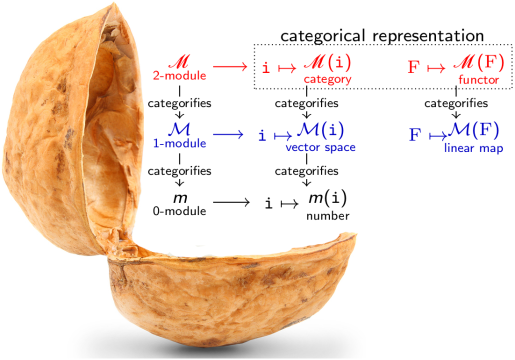
A primer on finitary 2-representation theory

Or: $\mathbb{N}_0$-matrices, my love



Joint with Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz and Xiaoting Zhang

January 2019

1. **Philosophy: "Categorifying" classical representation theory**
   - Some classical results
   - Some categorical results

2. **The decategorified story**
   - $\mathbb{N}_0$-representation theory
   - How cell theory helps

3. **The categorified story**
   - Finitary 2-representation theory
   - How cell theory helps

Pioneers of representation theory.

Let A be a finite-dimensional algebra.

Frobenius ∼1895++, Burnside ∼1900++, Noether ∼1928++. Representation theory is the useful? study of algebra actions

$$\mathcal{M}\colon \mathrm{A} \longrightarrow \mathcal{E}\mathrm{nd}(\mathtt{V}), \qquad \text{“}\mathcal{M}(a) = \text{a matrix in } \mathcal{E}\mathrm{nd}(\mathtt{V})\text{”}$$

with $\mathtt{V}$ being some vector space. (Called modules or representations.)

The “atoms” of such an action are called simple.

Maschke ∼1899, Noether, Schreier ∼1928. All modules are built out of simples (“Jordan–Hölder filtration”).

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Representation theo ... ns

A main goal of representation theory.
Classify simples.

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We want to have a
categorical version of this.

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I am going to explain what we can do at present.

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Dihedral groups as Coxeter groups.

I should do the Hecke case, but I will keep it easy.

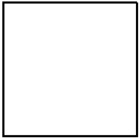
The dihedral groups are of Coxeter type I_{2n}:

$$D_{2n} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \overline{\mathtt{s}}_n = \underbrace{\ldots \mathtt{sts}}_{n} = w_0 = \underbrace{\ldots \mathtt{tst}}_{n} = \overline{\mathtt{t}}_n \rangle,$$

$$e.g.\ \ D_8 = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \mathtt{tsts} = w_0 = \mathtt{stst} \rangle$$

Example. A finite ▸ Coxeter group is the symmetry group of a (semi)regular polyhedron, *e.g.* for I_8 we have a 4-gon:

Idea (Coxeter ∼1934++).



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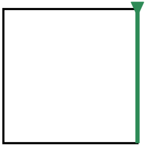

Fact. The symmetries are given by exchanging flags.

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Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

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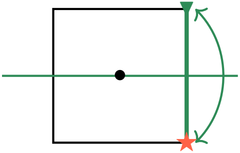

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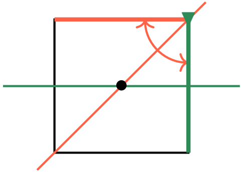

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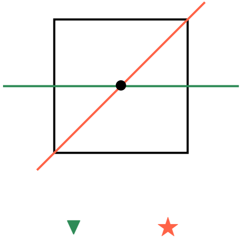

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Write a vertex i for each H_i.

Connect i, j by an n-edge for H_i, H_j having angle $\cos(\pi/n)$.

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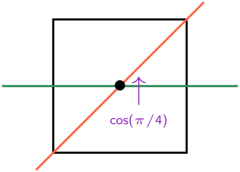

This gives a generator-relation presentation.

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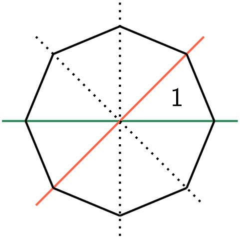

To write down the elements use the Coxeter complex.

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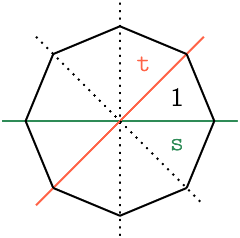

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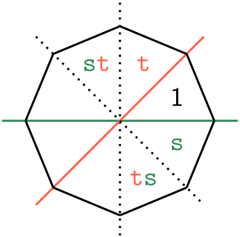

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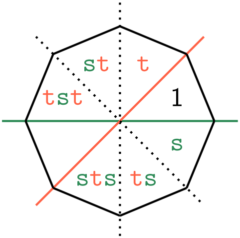

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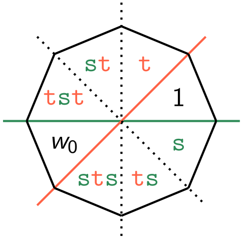

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Dihedral representation theory on one slide.

One-dimensional modules. $\mathcal{M}_{\lambda_{s},\lambda_{t}}, \lambda_{s}, \lambda_{t} \in \mathbb{C}, s \mapsto \lambda_{s}, t \mapsto \lambda_{t}$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathcal{M}_{-1,-1}, \mathcal{M}_{1,-1}, \mathcal{M}_{-1,1}, \mathcal{M}_{1,1}$	$\mathcal{M}_{-1,-1}, \mathcal{M}_{1,1}$

Two-dimensional modules. $\mathcal{M}_{z}, z \in \mathbb{C}, s \mapsto \begin{pmatrix} 1 & z \\ 0 & -1 \end{pmatrix}, t \mapsto \begin{pmatrix} -1 & 0 \\ \bar{z} & 1 \end{pmatrix}$.

$n \equiv 0 \bmod 2$	$n \not\equiv 0 \bmod 2$
$\mathcal{M}_{z}, z \in V(n) - \{0\}$	$\mathcal{M}_{z}, z \in V(n)$

$V(n) = \{2\cos(\pi k/n{-}1) \mid k = 1, \ldots, n-2\}$.

Dihedral representation theory on one slide.

One-dimensiona ... $\mathtt{t}$.

Proposition (Lusztig?).

The list of one- and two-dimensional D_{2n}-modules is a complete, irredundant list of simples.

