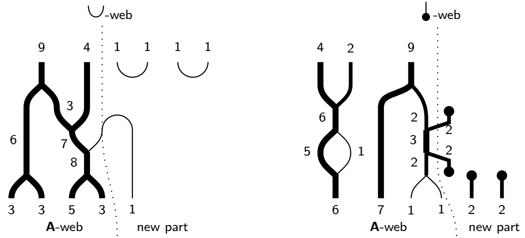
Webs and q-Howe dualities in types **BCD**

Or: A story about "howe" I failed

Daniel Tubbenhauer



Joint work with Antonio Sartori

April 2017

1. The type **A** story
   - Classical Schur-Weyl duality
   - Howe's dualities in type **A**
2. The type **BCD** story
   - Classical Schur-Weyl-Brauer duality
   - Howe's dualities in types **BCD**
3. The quantum story
   - Various quantizations
   - Concluding remarks

A pioneer of representation theory

▸ Schur's remarkable relationship between $\mathfrak{gl}_n$ and the symmetric group S_k:

Schur ∼1901. Let $\mathrm{V} = \mathrm{V}^{\mathfrak{gl}} = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{gl}_n) \circlearrowright \underbrace{\mathrm{V} \otimes \cdots \otimes \mathrm{V}}_{k \text{ times}} \circlearrowleft \mathbb{C}[S_k]$$

generating each other's centralizer. The $\mathbf{U}(\mathfrak{gl}_n)$-$\mathbb{C}[S_k]$-bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} \mathrm{L}(\mathfrak{gl}_n, \lambda) \otimes \mathrm{L}(S_k, \lambda^{\mathrm{T}}).$$

The λ's are partitions (Young diagrams) of k with at most n rows.

Schur's remarkable relationship between $\mathfrak{gl}_n$ and the symmetric group S_k:

First statement

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$$\bigoplus_{\lambda \in \mathfrak{P}}$$

The precise form does not matter for today. It is only important that one can make it explicit.

The λ's are partitions (Young diagrams) of k with at most n rows.

The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowright \underbrace{\mathrm{V}\otimes\cdots\otimes\mathrm{V}}_{k \text{ times}} \circlearrowleft \overset{\text{use}}{\mathbb{C}[S_k]}$$

Schur's first statement gives a functor

Categorical version of the symmetric group $\boldsymbol{\mathcal{S}} \xrightarrow{\Phi} \boldsymbol{\mathcal{R}}\mathbf{ep}(\mathfrak{gl}_n)$

The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowright \underbrace{\mathrm{V} \otimes \cdots \otimes \mathrm{V}}_{k \text{ times}} \circlearrowleft \overset{\text{use}}{\mathbb{C}[S_k]}$$

Schur's second statement gives a full functor

$$\mathcal{S} \xrightarrow[\text{full}]{\Phi} \mathcal{R}\mathbf{ep}(\mathfrak{gl}_n)$$

The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowright \underbrace{\mathrm{V} \otimes \cdots \otimes \mathrm{V}}_{k \text{ times}} \circlearrowleft \overset{\text{use}}{\mathbb{C}[S_k]}$$

Schur's third statement gives a full functor

$$\mathcal{S} \xrightarrow[\text{full}]{\Phi} \mathcal{R}\mathbf{ep}(\mathfrak{gl}_n)$$

$$\mathcal{S}/\text{“}\ker(\Phi)\text{”} \xrightarrow[\text{fully faithful}]{\Phi} \mathcal{R}\mathbf{ep}(\mathfrak{gl}_n)$$

whose “kernel $\ker(\Phi)$” can be calculated.

Hence, up to taking duals and Karoubi closures, Schur gave us a ▸ diagrammatic presentation of the representation category $\mathcal{R}\mathbf{ep}(\mathfrak{gl}_n)$ of $\mathfrak{gl}_n$.

