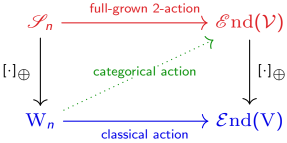
2-representation theory of Coxeter groups: a biased survey

Or: The "next generation" of representation theory of Coxeter groups!?

Daniel Tubbenhauer



Joint work with Marco Mackaay, Volodymyr Mazorchuk and Vanessa Miemietz

April 2017

1. Classical representation theory
   - The origin of representation theory
   - Some classical results

2. "Higher" representation theory
   - Categorification in a nutshell
   - "Lifting" some classical results

3. Categorical representation theory of Coxeter groups
   - The "How to"
   - Classification for symmetric and dihedral groups

Pioneers of representation theory

Let G be a finite group.

Frobenius ∼1895++, Burnside ∼1900++. Representation theory is the study of linear group actions: ▸ useful?

$$\mathrm{M}\colon G \longrightarrow \mathrm{End}(\mathrm{V}),$$ "$\mathrm{M}(g)$ = a matrix in $\mathrm{End}(\mathrm{V})$"

with V being some $\mathbb{C}$-vector space. We call V a module or a representation.

The "atoms" of such an action are called simple.

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Distant future goal: We want to have a categorical version of this!

Pioneers of representation theory

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Main future goal: We want to have a categorical version of this.
I am going to explain what we can do at present.

Some basic theorems in classical representation theory

- ▷ All G-modules are built out of simples.
- ▷ The ▸ character of a simple G-module determines it.
- ▷ There is a one-to-one correspondence

$$|\{\text{simple } G\text{-modules}\}/\text{iso}| \xleftrightarrow{1:1} |\{\text{conjugacy classes in } G\}|.$$

- ▷ All simples can be constructed intrinsically using the regular G-module.

Some basic theorems in classical representation theory

- ▷ All G-modules are built out of simples.
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The character only remembers the traces of the acting matrices.

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"Regular representation $=$ G acting on itself."

- ▷ All simples can be constructed intrinsically using the regular G-module.

Some basic theorems in classical representation theory

Goal: Find categorical versions of these facts.

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Finite Coxeter groups

A family of groups with interesting representation theory are the ▸ finite Coxeter groups.

These have two different interesting representations: These are usually not integral.

- ▷ **Frobenius & many others ∼1895++.** The simples.
- ▷ **Kazhdan–Lusztig ∼1979++.** The cell representations.

These are always integral.

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Example. In case of the symmetric group S_n "simples = cells".

They have been constructed by Frobenius ∼1895++, Young ∼1900++ and Schur ∼1901++, and correspond to integer partitions of n.

$$\text{e.g.: } S_3 = \langle s, t \mid s^2 = t^2 = 1, tst = w_0 = sts \rangle$$

$$\text{simples} \xleftrightarrow{1:1} \{C(1),\ C(sts),\ C(st)\} \xleftrightarrow{1:1} \{1{+}1{+}1,\ 2{+}1{+}0,\ 3{+}0{+}0\}$$

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Example. In case of the dihedral group W_n the simples are either one-dimensional or two-dimensional:

$$\mathrm{V}_{\pm\pm} = \mathbb{C}; \quad s \rightsquigarrow +1, -1; t \rightsquigarrow +1, -1,$$

$$\mathrm{V}_k = \mathbb{C}^2; \quad s \rightsquigarrow \begin{pmatrix} \cos(^{2\pi k}/_n) & \sin(^{2\pi k}/_n) \\ \sin(^{2\pi k}/_n) & -\cos(^{2\pi k}/_n) \end{pmatrix}; t \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \cong \mathrm{V}_k.$$

Here n is even and $k = \{1, 2, \dots, {}^{n-2}/_2\}$.

The case for n odd works similar.

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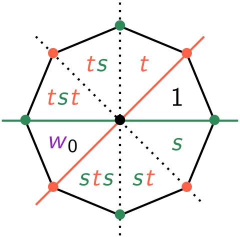

The definition of the cells is a bit involved, using Kazhdan–Lusztig combinatorics. I skip it for today.