$\mathcal{M}_{\text{-}1,\text{-}1}, \mathcal{M}_{1,\text{-}1}, \mathcal{M}_{\text{-}1,1}, \mathcal{M}_{1,1}$ | $\mathcal{M}_{\text{-}1,\text{-}1}, \mathcal{M}_{1,1}$

I learned this construction from Mackaay in 2017.

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Note that this requires complex parameters.
In particular, this does not work over $\mathbb{Z}$.

$\mathcal{M}_z, z \in V(n) - \{0\}$ | $\mathcal{M}_z, z \in V(n)$

$V(n) = \{2\cos(\pi k/n-1) \mid k = 1, \ldots, n-2\}$.

Pioneers of 2-representation theory.

Let $\mathscr{C}$ be a finitary 2-category.

Slogan (finitary).
Everything that could be finite is finite.

Etingof–Ostrik, Chuang–Rouquier, many others ∼2000++. Higher representation theory is the useful? study of actions of 2-categories:

$$\mathscr{M}\colon \mathscr{C} \longrightarrow \mathscr{E}\mathrm{nd}(\mathcal{V}), \quad \text{“}\mathscr{M}(\mathrm{F}) = \text{a functor in } \mathscr{E}\mathrm{nd}(\mathcal{V})\text{”}$$

with $\mathcal{V}$ being some finitary category. (Called 2-modules or 2-representations.)

The “atoms” of such an action are called 2-simple.

Mazorchuk–Miemietz ∼2014. All (suitable) 2-modules are built out of 2-simples (“weak 2-Jordan–Hölder filtration”).

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A main goal of 2-representation theory.
Classify 2-simples.

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Pioneers of 2-representation theory

Main examples to keep in mind.

Let $\mathscr{C}$ be a finitary 2-category.

Example. $\mathscr{C} = \mathcal{V}\mathrm{ec}_G$ or $\mathcal{R}\mathrm{ep}(G)$.
Features. Semisimple, classification of 2-simples well-understood.
Comments. I will discuss the classification "in real time".

Etingof–O

representation theory is the useful? study of actions of 2-categories:

Example. $\mathscr{C} = \mathcal{R}\mathrm{ep}_q^{sesi}(g)_{\text{level } n}$.
Features. Semisimple, finitely many 2-simples,
classification of 2-simples only known for $g = \mathrm{Sl}_2$, some guesses for general g.
Comments. The classification of 2-simples is related to Dynkin diagrams.

with

The "atoms" of such an action are called 2-simple.

Ma

2-s

Example. $\mathscr{C} =$ Hecke category.
Features. Non-semisimple, not known whether there are finitely many 2-simples,
classification of 2-simples only known in special cases.
Comments. Hopefully, by the end of the year we have a classification
by reducing the problem to the above examples.

2-modules of dihedral groups.

The dihedral group D_{2n} of the regular n-gon has a Kazhdan–Lusztig (KL) basis.

$$\text{Consider:} \quad \theta_w = \textstyle\sum_{w' \leq w} w', \quad \textit{e.g. } \theta_{\mathtt{st}} = \mathtt{st} + \mathtt{s} + \mathtt{t} + 1.$$

Motivation. The KL basis has some neat integral properties and exists for any Coxeter group. (It isn't as easy to write down, but exists.)

We want a categorical action. So we need:

- ▷ A category $\mathcal{V}$ to act on.
- ▷ Endofunctors acting on $\mathcal{V}$ for the (fixed!) KL basis.
- ▷ The relations of the KL basis have to be satisfied by the functors.
- ▷ A coherent choice of natural transformations. ($\mathscr{C}$ = Hecke category.)

2-modules of dihedral groups.

The dihedral [...] (KL) basis.

C[...] 1.

Motivation. [...]ts for any Coxeter grou[...]

Theorem ∼2016.

Fixing the KL basis, there is a one-to-one correspondence

$$\{(\text{non-trivial})\ \text{2-simple } \mathrm{D}_{2n}\text{-modules}\}/\text{2-iso} \xleftrightarrow{1:1} \{\text{bicolored ADE Dynkin diagrams with Coxeter number } n\}.$$

Thus, its easy to write down a list.

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An algebra P with a fixed basis $\mathrm{B^P}$ with $1 \in \mathrm{B^P}$ is called a $\mathbb{N}_0$-algebra if

$$\mathrm{xy} \in \mathbb{N}_0\mathrm{B^P} \quad (\mathrm{x}, \mathrm{y} \in \mathrm{B^P}).$$

A P-module M with a fixed basis $\mathrm{B^M}$ is called a $\mathbb{N}_0$-module if

$$\mathrm{xm} \in \mathbb{N}_0\mathrm{B^M} \quad (\mathrm{x} \in \mathrm{B^P}, \mathrm{m} \in \mathrm{B^M}).$$

These are $\mathbb{N}_0$-equivalent if there is a $\mathbb{N}_0$-valued change of basis matrix.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}_0$-equivalence comes from 2-equivalence.

Example.

Group algebras of finite groups with basis given by group elements are $\mathbb{N}_0$-algebras.

The regular module is a $\mathbb{N}_0$-module.

A

A P-module M with a fixed basis B^{M} is called a $\mathbb{N}_0$-module if

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A P-m

Example.

The regular module of a group algebra decomposes over $\mathbb{C}$ into simples.

However, this decomposition is almost never an $\mathbb{N}_0$-equivalence.
(I will come back to this in a second.)

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These

Examp ation of 2-categ

Example.

Hecke algebras of (finite) Coxeter groups with their KL basis are $\mathbb{N}_0$-algebras.

For the symmetric group a ▸ miracle happens: all simples are $\mathbb{N}_0$-modules.

Clifford, Munn, Ponizovskiĭ ∼1942++, Kazhdan–Lusztig ∼1979. $x \leq_L y$ if x appears in zy with non-zero coefficient for $z \in B^P$. $x \sim_L y$ if $x \leq_L y$ and $y \leq_L x$. $\sim_L$ partitions P into left cells L. Similarly for right R, two-sided cells J or $\mathbb{N}_0$-modules.

