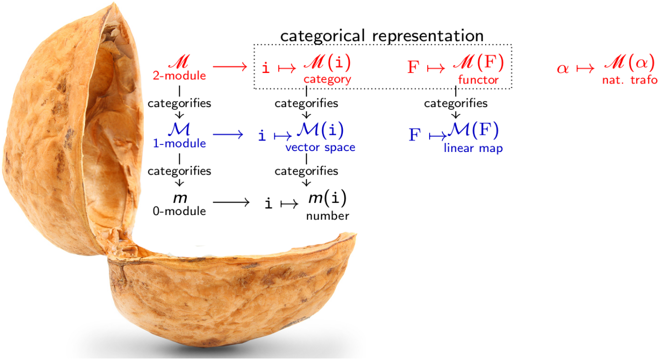
What is...(2-)representation theory?

Or: A (fairy) tale of matrices and functors

Daniel Tubbenhauer



October 2018

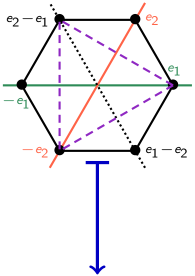
1. **Classical representation theory**
   - Main ideas
   - Some classical results
   - Some examples

2. **Categorical representation theory**
   - Main ideas
   - Some categorical results
   - An example

A linearization of group theory

Slogan. Representation theory is group theory in vector spaces.

symmetries of n-gons $\subset \mathcal{A}\mathrm{ut}(\mathbb{R}^2)$



$$\left\{ \qquad \right\}$$

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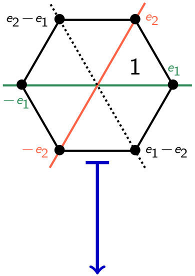

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \right\}$$

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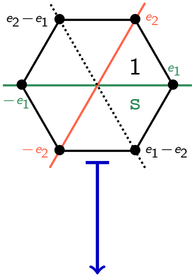

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad \right\}$$

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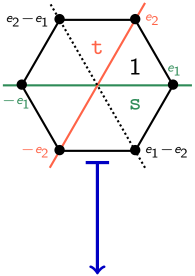

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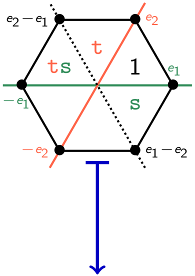

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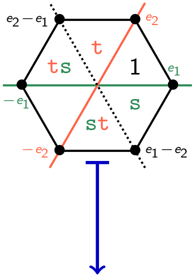

$$\left\{ \underset{1}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{s}{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}}, \underset{t}{\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}}, \underset{ts}{\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}}, \underset{st}{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}}, \qquad \right\}$$

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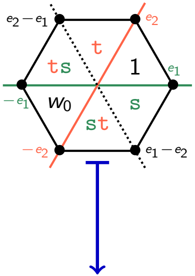

$$\left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{1}, \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}}_{\mathtt{s}}, \underbrace{\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}}_{\mathtt{t}}, \underbrace{\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}}_{\mathtt{ts}}, \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}}_{\mathtt{st}}, \underbrace{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}_{\mathtt{sts}=\mathtt{tst}\ (w_0)} \right\}$$

A linearization of group theory

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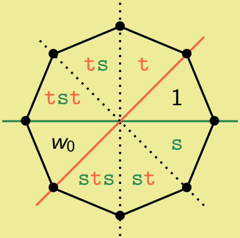
symmetries of n-gons $\subset Aut(\mathbb{R}^2)$

These symmetry groups of the regular n-gons are the so-called dihedral groups

$$\mathrm{D}_{2n} = \langle s, t \mid s^2 = t^2 = 1, \underbrace{\ldots tsts}_{n} = w_0 = \underbrace{\ldots stst}_{n} \rangle$$

which are the easiest examples of Coxeter groups.

Example $n = 4$; its Coxeter complex.



w_0

Pioneers of representation theory

Let G be a finite group.

Frobenius ∼1895++, Burnside ∼1900++. Representation theory is the study of linear group actions

useful?

$$\mathcal{M}\colon \mathrm{G} \longrightarrow \mathcal{A}\mathrm{ut}(\mathtt{V}),$$

"$\mathcal{M}(g) =$ a matrix in $\mathcal{A}\mathrm{ut}(\mathtt{V})$"

with $\mathtt{V}$ being some vector space. (Called modules or representations.)

The "atoms" of such an action are called simple. A module is called semisimple if it is a direct sum of simples.

Maschke ∼1899. All modules are built out of simples ("Jordan–Hölder" filtration).

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We want to have a categorical version of this!

Pioneers of representation theory

Let A be a finite-dimensional algebra.

Noether $\sim$1928++. Representation theory is the useful? study of algebra actions

$$\mathcal{M}\colon \mathrm{A} \longrightarrow \mathcal{E}\mathrm{nd}(\mathtt{V}),$$

"$\mathcal{M}(a)$ = a matrix in $\mathcal{E}\mathrm{nd}(\mathtt{V})$"

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We want to have a categorical version of this.

I am going to explain what we can do at present.

Life is non-semisimple

collection ("category") of modules $\leftrightsquigarrow$ the world

modules $\leftrightsquigarrow$ chemical compounds

simples $\leftrightsquigarrow$ elements

semisimple $\leftrightsquigarrow$ only trivial compounds

non-semisimple $\leftrightsquigarrow$ non-trivial compounds

Main goal of representation theory. Find the periodic table of simples.

Life...

colle...

mod...

simp...

semi...

Example.

Back to the dihedral group, an invariant of the module is the ▸ character χ which only remembers the traces of the acting matrices:

$$\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&1\\0&-1\end{pmatrix},\begin{pmatrix}-1&0\\1&1\end{pmatrix},\begin{pmatrix}-1&-1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\-1&-1\end{pmatrix},\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\right\}$$

1	s	t	ts	st	sts=tst (w_0)
$\chi=2$	$\chi=0$	$\chi=0$	$\chi=-1$	$\chi=-1$	$\chi=0$

non-semisimple ⇝ non-trivial compounds

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$\chi = 2$	$\chi = 0$	$\chi = 0$	$\chi = -1$	$\chi = -1$	$\chi = 0$

Fact.

Semisimple case: the character determines the module

⇝

mass determines the chemical compound.

Life is non-semisimple

Example.

$$\mathbb{Z}/2\mathbb{Z} \to \mathcal{A}\mathrm{ut}(\mathbb{C}^2),\ 0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ \&\ 1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Common eigenvectors: $(1,1)$ and $(1,-1)$ and base change gives

$$0 \mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right)\ \&\ 1 \mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right)$$

and the module decomposes.

collection (

modules ⇜

simples ⇜

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and the module is non-simple, yet does not decompose.