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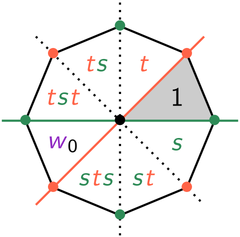

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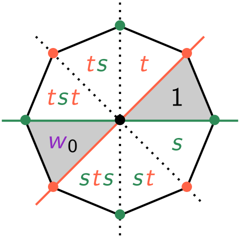

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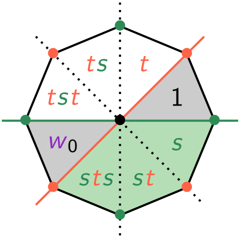

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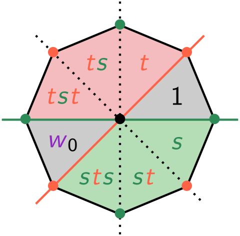
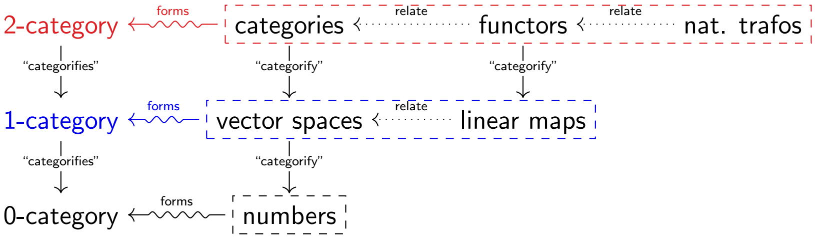
The cells partition the Coxeter group in question.



Categorification: A picture to keep in mind



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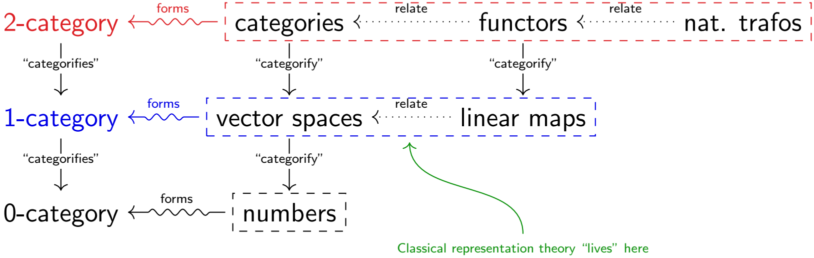

An algebra A can be viewed as an one-object category $\mathcal{C}$, and a representation as a functor from $\mathcal{C}$ into the one-object category $\mathcal{E}\mathrm{nd}(\mathrm{V})$, i.e.

$$\mathrm{M}\colon \mathcal{C} \longrightarrow \mathcal{E}\mathrm{nd}(\mathrm{V}).$$

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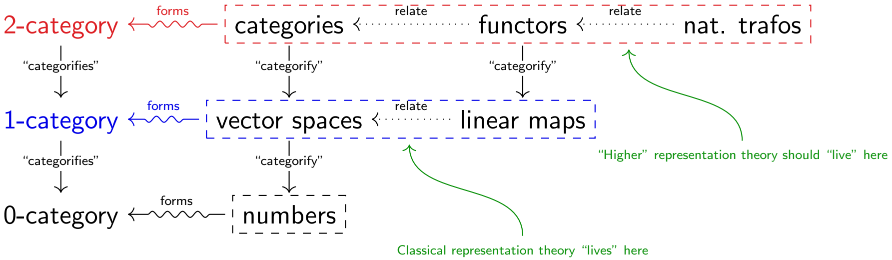

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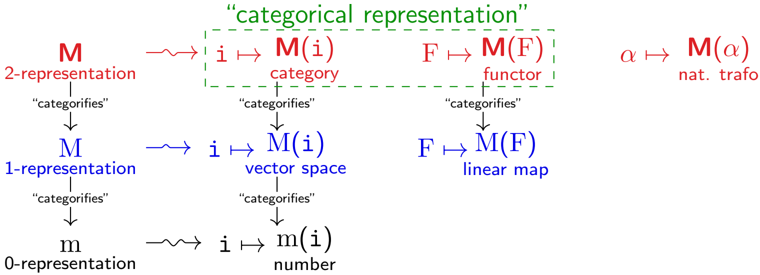
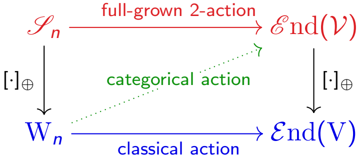


I only show you "the weak story" today, but we actually study the so-called strong version. Roughly, Coxeter groups can be categorified using Soergel bimodules, and studying their 2-representation theory fixes the higher structure.