A $\mathbb{N}_0$-module M is transitive if all basis elements belong to the same $\sim_L$ equivalence class. An apex of M is a maximal two-sided cell not killing it.

Fact. Each transitive $\mathbb{N}_0$-module has a unique apex.

Hence, one can study them cell-wise.

Example. Transitive $\mathbb{N}_0$-modules arise naturally as the decategorification of simple 2-modules.

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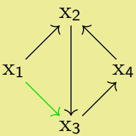
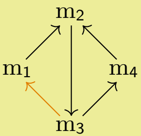
Example. Tran … ication of simple 2-modul …

Philosophy.

Imagine a graph whose vertices are the x's or the m's.
$v_1 \to v_2$ if v_1 appears in zv_2.





cells = connected components
transitive = one connected component

"The atoms of $\mathbb{N}_0$-representation theory".

Question ($\mathbb{N}_0$-representation theory). Classify them!

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Example.

Group algebras with the group element basis have only one cell, G itself.

Transitive $\mathbb{N}_0$-modules are $\mathbb{C}[G/H]$ for $H \subset G$ subgroup/conjugacy. The apex is G.

C ... x ap ... x. $\sim$ ... $\mathbb{N}$

A $\mathbb{N}_0$-module M is transitive if all basis elements belong to the same $\sim_L$ equivalence class. An apex of M is a maximal two-sided cell not killing it.

Fact. Each transitive $\mathbb{N}_0$-module has a unique apex.

Hence, one can study them cell-wise.

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Hecke algebras for the symmetric group with KL basis have ▸ cells coming from the Robinson–Schensted correspondence.

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Example.

Take $G = \mathbb{Z}/3\mathbb{Z}$. Then G has three conjugacy classes and three associated simples. These are given by specifying a third root of unity. ▸ (We do not like these!)

G has two subgroups; $\{e\}$ and G. The associated $\mathbb{N}_0$-modules are the regular and the trivial G-module.

▸ Another example.

Natural, and computable, examples of transitive $\mathbb{N}_0$-modules are the so-called cell modules which, in some sense, play the role of regular modules.

Fix a left cell L. Let $\mathrm{M}(\geq_{\mathsf{L}})$, respectively $\mathrm{M}(>_{\mathsf{L}})$, be the $\mathbb{N}_0$-modules spanned by all $\mathrm{x} \in \mathrm{B}^{\mathrm{P}}$ in the union $\mathsf{L}' \geq_{\mathsf{L}} \mathsf{L}$, respectively $\mathsf{L}' >_{\mathsf{L}} \mathsf{L}$.
We call $\mathrm{C}_{\mathsf{L}} = \mathrm{M}(\geq_{\mathsf{L}})/\mathrm{M}(>_{\mathsf{L}})$ the (left) cell module for L.

Fact. "Cell $\Rightarrow$ transitive $\mathbb{N}_0$-module".

Empirical fact. In well-behaved cases "Cell $\Leftrightarrow$ transitive $\mathbb{N}_0$-module", and classification of transitive $\mathbb{N}_0$-modules is fairly easy.

Question. Are there natural examples where "Cell $\not\Leftarrow$ transitive $\mathbb{N}_0$-module"?

Example. Decategorifications of cell 2-modules are key examples of cell modules.

Example.

$\mathbb{C}[G]$ with the group element basis has only one cell module, the regular module.

However, the transitive $\mathbb{N}_0$-modules $\mathbb{C}[G/H]$ are cell modules for G/H if $H \triangleleft G$.
So morally, "Cell $\Leftrightarrow$ transitive $\mathbb{N}_0$-module".

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For Hecke algebras of the symmetric group with KL basis "Cell $\Leftrightarrow$ transitive $\mathbb{N}_0$-module". ▸ Example.

In general, for Hecke algebras the cell modules are Lusztig's cell modules studied in connection with reductive groups in characteristic p.

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Example (dihedral case).

Cells:

cell	0	1	2
size	1	$2n-2$	1
sr	yes	no	yes

1 for n even:

$\frac{n}{2}$	$\frac{n-2}{2}$
$\frac{n-2}{2}$	$\frac{n}{2}$

1 for n odd:

$\frac{n-1}{2}$	$\frac{n-1}{2}$
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In the dihedral case the DE-modules are not cell modules.

An additive, $\mathbb{k}$-linear, idempotent complete, Krull–Schmidt category $\mathcal{C}$ is called finitary if it has only finitely many isomorphism classes of indecomposable objects and the morphism sets are finite-dimensional. A 2-category $\mathscr{C}$ with finitely many objects is finitary if its hom-categories are finitary, $\circ_h$-composition is additive and linear, and identity 1-morphisms are indecomposable.

A simple transitive 2-module (2-simple) of $\mathscr{C}$ is an additive, $\mathbb{k}$-linear 2-functor

$$\mathscr{M}\colon \mathscr{C} \to \mathscr{A}^{\mathrm{f}}(= \text{2-cat of finitary cats}),$$

such that there are no non-zero proper $\mathscr{C}$-stable ideals.

There is also the notion of 2-equivalence.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}_0$-equivalence comes from 2-equivalence.

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Mazorchuk–Miemietz ∼2014.

2-Simples $\leftrightsquigarrow$ simples (*e.g.* weak 2-Jordan–Hölder filtration),

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Chan–Mazorchuk ∼2016.

Every 2-simple has an associated apex not killing it.

Thus, we can again study them separately for different cells.

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$\mathrm{B}\text{-p}\mathcal{M}\mathrm{od}$ (with B finite-dimensional) is a prototypical object of $\mathscr{A}^{\mathrm{f}}$.

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Example.

G can be (naively) categorified using G-graded vector spaces $\mathcal{V}\mathrm{ec}_G \in \mathscr{A}^{\mathrm{f}}$.

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Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules arise naturally as the decategorification of 2...

Example (Mazorchuk–Miemietz & Chuang–Rouquier & Khovanov–Lauda & ...).

2-Kac–Moody algebras (+fc) are finitary 2-categories.

Their 2-simples are categorifications of the simples.

An additive, $\mathbb{k}$-linear, idempotent complete, Krull–Schmidt category $\mathcal{C}$ is called finitary if it has only finitely many isomorphism classes of indecomposable objects and the morphism sets are finite dimensional. A 2-category $\mathscr{C}$ with finitely many

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2-categories and 2-modules, and $\mathbb{N}_0$-equivalence comes from 2-equivalence.