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Morally: representation theory over $\mathbb{Z}$ is never semisimple.

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The strategy

"Groups, as men, will be known by their actions." – Guillermo Moreno

The study of group actions is of fundamental importance in mathematics and related field. Sadly, it is also very hard.

Representation theory approach. The analog linear problem of classifying G-modules has a satisfactory answer for many groups.

Problem involving
a group action
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Problem involving a group action $\mathrm{G} \circlearrowright \mathtt{X}$	$\cdots\cdots\cdots\rightarrow$	Problem involving a linear group action $\mathbb{k}[\mathrm{G}] \circlearrowright \mathbb{k}\mathtt{X}$

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⇣

"Decomposition of the problem" $\mathbb{k}[\mathrm{G}] \circlearrowright \bigoplus \mathtt{V}_i$

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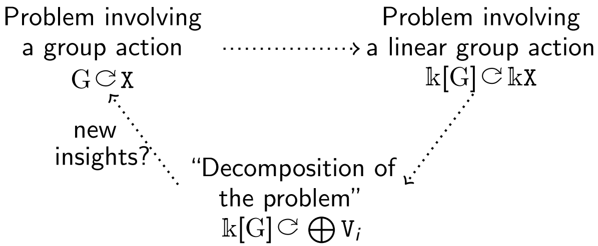

Philosophy. Turn problems into linear algebra.

Some theorems in classical representation theory

- ▷ All G-modules are built out of simples.
- ▷ The character of a simple G-module is an invariant.
- ▷ There is an injection

$$\{\text{simple } \mathrm{G}\text{-modules}\}/\text{iso} \hookrightarrow \{\text{conjugacy classes in } \mathrm{G}\},$$

which is $1:1$ in the semisimple case.

- ▷ All simples can be constructed intrinsically using the regular G-module.

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"Regular G-module $=$ G acting on itself."

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Some theorems in classical representation theory

Find categorical versions of these facts.

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Dihedral representation theory on one slide

One-dimensional modules. $\mathcal{M}_{\lambda_{s},\lambda_{t}}, \lambda_{s}, \lambda_{t} \in \mathbb{C}, s \mapsto \lambda_{s}, t \mapsto \lambda_{t}$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathcal{M}_{-1,-1}$, $\mathcal{M}_{1,-1}$, $\mathcal{M}_{-1,1}$, $\mathcal{M}_{1,1}$	$\mathcal{M}_{-1,-1}$, $\mathcal{M}_{1,1}$

Two-dimensional modules. $\mathcal{M}_{z}, z \in \mathbb{C}, s \mapsto \begin{pmatrix} 1 & z \\ 0 & -1 \end{pmatrix}, t \mapsto \begin{pmatrix} -1 & 0 \\ \bar{z} & 1 \end{pmatrix}$.

$n \equiv 0 \bmod 2$	$n \not\equiv 0 \bmod 2$
$\mathcal{M}_{z}, z \in V(n) - \{0\}$	$\mathcal{M}_{z}, z \in V(n)$

$V(n) = \{2\cos(\pi k/n-1) \mid k = 1, \ldots, n-2\}$.

Dihedral representation theory on one slide

One-dimensiona ... $\mathtt{t}$.

Proposition (Lusztig?).

The list of one- and two-dimensional D_{2n}-modules is a complete, irredundant list of simples.

$\mathcal{M}_{-1,-1}$, $\mathcal{M}_{1,-1}$, $\mathcal{M}_{-1,1}$, $\mathcal{M}_{1,1}$	$\mathcal{M}_{-1,-1}$, $\mathcal{M}_{1,1}$

I learned this construction from Mackaay in 2017.

Two-dimensional modules. $\mathcal{M}_z, z \in \mathbb{C}, \mathtt{s} \mapsto \begin{pmatrix} 1 & z \\ 0 & -1 \end{pmatrix}, \mathtt{t} \mapsto \begin{pmatrix} -1 & 0 \\ \overline{z} & 1 \end{pmatrix}$.

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Note that this requires complex parameters.
In particular, this does not work over $\mathbb{Z}$.

$\mathcal{M}_z, z \in V(n) - \{0\}$ | $\mathcal{M}_z, z \in V(n)$

$V(n) = \{2\cos(\pi k/n-1) \mid k = 1, \dots, n-2\}$.

Beware of infinite dimensions

Take the infinite-dimensional Weyl algebra $\mathrm{W} = \mathbb{C}\langle x, \delta \mid \delta x = 1 + x\delta \rangle$.

It has a very nice infinite-dimensional module

$$\mathrm{W} \to \mathcal{E}\mathrm{nd}(\mathbb{C}[X]),\ x \mapsto \cdot X,\ \delta \mapsto d/dX,$$

and $\delta x = 1 + x\delta$ just becomes Leibniz' product rule.

However, the classification of simples is not so easy. For example, W does not have any finite-dimensional module.

Why? Assume it has and $x \mapsto$ some matrix M; $\delta \mapsto$ some matrix N. Then:

$$\mathrm{tr}(MN) = \mathrm{tr}(NM) = 1 + \mathrm{tr}(MN) \quad \Rightarrow \quad 0 = 1.$$

But even there representation theory help

Take the infinite Artin–Tits group $\mathrm{B}(\mathrm{C}) = \langle b_i \mid \dots \underbrace{b_j b_i b_j}_{m_{ij}} = \underbrace{b_i b_j b_i}_{m_{ij}} \rangle$. ▸ Example

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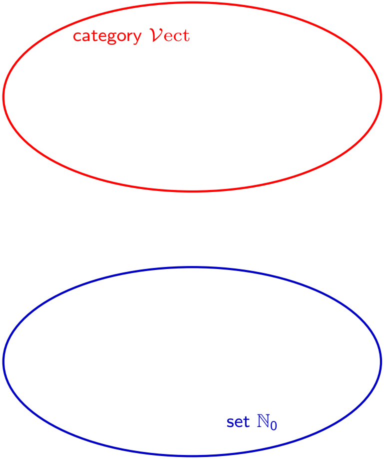
One can easily cook-up finite-dimensional modules which help to distinguish the elements of $\mathrm{B}(\mathrm{C})$.

However, it is very hard and not known in general how to find faithful ("injective") finite-dimensional modules.