Categorification: A picture to keep in mind

$\mathbf{M}$ (2-representation) $\rightsquigarrow$ $\mathtt{i} \mapsto \mathbf{M}(\mathtt{i})$ (category), $\mathrm{F} \mapsto \mathbf{M}(\mathrm{F})$ (functor), $\alpha \mapsto \mathbf{M}(\alpha)$ (nat. trafo)

↓ "categorifies"

M (1-representation) $\rightsquigarrow$ $\mathtt{i} \mapsto \mathrm{M}(\mathtt{i})$ (vector space), $\mathrm{F} \mapsto \mathrm{M}(\mathrm{F})$ (linear map)

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m (0-representation) $\rightsquigarrow$ $\mathtt{i} \mapsto \mathrm{m}(\mathtt{i})$ (number)

(Khovanov) Homology is an example in this spirit.

$\mathbf{M}$ (2-representation) $\rightsquigarrow$ $\mathtt{i} \mapsto \mathbf{M}(\mathtt{i})$ (category) $\quad \mathrm{F} \mapsto \mathbf{M}(\mathrm{F})$ (functor) $\quad \alpha \mapsto \mathbf{M}(\alpha)$ (nat. trafo)

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m

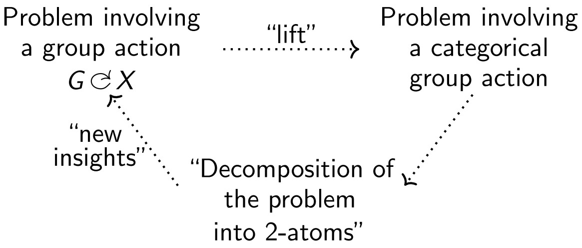
0-representati

Mazorchuk–Miemietz ∼2014.
Notion of a "2-atom" called simple transitive:
an appropriate 2-analog of simple modules.

(Khovanov) Homology is an ▸ example in this spirit.

Categorification: A picture to keep in mind

What one can hope for:



Example(Khovanov–Seidel & others 2000++).
Faithfulness of "categorical representations" of braid groups – this is a huge open problem in the classical case.

2-r α) rafo
1-r
0-r
(Khovanc

"Lifting" classical representation theory

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"Lifting" classical representation theory

- ▷ **Mazorchuk–Miemietz ∼2014.** All (suitable) 2-representations are built out of 2-atoms.

Note that we have a very particular notion what a "suitable" 2-representation is.

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- ▷ There is a one-to-one What characters were for Frobenius are these matrices for us.

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$$|\{\text{2-atoms of } \mathscr{C}\}/\text{equi.}|$$
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- ▷ **Mazorchuk–Miemietz ∼2014.** There exists principal 2-representations lifting the regular representation of Coxeter groups.
  Several authors including myself ∼2016. But even in well-behaved cases there are 2-atoms which do not arise in this way.

These turned out to be very interesting
since their importance is only visible via categorification.

Basic philosophy: Work with matrices long as possible!

Classifying "higher" representations of Coxeter groups:

1. list of candidates
2. reduce the list
3. construct the remaining ones

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Classifying "higher" representations of Coxeter groups:

Everything depends on the choice of generators and relations.

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Classifying "higher" representations of Coxeter groups:

1. list of candidates $\xleftarrow{\text{give}}$ relations among the generators
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Classifying "higher" representations of Coxeter groups:

Steps 1 and 2 only deal with matrices.

1 list of candidates $\xleftarrow{\text{give}}$ relations among the generators

2 reduce the list $\xleftarrow{\text{give}}$ assumptions on the 2-modules

3 construct the remaining ones $\overset{?}{\cdots\cdots}$ no general procedure

Step 3 needs "higher treatment".

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Classifying "higher" representations of Coxeter groups:

1. list of candidates $\xleftarrow{\text{give}}$ relations among the generators
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The best we have for the construction in general is
Mackaay–Mazorchuk–Miemietz–T.'s (co)algebra approach.

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---

Mazorchuk–Miemietz ∼2010. There are so-called cell 2-representations $\mathbf{C}_{\mathcal{L}}$. These work for any Coxeter group and categorify the cell representations of Kazhdan–Lusztig. All cells can be categorified.

State of the arts

Classification results are rare at the moment. But:

- ▷ **Mazorchuk–Miemietz $\sim$2014.** There is a classification in Coxeter type A.
- ▷ **Several authors including myself $\sim$2016.** There is a classification in dihedral Coxeter type.