Example.

Fusion or modular categories are semisimple examples
of finitary 2-categories. (Example. $\mathcal{R}\mathrm{ep}_q^{sesi}(g)_n$.)
Their 2-modules play a prominent role in quantum algebra and topology.

An additive, $\mathbb{k}$-linear, idempotent complete, Krull–Schmidt category $\mathcal{C}$ is called finitary if it has only finitely many isomorphism classes of indecomposable objects and the morphism sets are finite-dimensional. A 2-category $\mathscr{C}$ with finitely many objects is finitary if its hom-categories are finitary, $\circ_h$-composition is additive and linear, and identity 1-morphisms are indecomposable.

A simple ... unctor

such that ...

There is ...

On the categorical level the impact of the choice of basis is evident:

These are the indecomposable objects in some 2-category,
and different bases are categorified by
potentially non-equivalent 2-categories.

So, of course, the 2-representation theory differs!

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}_0$-equivalence comes from 2-equivalence.

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Question ("2-representation theory").

Classify all 2-simples of a fixed finitary 2-category.

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This is the categorification of

'Classify all simples a fixed finite-dimensional algebra',

but much harder, *e.g.* it is unknown whether
there are always only finitely many 2-simples (probably not).

One can do even better than just reducing the theory to a fixed apex; one can reduce to the diagonal. Roughly:

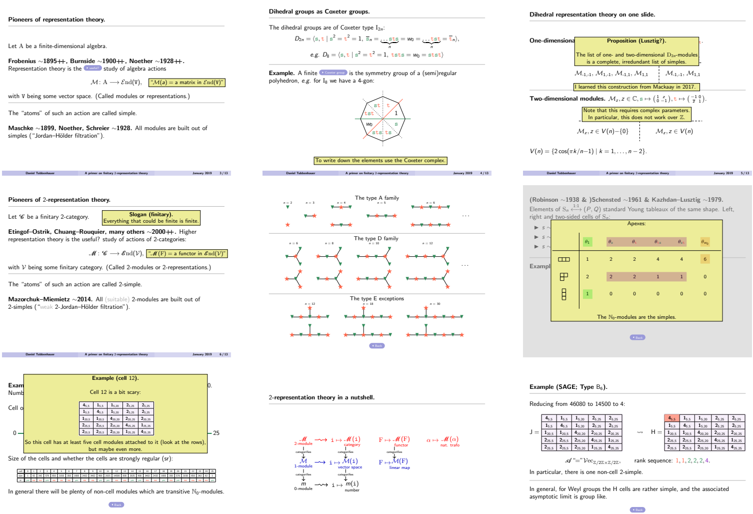
For each two-sided cell J fix a left cell L and consider the diagonal cell $\mathrm{H} = \mathrm{L} \cup \mathrm{L}^{*}$.

Green ∼1951, Mackaay–Mazorchuk–Miemietz–Zhang ∼2018. For any fiat 2-category $\mathscr{C}$ there exists a fiat 2-subcategory $\mathscr{A}$ such that

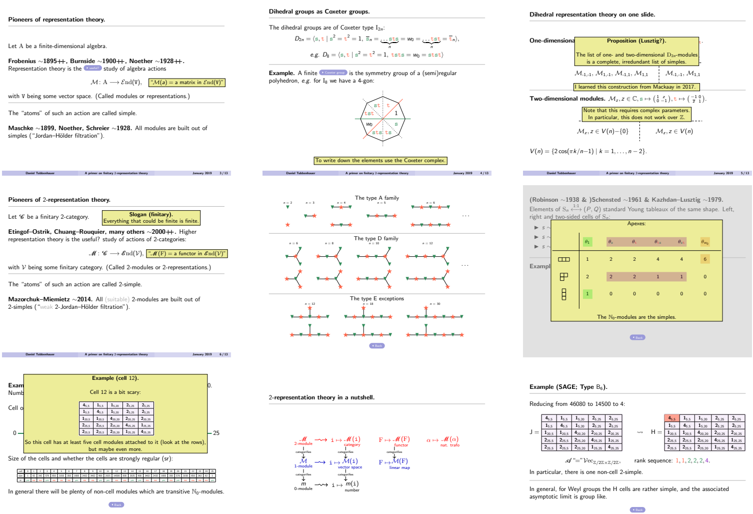
$$\left\{\begin{matrix}\text{2-simples of } \mathscr{C} \\ \text{with apex J}\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{2-simples of } \mathscr{A} \\ \text{with apex H}\end{matrix}\right\}$$

This ▸ reduces the classification to the diagonal H.

We ▸ hope that this will finally lead to a classification of 2-simples for Soergel bimodules using asymptotic Hecke algebras and categories. (At the moment this is widely open.)



There is still much to do...



Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

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Nowadays representation theory is pervasive across mathematics, and beyond.

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But this wasn't clear at all when Frobenius started it.

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samen Factor f abgesehen) einen relativen Charakter von $\mathfrak{H}$, und umgekehrt lässt sich jeder relative Charakter von $\mathfrak{H}$, $\chi_0, \cdots \chi_{k-1}$, auf eine oder mehrere Arten durch Hinzufügung passender Werthe $\chi_k, \cdots \chi_{k'-1}$ zu einem Charakter von $\mathfrak{H}'$ ergänzen.

§ 8.

Ich will nun die Theorie der Gruppencharaktere an einigen Beispielen erläutern. Die geraden Permutationen von 4 Symbolen bilden eine Gruppe $\mathfrak{H}$ der Ordnung $h = 12$. Ihre Elemente zerfallen in 4 Classen, die Elemente der Ordnung 2 bilden eine zweiseitige Classe (1), die der Ordnung 3 zwei inverse Classen (2) und (3) = (2′). Sei ρ eine primitive cubische Wurzel der Einheit.

Tetraeder. $h = 12$.

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	h_α
χ_0	1	3	1	1	1
χ_1	1	−1	1	1	3
χ_2	1	0	ρ	ρ^2	4
χ_3	1	0	ρ^2	ρ	4

Figure: "Über Gruppencharaktere (characters of groups)" by Frobenius (1896). Bottom: first published character table.