Categorification in a nutshell



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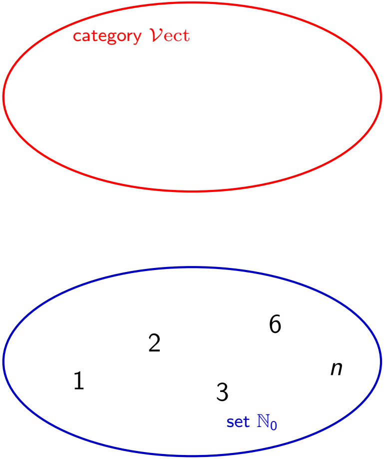

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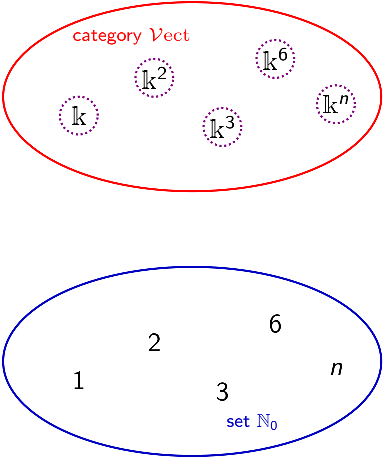

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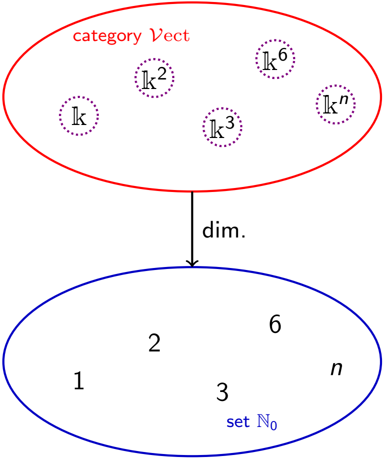

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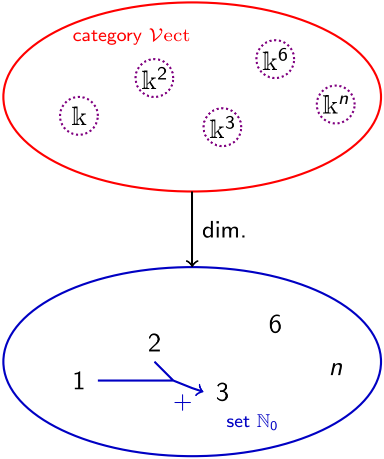

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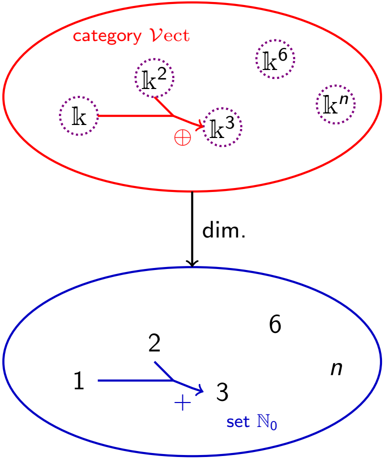

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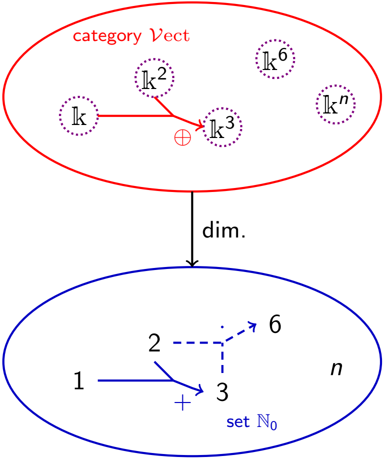

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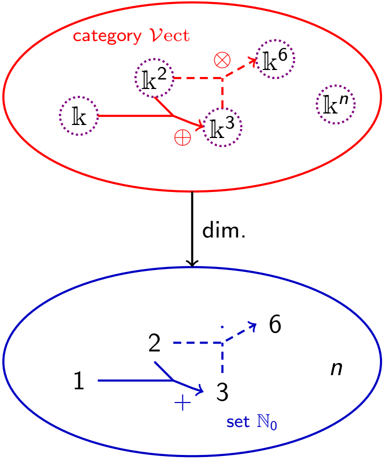

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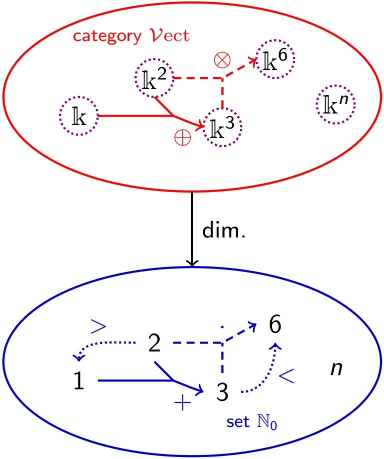

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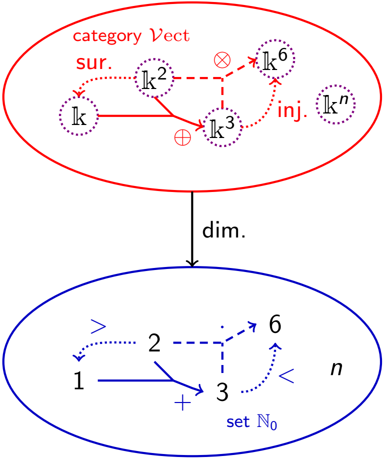

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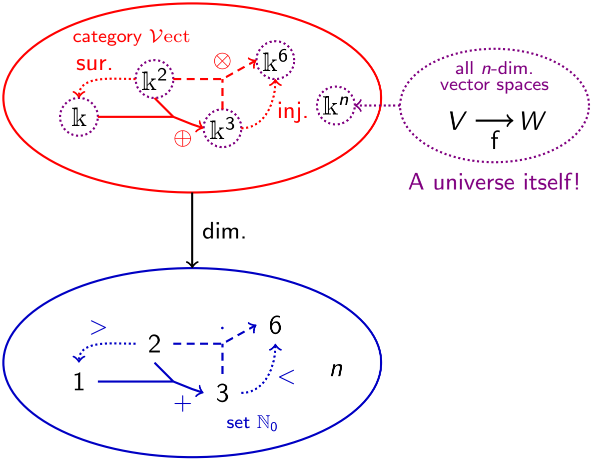

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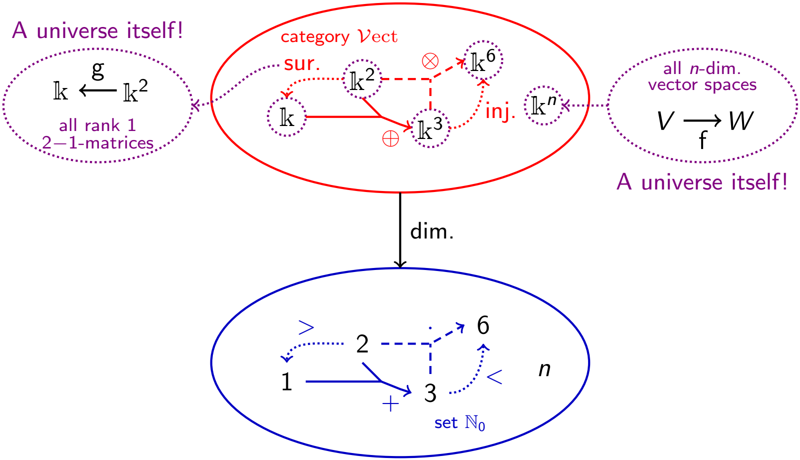