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	Type A	Type $\mathrm{I}_2(n)$
All simples are “categorifyable”	✓	✗
All cells are “categorifyable”	✓	✓
All 2-atoms are 2-cells	✓	✗
“Uniqueness” of 2-atoms	✓	✗

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"Uniqueness" of 2-atoms	✓	✗

For the symmetric groups the uncategorified and the categorified story are completely parallel.

But this is misleading and purely a type A phenomenon.

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All simples are "categorifyable"	✓	✗	Most of them are not "categorifyable".
All cells are "categorifyable"	✓	✓	
All 2-atoms are 2-cells	✓	✗	More on the next slide.
"Uniqueness" of 2-atoms	✓	✗	This is very new and has not shown up in categorification yet.

For the dihedral groups the uncategorified and the categorified story are very different.

Towards 1, 2 and 3

Assume one has a category $\mathcal{V}$ and a "categorical action $\mathbf{m}\colon \mathbb{C}[W_n] \to \mathscr{E}\mathrm{nd}(\mathcal{V})$". Then there is a graph $G_{\mathbf{m}}$ together with a two-coloring associated to $\mathbf{m}$, called the principal graph of $\mathbf{m}$.

Several authors including myself ∼2016. A $\mathcal{V}$ and a 2-atom $\mathbf{m}$ can only exist if $G_{\mathbf{m}}$ is of ADE Dynkin type. Hereby, the Coxeter number of $G_{\mathbf{m}}$ is $n-2$.

Thus, it is easy to write down the ▸ list of all candidates.

Mackaay–T., Mackaay–Mazorchuk–Miemietz–T. ∼2016. We can also construct all of these and say whether these are equivalent, which completes the (graded) classification.

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Hence, for fixed n, there are only up to six 2-atoms.

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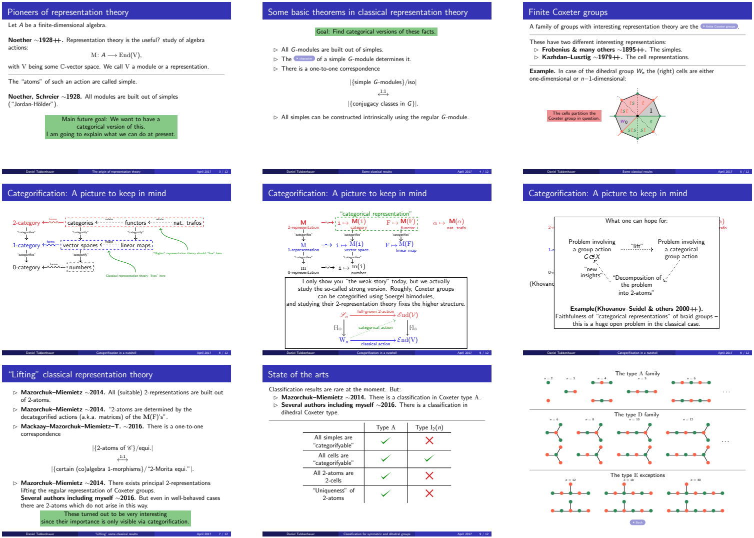
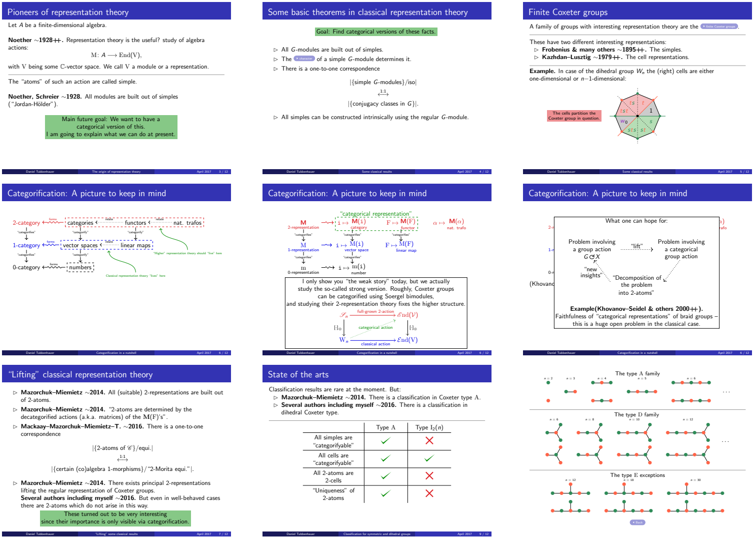
Concluding remarks—let me dream a bit