Note the root of unity ρ!

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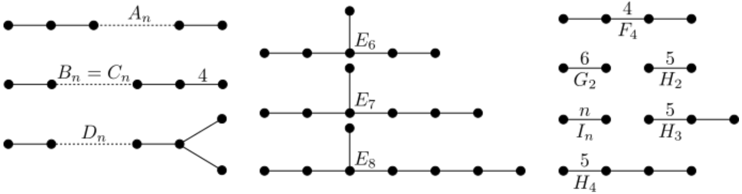

Figure: The connected Coxeter diagrams of finite type. Their numbers ordered by dimension: $1, \infty, 3, 5, 3, 4, 4, 4, 3, 3, 3, 3, 3, \dots$.

Examples.
Type A_3 ⇜⇝ tetrahedron ⇜⇝ symmetric group S_4.
Type B_3 ⇜⇝ cube/octahedron ⇜⇝ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$.
Type H_3 ⇜⇝ dodecahedron/icosahedron ⇜⇝ exceptional Coxeter group.

(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

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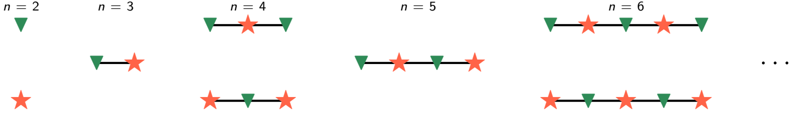
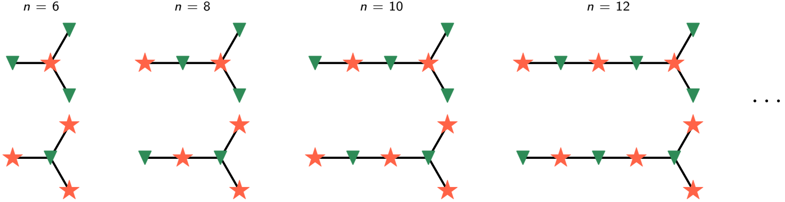
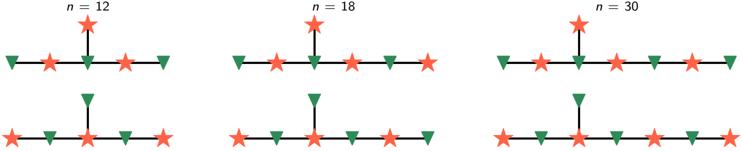
The type A family



The type D family



The type E exceptions



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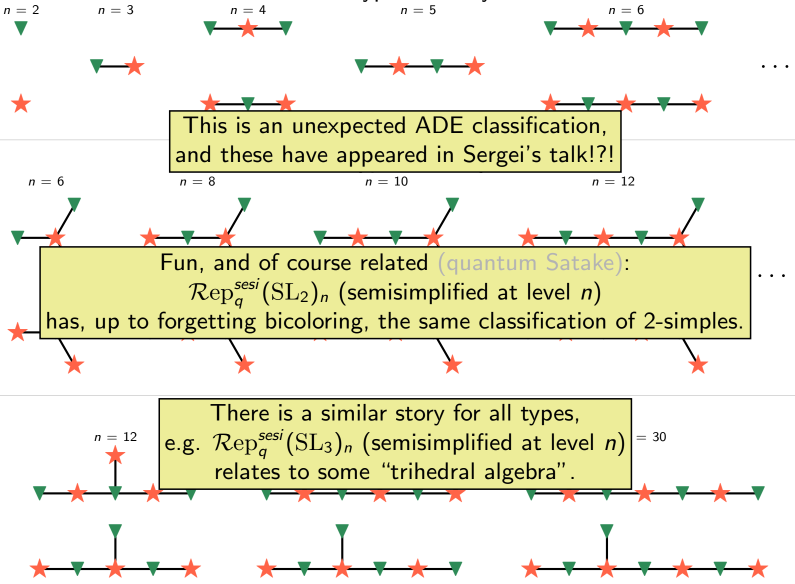


This is an unexpected ADE classification, and these have appeared in Sergei's talk!?!

Fun, and of course related (quantum Satake): $\mathcal{R}\mathrm{ep}_q^{sesi}(\mathrm{SL}_2)_n$ (semisimplified at level n) has, up to forgetting bicoloring, the same classification of 2-simples.

There is a similar story for all types, e.g. $\mathcal{R}\mathrm{ep}_q^{sesi}(\mathrm{SL}_3)_n$ (semisimplified at level n) relates to some "trihedral algebra".

Back

The KL basis elements for S_3 with $\mathtt{s} = (1,2), \mathtt{t} = (2,3)$ and $\mathtt{sts} = w_0 = \mathtt{tst}$ are:

$$\theta_1 = 1, \quad \theta_{\mathtt{s}} = \mathtt{s} + 1, \quad \theta_{\mathtt{t}} = \mathtt{t} + 1, \quad \theta_{\mathtt{ts}} = \mathtt{ts} + \mathtt{s} + \mathtt{t} + 1,$$
$$\theta_{\mathtt{st}} = \mathtt{st} + \mathtt{s} + \mathtt{t} + 1, \quad \theta_{w_0} = w_0 + \mathtt{ts} + \mathtt{st} + \mathtt{s} + \mathtt{t} + 1.$$

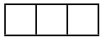
	1	s	t	ts	st	w_0
	1	1	1	1	1	1
	2	0	0	-1	-1	0
	1	-1	-1	1	1	-1

Figure: The character table of S_3.

Back

The KL basis elements for S_3 with $s = (1,2)$, $t = (2,3)$ and $sts = w_0 = tst$ are:

$$\theta_1 = 1, \quad \theta_s = s + 1, \quad \theta_t = t + 1, \quad \theta_{ts} = ts + s + t + 1,$$
$$\theta_{st} = st + s + t + 1, \quad \theta_{w_0} = w_0 + ts + st + s + t + 1.$$

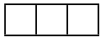
	θ_1	θ_s	θ_t	θ_{ts}	θ_{st}	θ_{w_0}
	1	2	2	4	4	6
	2	2	2	1	1	0
	1	0	0	0	0	0

Figure: The character table of S_3.