"Thick" Schur-Weyl duality

One of Howe's remarkable relationships between $\mathfrak{gl}_n$ and $\mathfrak{gl}_k$:

Howe ∼1975. Let $\mathrm{V} = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{gl}_n) \circlearrowright \underbrace{{\textstyle\bigwedge}^{\bullet}\mathrm{V} \otimes \cdots \otimes {\textstyle\bigwedge}^{\bullet}\mathrm{V}}_{k \text{ times}} \circlearrowleft \mathbf{U}(\mathfrak{gl}_k)$$

generating each other's centralizer, and ${\textstyle\bigwedge}^{a_1}\mathrm{V} \otimes \cdots \otimes {\textstyle\bigwedge}^{a_k}\mathrm{V}$ is the $(a_1, \dots, a_k)$th weight space as regards $\mathbf{U}(\mathfrak{gl}_k)$. The $\mathbf{U}(\mathfrak{gl}_n)$-$\mathbf{U}(\mathfrak{gl}_k)$-bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} \mathrm{L}(\mathfrak{gl}_n, \lambda) \otimes \mathrm{L}(\mathfrak{gl}_k, \lambda^{\mathrm{T}}).$$

The λ's are partitions with at most k columns and n rows.

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$1^1/_2$th statement

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Again: The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowright \underbrace{\textstyle\bigwedge^{\bullet}\mathrm{V} \otimes \cdots \otimes \bigwedge^{\bullet}\mathrm{V}}_{k \text{ times}} \circlearrowleft \overset{\text{use}}{\mathbf{U}(\mathfrak{gl}_k)}$$

Howe's first statement gives a functor

Dot version generated by weight space idempotents 1_λ, and E_i and F_i

$$\dot{\mathbf{U}}(\mathfrak{gl}_k) \xrightarrow{\Phi^{\mathrm{ext}}_{\mathbf{A}}} \mathcal{R}\mathbf{ep}(\mathfrak{gl}_n)$$

Again: The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowright \underbrace{\textstyle\bigwedge^{\bullet}\mathrm{V} \otimes \cdots \otimes \bigwedge^{\bullet}\mathrm{V}}_{k \text{ times}} \circlearrowleft \overset{\text{use}}{\mathbf{U}(\mathfrak{gl}_k)}$$

Howe's second statement gives a full functor

$$\dot{\mathbf{U}}(\mathfrak{gl}_k) \xrightarrow[\text{full}]{\Phi_{\mathbf{A}}^{\mathrm{ext}}} \mathcal{R}\mathbf{ep}(\mathfrak{gl}_n)$$

Again: The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowright \underbrace{\textstyle\bigwedge^{\bullet}\mathrm{V} \otimes \cdots \otimes \bigwedge^{\bullet}\mathrm{V}}_{k \text{ times}} \circlearrowleft \overset{\text{use}}{\mathbf{U}(\mathfrak{gl}_k)}$$

Howe's third statement gives a full functor

$$\dot{\mathbf{U}}(\mathfrak{gl}_k) \xrightarrow[\text{full}]{\Phi_{\mathbf{A}}^{\mathrm{ext}}} \mathcal{R}\mathrm{ep}(\mathfrak{gl}_n)$$

$$\dot{\mathbf{U}}(\mathfrak{gl}_k)/\text{“}\ker(\Phi_{\mathbf{A}}^{\mathrm{ext}})\text{”} \xrightarrow[\text{fully faithful}]{\Phi_{\mathbf{A}}^{\mathrm{ext}}} \mathcal{R}\mathrm{ep}(\mathfrak{gl}_n)$$

whose “kernel $\ker(\Phi_{\mathbf{A}}^{\mathrm{ext}})$” we can calculate.

Again: The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowright \underbrace{\textstyle\bigwedge^{\bullet}\mathrm{V} \otimes \cdots \otimes \bigwedge^{\bullet}\mathrm{V}}_{k \text{ times}} \circlearrowleft \overset{\text{use}}{\mathbf{U}(\mathfrak{gl}_k)}$$

Howe's $1^1/_2$th statement ▸ defines a ▸ diagrammatic category $\mathcal{W}\mathbf{eb}^{\curlywedge}$ such that

$$\begin{tikzcd}
\dot{\mathbf{U}}(\mathfrak{gl}_k) \arrow[rr, "\Phi^{\mathrm{ext}}_{\mathbf{A}}", "\text{full}"'] \arrow[drr] & & \mathcal{R}\mathbf{ep}(\mathfrak{gl}_n) \\
\mathcal{S} \arrow[rr, "\beta_{\mathbf{A}}", "\text{fully faithful}"'] & & \mathcal{W}\mathbf{eb}^{\curlywedge} \arrow[u, "\text{full}", "\Gamma^{\mathrm{ext}}_{\mathbf{A}}"']
\end{tikzcd}$$

commutes. In particular, $\mathcal{W}\mathbf{eb}^{\curlywedge}$ is a ▸ thick version of the symmetric group.