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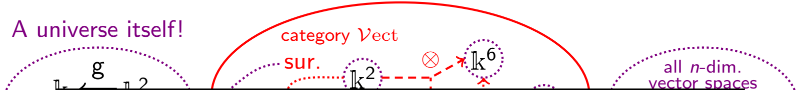
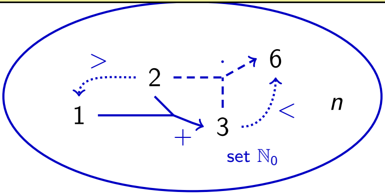


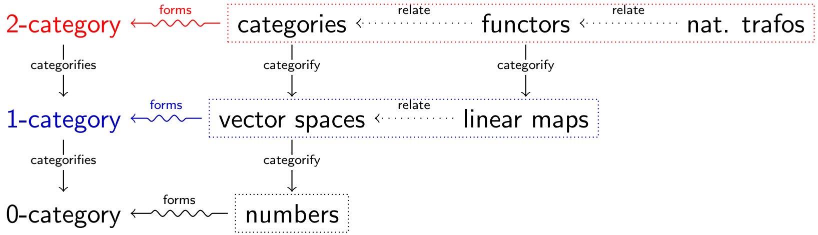
The point.

The category $\mathcal{V}\mathrm{ect}$ has the whole power of linear algebra at hand!

There is nothing comparable for $\mathbb{N}_0$:
$\mathbb{N}_0$ is just a shadow of $\mathcal{V}\mathrm{ect}$.

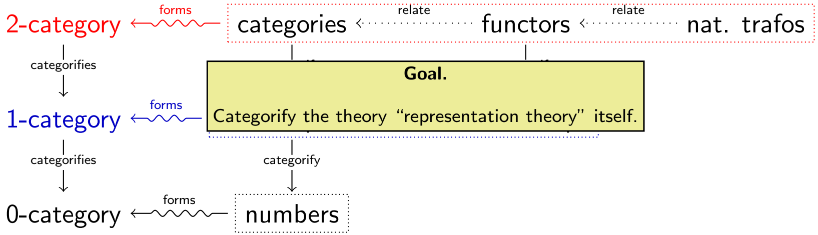


2-representation theory in a nutshell



The ladder of categorification: in each step there is a new layer of structure which is invisible on the ladder rung below.

2-representation theory in a nutshell



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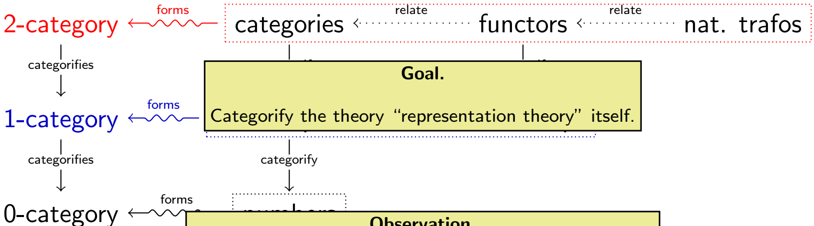


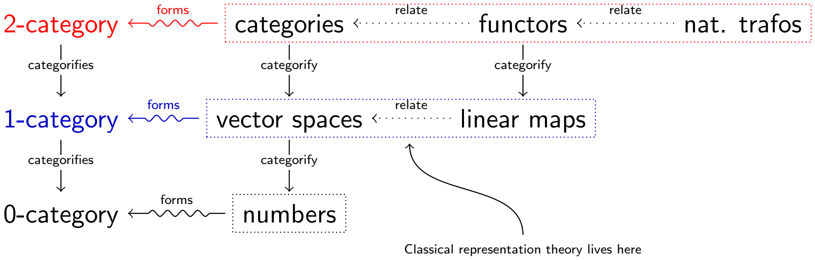
Goal.

Categorify the theory “representation theory” itself.

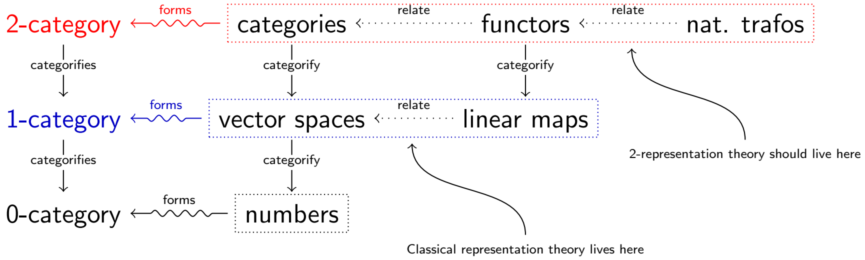
Observation.

A group G can be viewed as a single-object category $\mathcal{G}$,
and a module as a functor from $\mathcal{G}$
into the single-object category $\mathcal{A}\mathrm{ut}(\mathtt{V})$, i.e.
$\mathcal{M}\colon \mathcal{G} \longrightarrow \mathcal{A}\mathrm{ut}(\mathtt{V})$.

2-representation theory in a nutshell



2-representation theory in a nutshell



2-representation theory in a nutshell

categorical module

$\mathscr{M}$ 2-module $\rightsquigarrow$ $\mathtt{i} \mapsto \mathscr{M}(\mathtt{i})$ category, $\mathrm{F} \mapsto \mathscr{M}(\mathrm{F})$ functor, $\alpha \mapsto \mathscr{M}(\alpha)$ nat. trafo

categorifies ↓ categorifies ↓ categorifies ↓

$\mathcal{M}$ 1-module $\rightsquigarrow$ $\mathtt{i} \mapsto \mathcal{M}(\mathtt{i})$ vector space, $\mathrm{F} \mapsto \mathcal{M}(\mathrm{F})$ linear map

categorifies ↓ categorifies ↓

m 0-module $\rightsquigarrow$ $\mathtt{i} \mapsto m(\mathtt{i})$ number

2-representation theory in a nutshell

$\mathscr{M}$ $\rightsquigarrow$ $\mathtt{i} \mapsto \mathscr{M}(\mathtt{i})$ $\quad$ $\mathrm{F} \mapsto \mathscr{M}(\mathrm{F})$ $\quad$ $\alpha \mapsto \mathscr{M}(\alpha)$

2-modu … at. trafo

categorifi

$\mathcal{M}$

1-modu

categorifi

m

0-modu

What one can hope for.