Back

The KL basis elements for S_3 with $\mathtt{s} = (1,2), \mathtt{t} = (2,3)$ and $\mathtt{sts} = w_0 = \mathtt{tst}$ are:

$$\theta_1 = 1, \quad \theta_{\mathtt{s}} = \mathtt{s} + 1, \quad \theta_{\mathtt{t}} = \mathtt{t} + 1, \quad \theta_{\mathtt{ts}} = \mathtt{ts} + \mathtt{s} + \mathtt{t} + 1,$$
$$\theta_{\mathtt{st}} = \mathtt{st} + \mathtt{s} + \mathtt{t} + 1, \quad \theta_{w_0} = w_0 + \mathtt{ts} + \mathtt{st} + \mathtt{s} + \mathtt{t} + 1.$$

	θ_1	$\theta_{\mathtt{s}}$	$\theta_{\mathtt{t}}$	$\theta_{\mathtt{ts}}$	$\theta_{\mathtt{st}}$	θ_{w_0}
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	2	2	2	1	1	0
	1	0	0	0	0	0

Remark.

This non-negativity of the KL basis
is true for all symmetric groups,
but not for most other Coxeter groups (*cf.* dihedral case).

Figure: The character table of S_3.

(Robinson ∼1938 &)Schensted ∼1961 & Kazhdan–Lusztig ∼1979.

Elements of $\mathrm{S}_n \xleftrightarrow{1:1} (P, Q)$ standard Young tableaux of the same shape. Left, right and two-sided cells of S_n:

- $s \sim_{\mathrm{L}} t$ if and only if $Q(s) = Q(t)$.
- $s \sim_{\mathrm{R}} t$ if and only if $P(s) = P(t)$.
- $s \sim_{\mathrm{J}} t$ if and only if $P(s)$ and $P(t)$ have the same shape.

Example ($n = 3$).

$1 \leftrightsquigarrow \begin{smallmatrix}1&2&3\end{smallmatrix}, \begin{smallmatrix}1&2&3\end{smallmatrix}$

$\mathtt{s} \leftrightsquigarrow \begin{smallmatrix}1&3\\2&\end{smallmatrix}, \begin{smallmatrix}1&3\\2&\end{smallmatrix}$ $\qquad$ $\mathtt{ts} \leftrightsquigarrow \begin{smallmatrix}1&2\\3&\end{smallmatrix}, \begin{smallmatrix}1&3\\2&\end{smallmatrix}$

$\mathtt{t} \leftrightsquigarrow \begin{smallmatrix}1&2\\3&\end{smallmatrix}, \begin{smallmatrix}1&2\\3&\end{smallmatrix}$ $\qquad$ $\mathtt{st} \leftrightsquigarrow \begin{smallmatrix}1&3\\2&\end{smallmatrix}, \begin{smallmatrix}1&2\\3&\end{smallmatrix}$

$w_0 \leftrightsquigarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\\3\end{smallmatrix}$

(Robinson ∼1938 &)Schensted ∼1961 & Kazhdan–Lusztig ∼1979.

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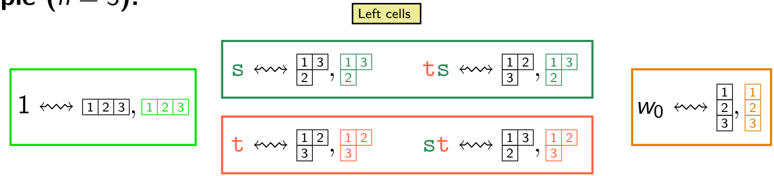
Example $(n = 3)$**.**



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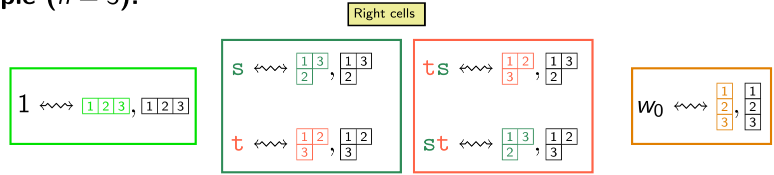
Example ($n = 3$).



Back

(Robinson $\sim$1938 &)Schensted $\sim$1961 & Kazhdan–Lusztig $\sim$1979.

Elements of $\mathrm{S}_n \xleftrightarrow{1:1} (P, Q)$ standard Young tableaux of the same shape. Left, right and two-sided cells of S_n:

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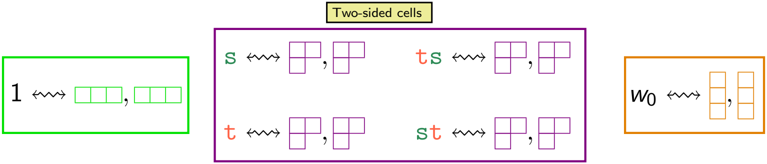
Example ($n = 3$).



(Robinson ∼1938 &)Schensted ∼1961 & Kazhdan–Lusztig ∼1979.

Elements of $\mathrm{S}_n \xleftrightarrow{1:1} (P, Q)$ standard Young tableaux of the same shape. Left, right and two-sided cells of S_n:

- $s \sim$
- $s \sim$
- $s \sim$

Exampl

Apexes:

	θ_1	θ_s	θ_t	θ_{ts}	θ_{st}	θ_{w_0}
	1	2	2	4	4	6
	2	2	2	1	1	0
	1	0	0	0	0	0

The $\mathbb{N}_0$-modules are the simples.

Back

The regular $\mathbb{Z}/3\mathbb{Z}$-module is

$$0 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad 1 \leftrightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \& \quad 2 \leftrightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Jordan decomposition over $\mathbb{C}$ with $\zeta^3 = 1$ gives

$$0 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad 1 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^{-1} \end{pmatrix} \quad \& \quad 2 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^{-1} & 0 \\ 0 & 0 & \zeta \end{pmatrix}$$

However, Jordan decomposition over $\mathbb{F}_3$ gives

$$0 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad 1 \leftrightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad 2 \leftrightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and the regular module does not decompose.

Back

Example ($G = \mathrm{D}_8$). Here we have three different notions of "atoms".

Classical representation theory. The simples from before.