The presentation functor

Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{gl}_n)$-intertwiners

$$\curlywedge_{a,b}^{a+b}\colon {\textstyle\bigwedge}^a\mathrm{V}\otimes{\textstyle\bigwedge}^b\mathrm{V}\twoheadrightarrow{\textstyle\bigwedge}^{a+b}\mathrm{V},\qquad \curlyvee_{a+b}^{a,b}\colon {\textstyle\bigwedge}^{a+b}\mathrm{V}\hookrightarrow{\textstyle\bigwedge}^a\mathrm{V}\otimes{\textstyle\bigwedge}^b\mathrm{V}$$

given by projection and inclusion.

The presentation functor is

$$\Gamma_{\mathbf{A}}^{\mathrm{ext}}\colon \mathcal{W}\mathrm{eb}^{\curlywedge}\to\mathcal{R}\mathrm{ep}(\mathfrak{gl}_n),\qquad a\mapsto{\textstyle\bigwedge}^a\mathrm{V},$$

$\mapsto \curlywedge_{a,b}^{a+b}$,

$\mapsto \curlyvee_{a+b}^{a,b}$

The presentation functor

Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{gl}_n)$-intertwiners

$$\curlywedge^{a+b}_{a,b}\colon \textstyle\bigwedge^a \mathrm{V} \otimes \bigwedge^b \mathrm{V} \twoheadrightarrow \bigwedge^{a+b} \mathrm{V}, \qquad \curlyvee^{a,b}_{a+b}\colon \textstyle\bigwedge^{a+b} \mathrm{V} \hookrightarrow \bigwedge^a \mathrm{V} \otimes \bigwedge^b \mathrm{V}$$

given by projection a

The (co)associativity relations say that $\bigwedge^\bullet \mathrm{V}$ is a (co)algebra with (co)multiplication $\curlywedge^{a+b}_{a,b}$ ($\curlyvee^{a,b}_{a+b}$).

The presentation functor is

$$\Gamma^{\mathrm{ext}}_{\mathbf{A}}\colon \mathcal{W}\mathrm{eb}^{\curlywedge} \to \mathcal{R}\mathrm{ep}(\mathfrak{gl}_n), \qquad a \mapsto \textstyle\bigwedge^a \mathrm{V},$$

$\mapsto \curlywedge^{a+b}_{a,b}$,

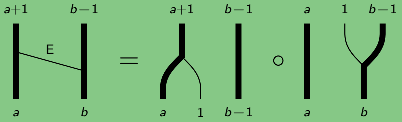
$\mapsto \curlyvee^{a,b}_{a+b}$

The presentation functor

Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{gl}_n)$-intertwiners

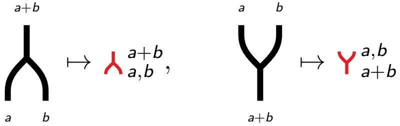
$$\curlywedge^{a+b}_{a,b}\colon \textstyle\bigwedge^a \mathrm{V} \otimes \bigwedge^b \mathrm{V} \twoheadrightarrow \bigwedge^{a+b} \mathrm{V}, \qquad \curlyvee^{a,b}_{a+b}\colon \bigwedge^{a+b} \mathrm{V} \hookrightarrow \bigwedge^a \mathrm{V} \otimes \bigwedge^b \mathrm{V}$$

We can play the game the other way around as well by defining Howe's action via:



etc.

$$\textstyle\bigwedge^a \mathrm{V} \otimes \bigwedge^b \mathrm{V} \to \bigwedge^a \mathrm{V} \otimes \mathrm{V} \otimes \bigwedge^{b-1} \mathrm{V} \to \bigwedge^{a+1} \mathrm{V} \otimes \bigwedge^{b-1} \mathrm{V}$$



Another pioneer of representation theory

Brauer's remarkable relationship between $\mathfrak{g}_n = \mathfrak{so}_n, \mathfrak{sp}_n$ and the Brauer algebra Br_n^k:

Brauer ∼1937. Let $\mathrm{V} = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{g}_n) \circlearrowright \underbrace{\mathrm{V} \otimes \cdots \otimes \mathrm{V}}_{k \text{ times}} \circlearrowleft \mathrm{Br}_n^k$$

generating each other's centralizer. The $\mathbf{U}(\mathfrak{g}_n)$-$\mathrm{Br}_n^k$-bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} \mathrm{L}(\mathfrak{g}_n, \lambda) \otimes \mathrm{L}(\mathrm{Br}_n^k, \lambda^{\mathrm{T}}).$$

The λ's are partitions of $k, k-2, k-4, \ldots$ whose precise form depend on $\mathfrak{g}_n$.

Another pioneer of representation theory

Be careful: One needs to work with $\mathfrak{o}_n$ in type **D**.
Today, I silently stay with $\mathfrak{so}_n$, and thus, in type **B**.

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The λ's are partitions of $k, k-2, k-4, \ldots$ whose precise form depend on $\mathfrak{g}_n$.

The diagrammatic presentation machine – it still works fine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{g}_n)} \circlearrowright \underbrace{\mathrm{V} \otimes \cdots \otimes \mathrm{V}}_{k \text{ times}} \circlearrowleft \overset{\text{use}}{\mathrm{Br}_n^k}$$

As usual, Brauer's insights give a full functor

Categorical version of the Brauer algebra $\mathcal{B}\mathbf{r}_n \xrightarrow[\text{full}]{\Phi} \mathcal{R}\mathbf{ep}(\mathfrak{g}_n)$

$$\mathcal{B}\mathbf{r}_n/\text{“}\mathrm{ker}(\Phi)\text{”} \xrightarrow[\text{fully faithful}]{\Phi} \mathcal{R}\mathbf{ep}(\mathfrak{g}_n)$$

whose “kernel $\mathrm{ker}(\Phi)$” can be calculated.

Hence, up to Spin's and Karoubi closures, Brauer gave us a diagrammatic presentation of the representation category $\mathcal{R}\mathbf{ep}(\mathfrak{g}_n)$ of $\mathfrak{g}_n$.

"Thick" Schur-Weyl-Brauer duality

Another one of Howe's remarkable relationships:

Howe ∼1975. Let $\mathrm{V} = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{so}_n) \circlearrowright \underbrace{{\textstyle\bigwedge}^{\bullet}\mathrm{V} \otimes \cdots \otimes {\textstyle\bigwedge}^{\bullet}\mathrm{V}}_{k \text{ times}} \circlearrowleft \mathbf{U}(\mathfrak{so}_{2k})$$

generating each other's centralizer, and ${\textstyle\bigwedge}^{a_1}\mathrm{V} \otimes \cdots \otimes {\textstyle\bigwedge}^{a_k}\mathrm{V}$ is the $(\overline{a}_1, \dots, \overline{a}_k)$th weight space of $\mathbf{U}(\mathfrak{so}_{2k})$. The $\mathbf{U}(\mathfrak{so}_n)$-$\mathbf{U}(\mathfrak{so}_{2k})$-bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} \mathrm{L}(\mathfrak{so}_n, \lambda) \otimes \mathrm{L}(\mathfrak{so}_{2k}, \textstyle\sum_{j=1}^{k} (\lambda_j^{\mathrm{T}} - n/2)\varepsilon_j).$$

The λ's again satisfy certain explicit conditions and $\overline{a}_i = a_i + n/2$.

"Thick" Schur-Weyl-Brauer duality

Another one of Howe's remarkable relationships:

Note that the action of $\mathbf{U}(\mathfrak{so}_{2k})$ is not as clear as it was for $\mathbf{U}(\mathfrak{gl}_k)$!

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$$\mathbf{U}(\mathfrak{so}_n) \circlearrowright \underbrace{\textstyle\bigwedge^{\bullet}\mathrm{V} \otimes \cdots \otimes \bigwedge^{\bullet}\mathrm{V}}_{k \text{ times}} \circlearrowleft \mathbf{U}(\mathfrak{so}_{2k})$$

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The λ's again satisfy certain explicit conditions and $\bar{a}_i = a_i + n/2$.