Problem involving a group action $\mathrm{G} \circlearrowright \mathtt{X}$ ···"lift"···> Problem involving a categorical group action

Problem involving a categorical group action ···> Decomposition of the problem into 2-simples

Decomposition of the problem into 2-simples ···new insights?···> Problem involving a group action $\mathrm{G} \circlearrowright \mathtt{X}$

The next ladder rung

Slogan. 2-representation theory is group theory in categories.

$$\mathrm{W} = \mathbb{C}\langle x, \delta \mid \delta x = 1 + x\delta \rangle$$

$$\longmapsto$$

$$\mathrm{W} \to \mathcal{E}\mathrm{nd}(\mathbb{C}[X]) \qquad x \mapsto \cdot X \qquad \delta \mapsto d/dX$$

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$$\mathrm{W} \to \mathcal{E}\mathrm{nd}(\textstyle\bigoplus_{i \in \mathbb{N}_0} \mathbb{C}\{X^i\}) \qquad x \mapsto \textstyle\bigoplus_{i \in \mathbb{N}_0} \cdot X \qquad \delta \mapsto \textstyle\bigoplus_{i \in \mathbb{N}_0} d/dX$$

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$$\downarrow$$

$$\mathrm{W} \to \mathscr{E}\mathrm{nd}(\textstyle\bigoplus_{i\in\mathbb{N}_0} \mathrm{N}_i\text{-}\mathcal{M}\mathrm{od}) \qquad x \mapsto \textstyle\bigoplus_{i\in\mathbb{N}_0} \cdot X \qquad \delta \mapsto \textstyle\bigoplus_{i\in\mathbb{N}_0} d/dX$$

Step 1.

Replace the vector spaces $\mathbb{C}\{X^i\}$ by appropriate categories N_i-$\mathcal{M}\mathrm{od}$.

Here N_i are certain algebras ("Nil Coxeter") which embed into each other $\mathrm{N}_i \hookrightarrow \mathrm{N}_{i+1}$, of which we think about as lifting $\mathbb{C}\{X^i\} \xrightarrow{\cdot X} \mathbb{C}\{X^{i+1}\}$.

The next ladder rung

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Step 2.

Replace the linear operators $\cdot X\colon \mathbb{C}\{X^i\} \to \mathbb{C}\{X^{i+1}\}$ by appropriate ("induction") functors $\mathcal{I}\mathrm{nd}_i^{i+1}\colon \mathrm{N}_i\text{-}\mathcal{M}\mathrm{od} \to \mathrm{N}_{i+1}\text{-}\mathcal{M}\mathrm{od}$.

The next ladder rung

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$$\big\downarrow$$

$$\mathrm{W} \to \mathscr{E}\mathrm{nd}\big(\textstyle\bigoplus_{i\in\mathbb{N}_0} \mathrm{N}_i\text{-}\mathcal{M}\mathrm{od}\big) \quad x \mapsto \textstyle\bigoplus_{i\in\mathbb{N}_0} \mathcal{I}\mathrm{nd}_i^{i+1} \quad \delta \mapsto \textstyle\bigoplus_{i\in\mathbb{N}_0} \mathcal{R}\mathrm{es}_{i+1}^{i}$$

Step 3.

Replace the linear operators $d/dX\colon \mathbb{C}\{X^{i+1}\} \to \mathbb{C}\{X^{i}\}$ by appropriate ("restriction") functors $\mathcal{R}\mathrm{es}_{i+1}^{i}\colon \mathrm{N}_i\text{-}\mathcal{M}\mathrm{od} \to \mathrm{N}_{i+1}\text{-}\mathcal{M}\mathrm{od}$.

The next ladder rung

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Step 4.

Check that everything works.

In particular, the reciprocity $\mathcal{R}\mathrm{es}_{i+1}^{i}\mathcal{I}\mathrm{nd}_i^{i+1} \cong \mathcal{I}\mathrm{d} \oplus \mathcal{I}\mathrm{nd}_i^{i-1}\mathcal{R}\mathrm{es}_{i-1}^{i}$ categorifies Leibniz' product rule.

Pioneers of 2-representation theory

Let G be a finite group.

Plus some coherence conditions which I will not explain.

Chuang–Rouquier & many others ∼2004++. Higher representation theory is the useful? study of (certain) categorical actions, e.g.

$$\mathscr{M}\colon \mathrm{G} \longrightarrow \mathscr{A}\mathrm{ut}(\mathcal{V}),$$ "$\mathscr{M}(g)$ = a functor in $\mathscr{A}\mathrm{ut}(\mathcal{V})$"

with $\mathcal{V}$ being some $\mathbb{C}$-linear category. (Called 2-modules or 2-representations.)

The "atoms" of such an action are called 2-simple.

Mazorchuk–Miemietz ∼2014. All (suitable) 2-modules are built out of 2-simples ("weak 2-Jordan–Hölder filtration").

Pioneers of 2-representation theory

Let $\mathscr{C}$ be a finitary 2-category.

Chuang–Rouquier & many others ∼2004++. Higher representation theory is the useful? study of actions of 2-categories:

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The three goals of 2-representation theory.
Improve the theory itself.
Discuss examples.
Find applications.

“Lifting” classical representation theory

- All G-modules are built out of simples.
- The character of a simple G-module is an invariant.
- There is an injection

$$\{\text{simple } \mathrm{G}\text{-modules}\}/\text{iso} \hookrightarrow \{\text{conjugacy classes in } \mathrm{G}\},$$

  which is $1:1$ in the semisimple case.
- All simples can be constructed intrinsically using the regular G-module.

Goal 1. Improve the theory itself.

“Lifting” classical representation theory

- ▷ All (suitable) 2-modules are built out of 2-simples.
- ▷ The character of a s

Note that we have a very particular notion what a “suitable” 2-module is.

- ▷ There is an injection

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"Lifting" classical representation theory

- All (suitable) 2-modules are built out of 2-simples.
- The decategorified actions (a.k.a. matrices) of the $\mathrm{M}(\mathrm{F})$'s are invariants.
- There is an injection

  What characters were for Frobenius are these matrices for us.

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$$\{\text{2-simples of } \mathscr{C}\}/\text{equi.} \hookrightarrow \{\text{certain (co)algebra 1-morphisms}\}/\text{“2-Morita equi.”},$$

There are some technicalities.

which is $1:1$ in well-behaved cases.
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Goal 1. Improve the theory itself.