	$\mathcal{M}_{-1,-1}$	$\mathcal{M}_{1,-1}$	$\mathcal{M}_{\sqrt{2}}$	$\mathcal{M}_{-1,1}$	$\mathcal{M}_{1,1}$
atom	sign		rotation		trivial
rank	1	1	2	1	1

Group element basis. Subgroups and ranks of $\mathbb{N}_0$-modules.

subgroup	1	$\langle st \rangle$	$\langle w_0 \rangle$	$\langle w_0, s \rangle$	$\langle w_0, sts \rangle$	G
atom	regular	$\mathcal{M}_{1,1} \oplus \mathcal{M}_{-1,-1}$	$\mathcal{M}_{\sqrt{2}} \oplus \mathcal{M}_{\sqrt{2}}$	$\mathcal{M}_{1,1} \oplus \mathcal{M}_{1,-1}$	$\mathcal{M}_{1,1} \oplus \mathcal{M}_{-1,1}$	trivial
rank	8	2	4	2	2	1

KL basis. ADE diagrams and ranks of $\mathbb{N}_0$-modules.

	bottom cell			top cell
atom	sign	$\mathcal{M}_{1,-1} \oplus \mathcal{M}_{\sqrt{2}}$	$\mathcal{M}_{-1,1} \oplus \mathcal{M}_{\sqrt{2}}$	trivial
rank	1	3	3	1

Back

Example (SAGE). The symmetric group on 4 strands. Number of elements: 24. Number of cells: 5, named 0 (trivial) to 4 (top).

Cell order:

$$0 — 1 — 2 — 3 — 4$$

Size of the cells:

cell	0	1	2	3	4
size	1	9	4	9	1

Left cells are rows, right cells are columns.

Cell 1 is *e.g.*

s_1	$s_2 s_1$	$s_3 s_2 s_1$
$s_1 s_2$	s_2	$s_3 s_2$
$s_1 s_2 s_3$	$s_2 s_3$	s_3

$\xrightarrow{\text{number of elements}}$

1	1	1
1	1	1
1	1	1

Such cells of square size are called strongly regular.

Back

Further example

Example (SAGE). The symmetric group on 4 strands. Number of elements: 24. Number of cells: 5, named 0 (trivial) to 4 (top).

Fact.

Each left-right-intersection contains at least one element.
So strongly regular cells are as easy as possible.

Cell order:

Size of the cells:

cell	0	1	2	3	4
size	1	9	4	9	1

Cell 1 is *e.g.*

s_1	s_2s_1	$s_3s_2s_1$
s_1s_2	s_2	s_3s_2
$s_1s_2s_3$	s_2s_3	s_3

$\xrightarrow{\text{number of elements}}$

1	1	1
1	1	1
1	1	1

Such cells of square size are called strongly regular.

Back

Further example

Example (SAGE). The symmetric group on 4 strands. Number of elements: 24. Number of cells: 5, named 0 (trivial) to 4 (top).

Fact.

"Cell $\Leftrightarrow$ transitive $\mathbb{N}_0$-module" holds
$\mathbb{N}_0$-algebras with only strongly regular cells.

Cell order:

Size of the cells:

cell	0	1	2	3	4
size	1	9	4	9	1

Cell 1 is *e.g.*

s_1	s_2s_1	$s_3s_2s_1$
s_1s_2	s_2	s_3s_2
$s_1s_2s_3$	s_2s_3	s_3

$\xrightarrow{\text{number of elements}}$

1	1	1
1	1	1
1	1	1

Such cells of square size are called strongly regular.

Back

Further example

Example (SAGE). The symmetric group on 4 strands. Number of elements: 24. Number of cells: 5, named 0 (trivial) to 4 (top).

Fact.

"Cell $\Leftrightarrow$ transitive $\mathbb{N}_0$-module" holds $\mathbb{N}_0$-algebras with only strongly regular cells.

Cell order:

Size of the cells:

cell	0	1	2	3	4
size	1	9	4	9	1

Cell 1 is *e.g.*

Fact.

For the symmetric group all cells are strongly regular.

s_1s_2	s_2	s_3s_2
$s_1s_2s_3$	s_2s_3	s_3

$\longrightarrow$

1	1	1
1	1	1

Such cells of square size are called strongly regular.

Back

Further example

Example (SAGE). The symmetric group on 4 strands. Number of elements: 24. Number of cells: 5, named 0 (trivial) to 4 (top).

Cell orde

Size of t

Example. There are three rows with three elements, so three cells modules of dimension three.

All of them are $\mathbb{N}_0$-equivalent and here is one of them:

$$s_1 \leftrightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } s_2 \leftrightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } s_3 \leftrightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Cell 1 is *e.g.*

s_1	s_2s_1	$s_3s_2s_1$
s_1s_2	s_2	s_3s_2
$s_1s_2s_3$	s_2s_3	s_3

$\xrightarrow{\text{number of elements}}$

1	1	1
1	1	1
1	1	1

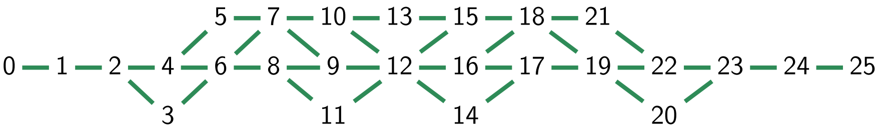
Such cells of square size are called strongly regular.

◂ Back

▸ Further example

Example (SAGE). The Weyl group of type B_6. Number of elements: 46080. Number of cells: 26, named 0 (trivial) to 25 (top).

Cell order:



Size of the cells and whether the cells are strongly regular (sr):

cell	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
size	1	62	342	576	650	3150	350	1600	2432	3402	900	2025	14500	600	2025	900	3402	2432	1600	350	576	3150	650	342	62	1
sr	yes	no	no	yes	no	no	no	yes	no	no	yes	yes	no	no	yes	yes	no	no	yes	no	yes	no	no	no	no	yes

In general there will be plenty of non-cell modules which are transitive $\mathbb{N}_0$-modules.

Back

Exam 0.

Numb

Cell o

Example (cell 12).