- ▷ Everything works graded as well, i.e. for Hecke algebras instead of Coxeter groups. In particular, with a bit more care, it works for braid groups.
- ▷ The dihedral story is just the tip of the iceberg. We hope that the general theory has impact beyond the dihedral case, e.g. for ▸ "generalized Coxeter–Dynkin diagrams" **à la Zuber** via **Elias'** quantum Satake.
- ▷ There are various connections:
  - ▶ To the theory of subfactors, fusion categories etc. **à la Etingof–Gelaki–Nikshych–Ostrik**,...
  - ▶ To quantum groups at roots of unity and their "subgroups" **à la Etingof–Khovanov, Ocneanu, Kirillov–Ostrik**,...
  - ▶ To web calculi **à la Kuperberg, Cautis–Kamnitzer–Morrison**,...
- ▷ More?



There is still much to do...



Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

Back

Frobenius' "character theory of the regular representation", e.g.:

$A(G = \mathbb{Z}/3\mathbb{Z}) =$

	X_0	X_1	X_2
$0 \circlearrowleft$	X_0	X_1	X_2
$1 \circlearrowleft$	X_1	X_2	X_0
$2 \circlearrowleft$	X_2	X_0	X_1

$$\Theta(G) = \det(A(G)) =$$

$\zeta = \exp({}^{2\pi i}/_3)$

$$(X_0 + X_1 + X_2)(X_0 + \zeta X_1 + \zeta^2 X_2)(X_0 + \zeta^2 X_1 + \zeta X_2).$$

The same decomposition into linear factors happens for all finite abelian groups.

Frobenius generalized this to arbitrary finite groups.

Nowadays we would say that each factor of $\Theta(G)$ corresponds to a simple G-module with dimension=degree. All simple characters arise in this way.

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$$\Theta(G) = \det(A(G)) =$$
$$(X_0 + X_1 + X_2)(X_0 + \zeta X_1 + \zeta^2 X_2)(X_0 + \zeta^2 X_1 + \zeta X_2).$$

The same decomposition … all finite abelian groups.

Using a more modern notation,
$\Theta(G) = \prod_{\text{simples } \mathrm{M}} \underbrace{\det(\sum_{g \in G} X_g \mathrm{M}(g))}_{\text{irr. factors}}.$

Frobenius generalized this to arbitrary finite groups.

Nowadays we would say that each factor of $\Theta(G)$ corresponds to a simple G-module with dimension=degree. All simple characters arise in this way.

Back

Frobenius' "character theory of the regular representation", e.g.:

$A(G = \mathbb{Z}/3\mathbb{Z}) =$

	X_0	X_1	X_2
$0 \circlearrowleft$	X_0	X_1	X_2
$1 \circlearrowleft$	X_1	X_2	X_0
$2 \circlearrowleft$	X_2	X_0	X_1

$$\Theta(G) = \det(A(G)) =$$

First simple. Second simple. Third simple.

$$(X_0 + X_1 + X_2)(X_0 + \zeta X_1 + \zeta^2 X_2)(X_0 + \zeta^2 X_1 + \zeta X_2).$$

The same decomposition into linear factors happens for all finite abelian groups.

Frobenius generaliz

Representation theory of finite abelian groups is "boring":
All simples are one-dimensional.
(Similarly in the categorical setup later on.)

Nowadays we would say that each factor of $\Theta(G)$ corresponds to a simple G-module with dimension=degree. All simple characters arise in this way.

Back

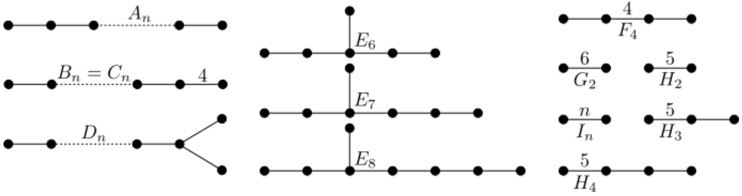


Figure: The Coxeter graphs of finite type.

Example. The type A family is given by the symmetric groups. The type $\mathrm{I}_2(n)$ family are the ▸ dihedral groups.