“Lifting” classical representation theory

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$$\{\text{2-simples of } \mathscr{C}\}/\text{equi.}$$
$$\hookrightarrow$$
$$\{\text{certain (co)algebra 1-morphisms}\}/\text{“2-Morita equi.”},$$

  which is $1:1$ in well-behaved cases.
- There exists principal 2-modules lifting the regular module.
  Even in well-behaved cases there are 2-simples which do not arise in this way.

These turned out to be very interesting,
since their importance is only visible via categorification.

Goal 1. Improve the theory itself.

2-modules of dihedral groups

$$\text{Consider:} \quad \theta_{s} = s + 1, \qquad \theta_{t} = t + 1.$$

(Motivation. The Kazhdan–Lusztig basis has some neat integral properties.)

These elements generate $\mathbb{C}[\mathrm{D}_{2n}]$ and their relations are fully understood:

$$\theta_{s}\theta_{s} = 2\theta_{s}, \qquad \theta_{t}\theta_{t} = 2\theta_{t}, \qquad \text{a relation for } \underbrace{\ldots sts}_{n} = \underbrace{\ldots tst}_{n}.$$

We want a categorical action. So we need:

- ▷ A category $\mathcal{V}$ to act on.
- ▷ Endofunctors Θ_{s} and Θ_{t} acting on $\mathcal{V}$.
- ▷ The relations of θ_{s} and θ_{t} have to be satisfied by the functors.
- ▷ A coherent choice of natural transformations. (Skipped today.)

Some details.

2-modules of dihedral groups

Consider: $\theta_s = s + 1, \qquad \theta_t = t + 1.$

(Motivation. ... perties.)

These elemen ... ood:

$\theta_s\theta_s$ ... $\underbrace{tst}_{n}$.

Theorem ∼2016.

There is a one-to-one correspondence

$$\{\text{(non-trivial) 2-simple } \mathrm{D}_{2n}\text{-modules}\}/\text{2-iso} \xleftrightarrow{1:1} \{\text{bicolored ADE Dynkin diagrams with Coxeter number } n\}.$$

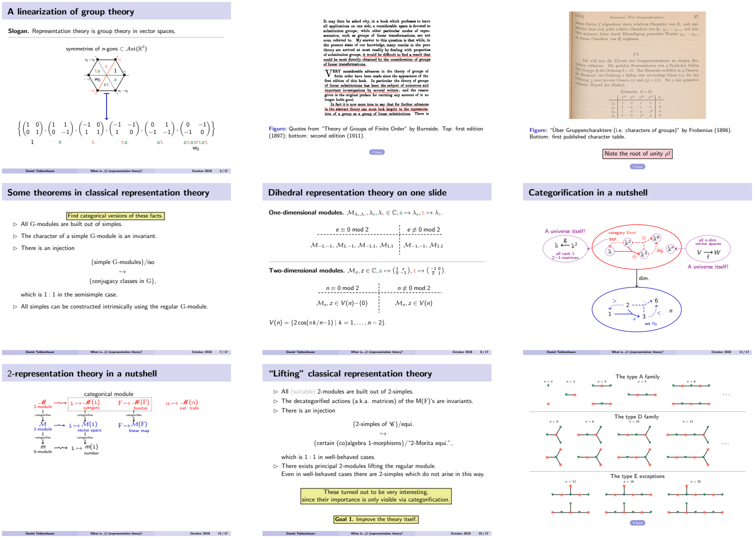
Thus, its easy to write down a list.

We want a ca ...

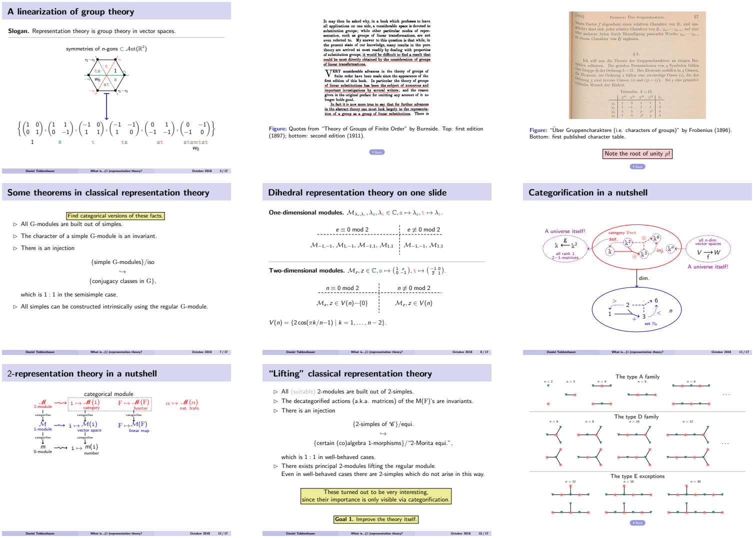
- ▷ A category $\mathcal{V}$ to act on.
- ▷ Endofunctors Θ_s and ...
- ▷ The relations of θ_s and θ_t have to be satisfied by the functors.
- ▷ A coherent choice of natural transformations. (Skipped today.)

Goal 2. Discuss examples.

Some details.



There is still much to do...



Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

Back

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Nowadays representation theory is pervasive across mathematics, and beyond.

VERY considerable advances in the theory of groups of

But this wasn't clear at all when Frobenius started it.

of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

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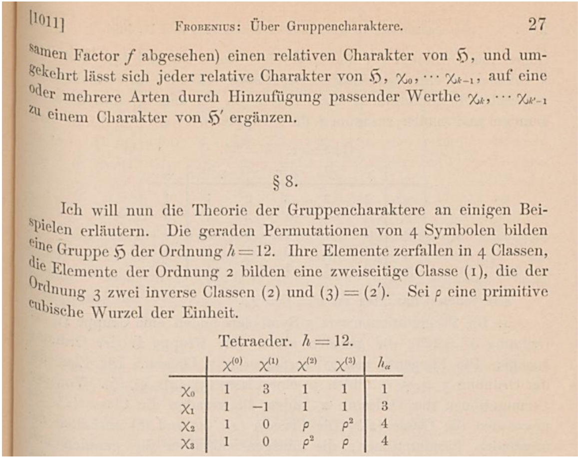
Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).


[1011] Frobenius: Über Gruppencharaktere. 27

samen Factor f abgesehen) einen relativen Charakter von $\mathfrak{H}$, und umgekehrt lässt sich jeder relative Charakter von $\mathfrak{H}$, $\chi_0, \cdots \chi_{k-1}$, auf eine oder mehrere Arten durch Hinzufügung passender Werthe $\chi_k, \cdots \chi_{k'-1}$ zu einem Charakter von $\mathfrak{H}'$ ergänzen.

