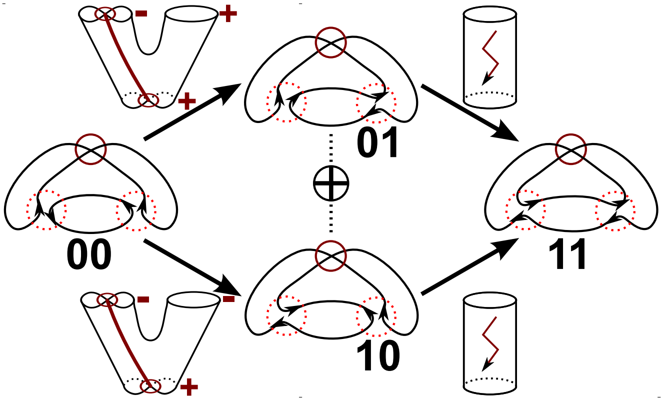
Now we have to translate the objects to graded vector spaces and the cobordisms to maps between them. The objects are $A \otimes A$, $A \oplus A$ and A.

Exempli gratia



The two right maps are 0 and the two multiplications are given by

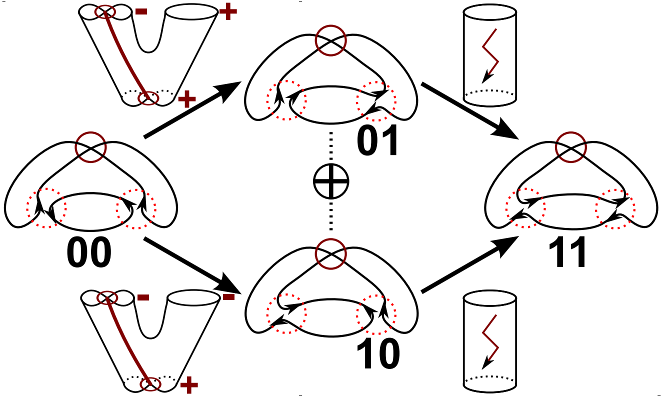
$$1 \otimes 1 \to 1, X \otimes 1 \to -X, 1 \otimes X \to -X \text{ and } X \otimes X \to 0$$



and

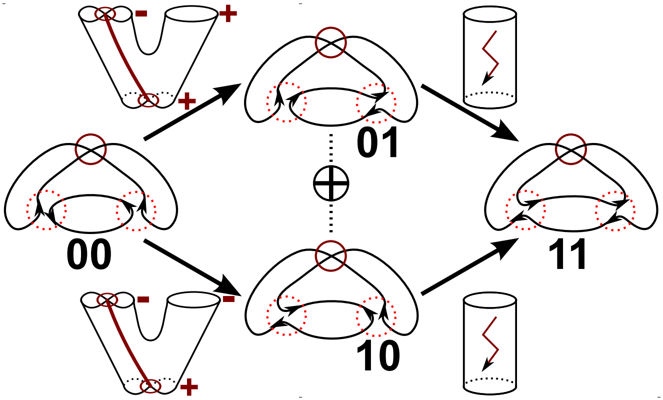
$$1 \otimes 1 \to 1, X \otimes 1 \to X, 1 \otimes X \to -X \text{ and } X \otimes X \to 0$$

for the upper and lower. Note that they are not the same.



Even a computer the homology now (with shifts). It turn out to be $t^0 + q^{-3}t^1 + qt^2 + q^2t^2$. Setting $t = -1$ gives the unnormalised virtual Jones polynomial.

Exempli gratia



Note that in every step we loose information. But even the virtual Khovanov complex is strict stronger then the virtual Jones polynomial.

What is to be done...

- There is a computer program for calculations. But it is to slow at the moment. One can use an extension of the construction to virtual tangles to drastic improve the calculation speed.
- Give a construction that works for the $U_q(\mathfrak{sl}_n)$ polynomial.
- An interpretation of the homology in terms of representations of $U_q(\mathfrak{sl}_2)$ and its categorification $\mathcal{U}(\mathfrak{sl}_2)$ is missing at the moment.
- Extend the Rasmussen invariant to virtual knots.
- Even more...

There is still much to do...

Thanks for your attention!