(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

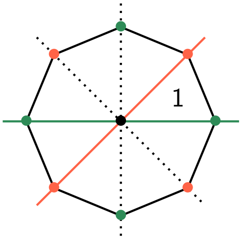
◂ Back

The dihedral groups are of Coxeter type $\mathrm{I}_2(n)$:

$$W_n = \langle s, t | s^2 = t^2 = 1, s_n = \underbrace{\dots sts}_{n} = w_0 = \underbrace{\dots tst}_{n} = t_n \rangle,$$

$$\text{e.g.: } W_4 = \langle s, t | s^2 = t^2 = 1, tsts = w_0 = stst \rangle$$

Example. These are the symmetry groups of regular n-gons, e.g. for $n = 4$ the Coxeter complex is:



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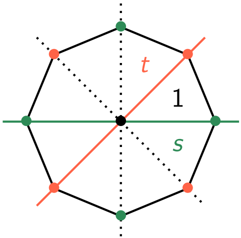

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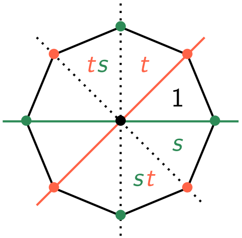

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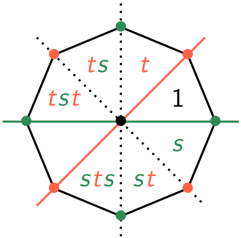

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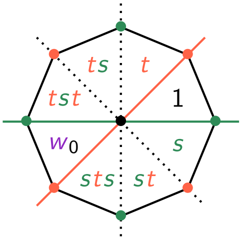
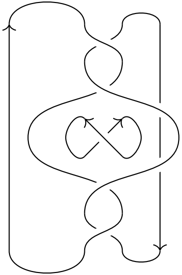


Back

Works for tangles as well,
fitting into the
2-categorical setup.



“Jones : Khovanov
$\leftrightsquigarrow$
Betti numbers : Homology”

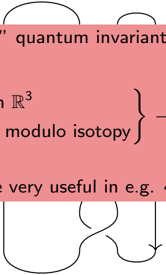
Quantum invariants of links: Jones & many others $\sim$1984++.

$$p(L) \text{ polynomial}\ , \quad L \cong L' \Rightarrow p(L) = p(L').$$

“Higher” quantum invariants of links: Khovanov & many others $\sim$1999++.

$$[\![L]\!] \text{ bigraded vector space}\ , \quad L \cong L' \Rightarrow [\![L]\!] \cong [\![L']\!]\ , \quad [\![L]\!] \xrightarrow[\text{characteristic}]{\text{graded Euler}} p(L).$$

Back



The main point about "higher" quantum invariants is that they provide functors:

$$\left\{ \begin{matrix} \text{link embeddings in } \mathbb{R}^3 \\ \text{link cobordisms in } \mathbb{R}^3 \times [0,1] \text{ modulo isotopy} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{bigraded vector spaces} \\ \text{homogeneous linear maps} \end{matrix} \right\}$$

which turned out to be very useful in e.g. 4-dimensional topology.

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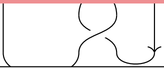
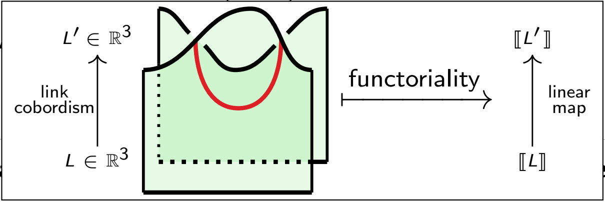


Quantum inv



“Higher” qu ... ers ∼1999++.

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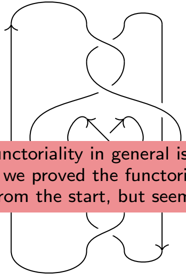
◂ Back



To prove functoriality in general is very hard.
In joint work with Ehrig–Wedrich ∼2017 we proved the functoriality of Khovanov–Rozansky's invariants.
(This was conjectured from the start, but seemed infeasible to prove.)

Quantum invariants of links: Jones & many others ∼1984++.

One of our main ingredients? "Higher" representation theory.