§ 8.

Ich will nun die Theorie der Gruppencharaktere an einigen Beispielen erläutern. Die geraden Permutationen von 4 Symbolen bilden eine Gruppe $\mathfrak{H}$ der Ordnung $h = 12$. Ihre Elemente zerfallen in 4 Classen, die Elemente der Ordnung 2 bilden eine zweiseitige Classe (1), die der Ordnung 3 zwei inverse Classen (2) und (3) = (2'). Sei ρ eine primitive cubische Wurzel der Einheit.

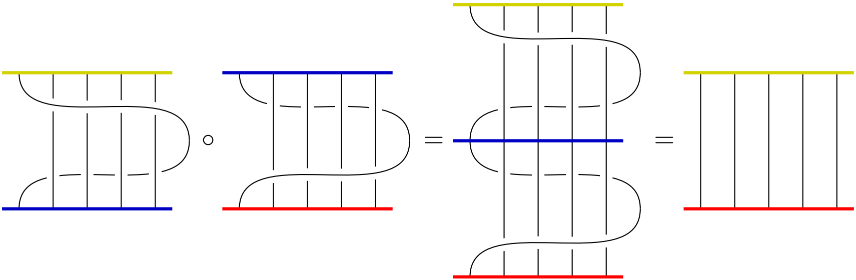
Tetraeder. $h = 12$.

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	h_α
χ_0	1	3	1	1	1
χ_1	1	−1	1	1	3
χ_2	1	0	ρ	ρ^2	4
χ_3	1	0	ρ^2	ρ	4

Figure: "Über Gruppencharaktere (i.e. characters of groups)" by Frobenius (1896). Bottom: first published character table.

Note the root of unity ρ!

Example. Prototypical braids in $\mathbb{R}^2 \times [0, 1]$ are



These form a(n infinite) group.

Theorem (Artin ∼1925). The braid group $\mathrm{B}(\mathrm{A})$ is an algebraic model of the group of braids in $\mathbb{R}^2 \times [0, 1]$.

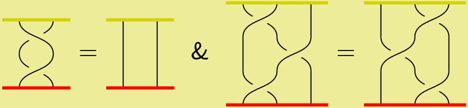
Example. Prototypical braids in $\mathbb{R}^2 \times [0,1]$ are

Proof (idea).

The generators b_i correspond to the simple braid swapping the i and the $i+1$ strands

$b_i \mapsto$ 

The relations boil down to



which gives a surjection.

Checking injectivity of this map is work.

group of braids in $\mathbb{R}^2 \times [0,1]$.

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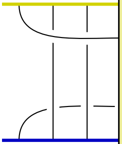
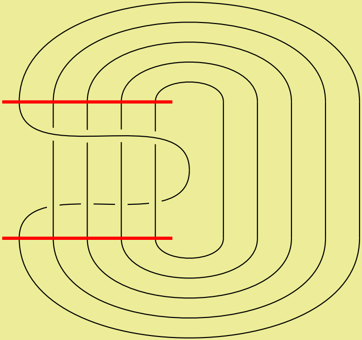


These form a

Theorem (A odel of the
group of braid

Observation (e.g. Alexander ∼1923, Markov ∼1935).

Identifying bottom and top gives you knots and links, e.g.



and the study of knots and links can be largely reduced to braids and their modules.

Example. Prototypical braids in $\mathbb{R}^2 \times [0,1]$ are

Example.

Here is a finite-dimensional module of $\mathrm{B}(\mathrm{A})$ for three strands:

$$\mathrm{B}(\mathrm{A}) \to \mathcal{A}\mathrm{ut}((\mathbb{C}(q,t))^3), \quad b_1 \mapsto \begin{pmatrix} -q^2t & 0 & q^2-q \\ 0 & 0 & q \\ 0 & 1 & 1-q \end{pmatrix} \ \& \ b_2 \mapsto \begin{pmatrix} 0 & q & 0 \\ 1 & 1-q & 0 \\ 0 & t(q^2-q) & -q^2t \end{pmatrix}$$

Theorem (Lawrence ∼1990, Bigelow & Kramer ∼2002).

This works in general for $\mathrm{B}(\mathrm{A})$ and the modules are faithful.
(Two braids are the same iff their matrices are the same.)

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Back

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However, for general Artin–Tits braid groups basically all questions are widely open.

Back

Khovanov & others ∼1999++. Knot homologies are instances of 2-representation theory. Low-dim. topology & Math. Physics

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Khovanov–Seidel & others ∼2000++. Faithful 2-modules of braid groups. Low-dim. topology & Symplectic geometry

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Chuang–Rouquier ∼2004. Proof of the Broué conjecture using 2-representation theory. p-RT of finite groups & Geometry & Combinatorics

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Elias–Williamson ∼2012. Proof of the Kazhdan–Lusztig conjecture using ideas from 2-representation theory. Combinatorics & RT & Geometry

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Riche–Williamson ∼2015. Tilting characters using 2-representation theory. p-RT of reductive groups & Geometry

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Many more...

Back

Khovanov & others ∼1999++. Knot homologies are instances of 2-representation theory. Low-dim. topology & Math. Physics

Goal 3. Find application.

Khovanov–Seidel & others ∼2000++. Faithful 2-modules of braid groups. Low-dim. topology & Symplectic geometry

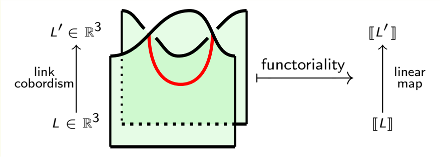
Chuang–Rou ... -representation theory. p-RT

Elias–William ... re using ideas from 2-represe

Riche–Willian ... on theory. p-RT of redu

Many more...

Functoriality of Khovanov–Rozansky's invariants ∼2017.

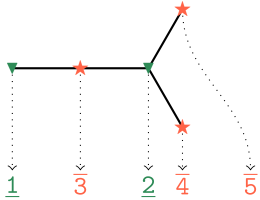


(This was conjectured from about 10 years, but seemed infeasible to prove, and has some impact on 4-dim. topology.)
The main ingredient?
2-representation theory.