Cell 12 is a bit scary:

$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{5,5}$	$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{20,5}$	$\mathbf{1}_{20,5}$	$\mathbf{4}_{20,20}$	$\mathbf{2}_{20,25}$	$\mathbf{2}_{20,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{4}_{25,25}$	$\mathbf{1}_{25,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{1}_{25,25}$	$\mathbf{4}_{25,25}$

So this cell has at least five cell modules attached to it (look at the rows), but maybe even more.

0 25

Size of the cells and whether the cells are strongly regular (sr):

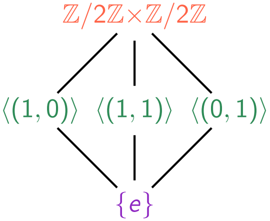
cell	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
size	1	62	342	576	650	3150	350	1600	2432	3402	900	2025	14500	600	2025	900	3402	2432	1600	350	576	3150	650	342	62	1
sr	yes	no	no	yes	no	no	no	yes	no	no	yes	yes	no	no	yes	yes	no	no	yes	no	yes	no	no	no	no	yes

In general there will be plenty of non-cell modules which are transitive $\mathbb{N}_0$-modules.

Back

Example ($G = \mathbb{Z}/2 \times \mathbb{Z}/2$).

Subgroups, Schur multipliers and 2-simples.



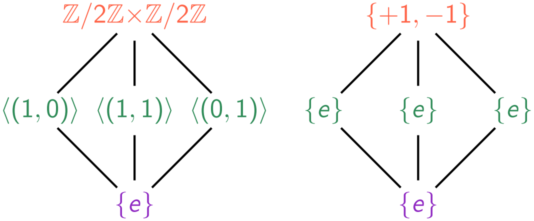
In particular, there are two categorifications of the trivial module, and the rank sequences read

$$\text{decat: } 1, 2, 2, 2, 4, \qquad \text{cat: } 1, 1, 2, 2, 2, 4.$$

Back

Example ($G = \mathbb{Z}/2\times\mathbb{Z}/2$).

Subgroups, Schur multipliers and 2-simples.



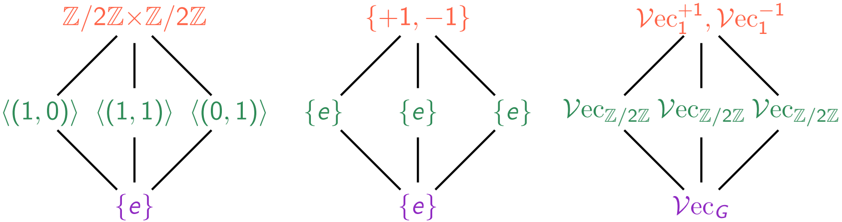
In particular, there are two categorifications of the trivial module, and the rank sequences read

$$\text{decat: } 1, 2, 2, 2, 4, \qquad \text{cat: } 1, 1, 2, 2, 2, 4.$$

Back

Example ($G = \mathbb{Z}/2 \times \mathbb{Z}/2$).

Subgroups, Schur multipliers and 2-simples.



In particular, there are two categorifications of the trivial module, and the rank sequences read

decat: $1, 2, 2, 2, 4,$ cat: $1, 1, 2, 2, 2, 4.$

Back

Example (Strongly regular cells).

For a strongly regular cell H consists only of one element:

$$\mathrm{J} = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \;\&\; \mathrm{L} = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \;\&\; \mathrm{L}^* = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \quad \rightsquigarrow \quad \mathrm{H} = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

and the associated 2-category $\mathscr{A}$ is has only one indecomposable. Not surprisingly, such a 2-category has only one 2-simple.

In particular, this reduces the classification of a potentially complicated 2-category to another classification problem for a trivial 2-category.

Back

Example (SAGE; Type B_6).

Reducing from 46080 to 14500 to 4:

$$J = \begin{array}{|c|c|c|c|c|} \hline \mathbf{4}_{5,5} & \mathbf{1}_{5,5} & \mathbf{1}_{5,20} & \mathbf{2}_{5,25} & \mathbf{2}_{5,25} \\ \hline \mathbf{1}_{5,5} & \mathbf{4}_{5,5} & \mathbf{1}_{5,20} & \mathbf{2}_{5,25} & \mathbf{2}_{5,25} \\ \hline \mathbf{1}_{20,5} & \mathbf{1}_{20,5} & \mathbf{4}_{20,20} & \mathbf{2}_{20,25} & \mathbf{2}_{20,25} \\ \hline \mathbf{2}_{25,5} & \mathbf{2}_{25,5} & \mathbf{2}_{25,20} & \mathbf{4}_{25,25} & \mathbf{1}_{25,25} \\ \hline \mathbf{2}_{25,5} & \mathbf{2}_{25,5} & \mathbf{2}_{25,20} & \mathbf{1}_{25,25} & \mathbf{4}_{25,25} \\ \hline \end{array} \quad \rightsquigarrow \quad H = \begin{array}{|c|c|c|c|c|} \hline \mathbf{4}_{5,5} & \mathbf{1}_{5,5} & \mathbf{1}_{5,20} & \mathbf{2}_{5,25} & \mathbf{2}_{5,25} \\ \hline \mathbf{1}_{5,5} & \mathbf{4}_{5,5} & \mathbf{1}_{5,20} & \mathbf{2}_{5,25} & \mathbf{2}_{5,25} \\ \hline \mathbf{1}_{20,5} & \mathbf{1}_{20,5} & \mathbf{4}_{20,20} & \mathbf{2}_{20,25} & \mathbf{2}_{20,25} \\ \hline \mathbf{2}_{25,5} & \mathbf{2}_{25,5} & \mathbf{2}_{25,20} & \mathbf{4}_{25,25} & \mathbf{1}_{25,25} \\ \hline \mathbf{2}_{25,5} & \mathbf{2}_{25,5} & \mathbf{2}_{25,20} & \mathbf{1}_{25,25} & \mathbf{4}_{25,25} \\ \hline \end{array}$$

$\mathscr{A}$ "=" $\mathcal{V}\mathrm{ec}_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$, rank sequence: 1, 1, 2, 2, 2, 4.

In particular, there is one non-cell 2-simple.

In general, for Weyl groups the H cells are rather simple, and the associated asymptotic limit is group like.

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