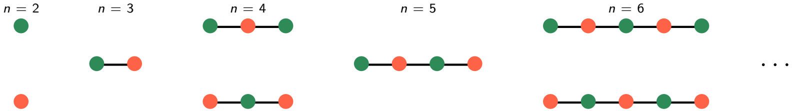
$$p(L) \text{ polynomial}, \quad L \cong L' \Rightarrow p(L) = p(L').$$

"Higher" quantum invariants of links: Khovanov & many others ∼1999++.

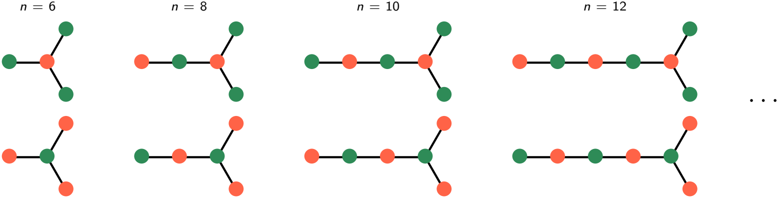
$$[\![L]\!] \text{ bigraded vector space}, \quad L \cong L' \Rightarrow [\![L]\!] \cong [\![L']\!], \quad [\![L]\!] \xrightarrow[\text{characteristic}]{\text{graded Euler}} p(L).$$

Back

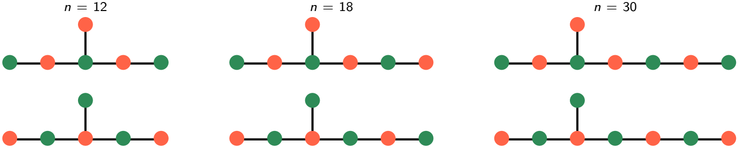
The type A family



The type D family



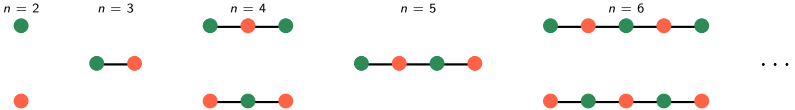
The type E exceptions



Back

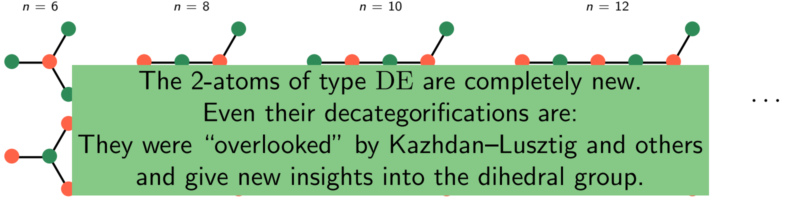
The type A family

2-cells.



The type D family

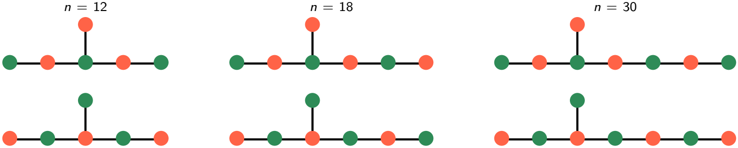
Not 2-cells.



The 2-atoms of type DE are completely new.
Even their decategorifications are:
They were "overlooked" by Kazhdan–Lusztig and others
and give new insights into the dihedral group.

The type E exceptions

Not 2-cells.



Back

SU(3)$_k$

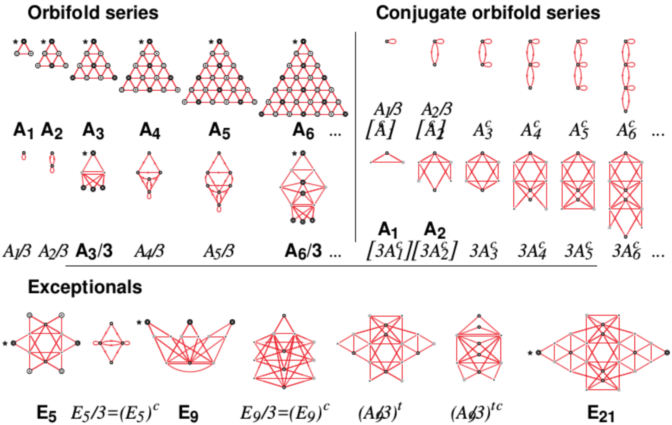


Figure: "Subgroups" of quantum $\mathrm{SU}(3)$.

(Picture from "The classification of subgroups of quantum $\mathrm{SU}(N)$", **Ocneanu** $\sim$**2000**.)

Back