Back

Construct a D_∞-module $\mathtt{V}$ associated to a bipartite graph G:

$$\mathtt{V} = \langle \underline{1}, \underline{2}, \overline{3}, \overline{4}, \overline{5} \rangle_{\mathbb{C}}$$



$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Back

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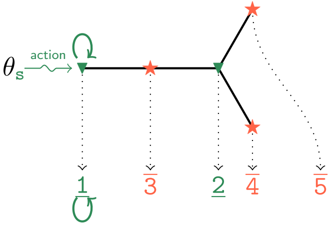

$$\theta_s \rightsquigarrow \mathrm{M}_s = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_t \rightsquigarrow \mathrm{M}_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Back

Construct a D_∞-module $\mathtt{V}$ associated to a bipartite graph G:

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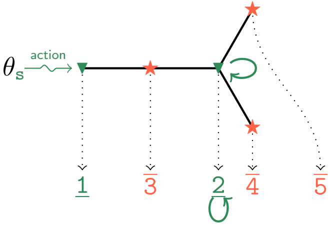

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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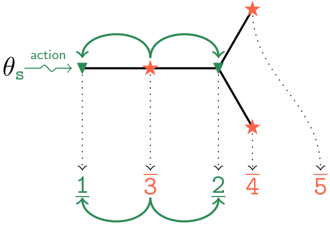

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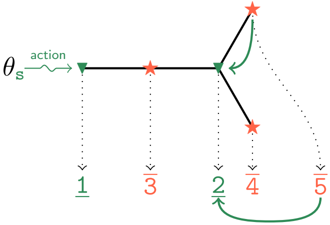

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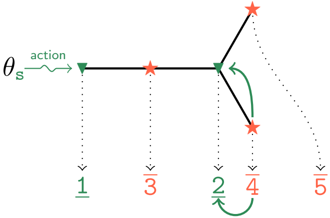

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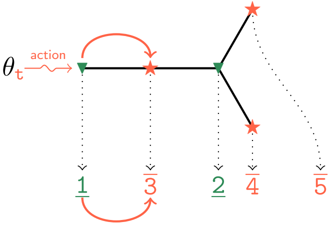

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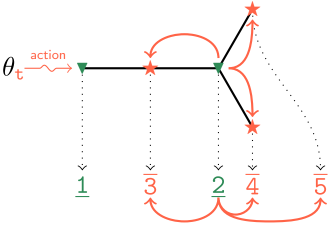

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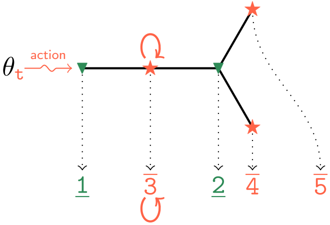

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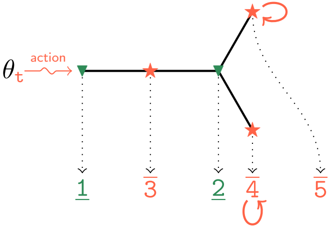

$$\theta_s \rightsquigarrow \mathrm{M}_s = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_t \rightsquigarrow \mathrm{M}_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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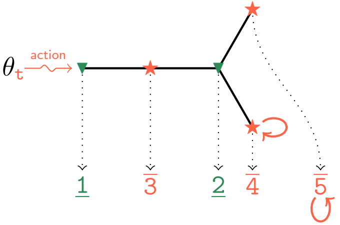

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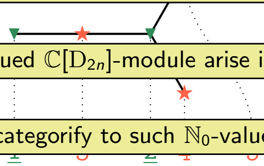
Lemma. For certain values of n these are $\mathbb{N}_0$-valued $\mathbb{C}[\mathrm{D}_{2n}]$-modules.



Lemma. All $\mathbb{N}_0$-valued $\mathbb{C}[\mathrm{D}_{2n}]$-module arise in this way.

Lemma. All 2-modules decategorify to such $\mathbb{N}_0$-valued $\mathbb{C}[\mathrm{D}_{2n}]$-module.

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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Categorification.

Category $\rightsquigarrow \mathcal{V} = \mathrm{Z}\text{-}\mathcal{M}\mathrm{od}$,
Z quiver algebra with underlying graph G.

Endofunctors $\rightsquigarrow$ tensoring with Z-bimodules.

Lemma. These satisfy the relations of $\mathbb{C}[\mathrm{D}_{2n}]$.

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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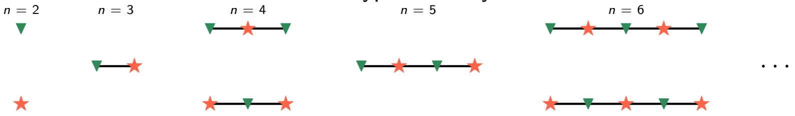
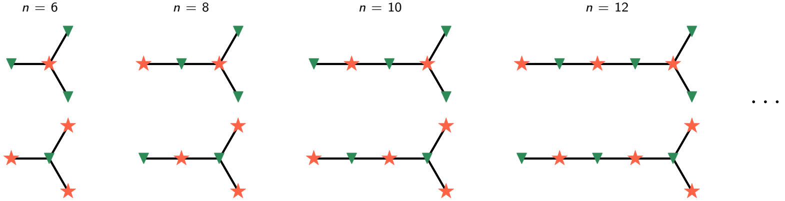
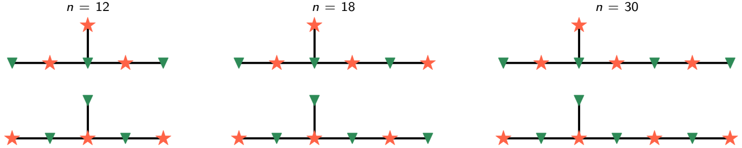
The type A family



The type D family



The type E exceptions



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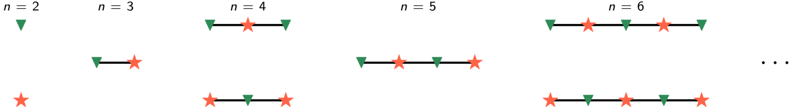
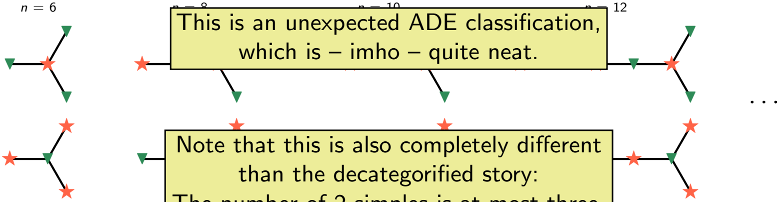
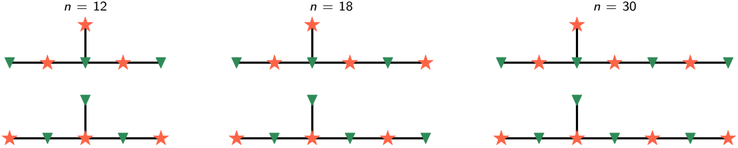


The type D family



This is an unexpected ADE classification, which is – imho – quite neat.

Note that this is also completely different than the decategorified story:
The number of 2-simples is at most three, but they grow in dimension when n grows.



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