Still alive: The diagrammatic presentation machine

$$
\begin{array}{ccccc}
\color{green}{\text{fix}} & & & & \color{blue}{\text{use}} \\
\mathbf{U}(\mathfrak{gl}_n) & \circlearrowright & {\textstyle\bigwedge}^{\bullet}\mathrm{V} \otimes \cdots \otimes {\textstyle\bigwedge}^{\bullet}\mathrm{V} & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_k) \\
\cup & & \| & & \cap \\
\mathbf{U}(\mathfrak{so}_n) & \circlearrowright & \underbrace{{\textstyle\bigwedge}^{\bullet}\mathrm{V} \otimes \cdots \otimes {\textstyle\bigwedge}^{\bullet}\mathrm{V}}_{k \text{ times}} & \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k})
\end{array}
$$

Howe's $1^1\!/_2$th statement ▸ defines a ▸ diagrammatic category $\boldsymbol{\mathcal{W}}\mathbf{eb}^{\cup}$ such that

$$
\begin{tikzcd}
\dot{\mathbf{U}}(\mathfrak{so}_{2k}) \arrow[rr, "\Phi^{\mathrm{ext}}_{\mathbf{BD}}", "\text{full}"'] \arrow[drr] & & \boldsymbol{\mathcal{R}}\mathbf{ep}(\mathfrak{so}_n) \\
\boldsymbol{\mathcal{B}}\mathbf{r}_n \arrow[rr, "\beta_{\cup}", "\text{fully faithful}"'] & & \boldsymbol{\mathcal{W}}\mathbf{eb}^{\cup} \arrow[u, "\text{full}", "\Gamma^{\mathrm{ext}}_{\mathbf{BD}}"']
\end{tikzcd}
$$

commutes. In particular, $\boldsymbol{\mathcal{W}}\mathbf{eb}^{\cup}$ is a ▸ thick version of the Brauer algebra.

Still alive: The diagrammatic presentation machine

Restricting the action on one side

$$\begin{array}{ccccc} \text{fix} & & & & \text{use} \\ \mathbf{U}(\mathfrak{gl}_n) & \circlearrowright & \bigwedge^{\bullet}\mathrm{V}\otimes\cdots\otimes\bigwedge^{\bullet}\mathrm{V} & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_k) \\ \cup & & \| & & \cap \\ \mathbf{U}(\mathfrak{so}_n) & \circlearrowright & \underbrace{\bigwedge^{\bullet}\mathrm{V}\otimes\cdots\otimes\bigwedge^{\bullet}\mathrm{V}}_{k\text{ times}} & \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k}) \end{array}$$

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commutes. In particular, $\mathcal{W}\mathbf{eb}^{\cup}$ is a ▸ thick version of the Brauer algebra.

Still alive: The diagrammatic presentation machine

Restricting the action on one side

$$\begin{array}{ccccc} & \text{fix} & & \text{use} & \\ & \mathbf{U}(\mathfrak{gl}_n) & \circlearrowright \bigwedge^{\bullet}\mathrm{V} \otimes \cdots \otimes \bigwedge^{\bullet}\mathrm{V} \circlearrowleft & \mathbf{U}(\mathfrak{gl}_k) & \\ & \cup & \| & \cap & \\ & \mathbf{U}(\mathfrak{so}_n) & \circlearrowright \underbrace{\bigwedge^{\bullet}\mathrm{V} \otimes \cdots \otimes \bigwedge^{\bullet}\mathrm{V}}_{k \text{ times}} \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k}) & \end{array}$$

Increases the centralizer on the other

Howe's $1^1/_2$th statement ‣ defines a ‣ diagrammatic category $\mathcal{W}\mathbf{eb}^{\cup}$ such that

$$\begin{tikzcd} \dot{\mathbf{U}}(\mathfrak{so}_{2k}) \arrow[rr, "\Phi^{\mathrm{ext}}_{\mathbf{BD}}", "\text{full}"'] \arrow[drr] & & \mathcal{R}\mathbf{ep}(\mathfrak{so}_n) \\ \mathcal{B}\mathbf{r}_n \arrow[rr, "\beta_{\cup}", "\text{fully faithful}"'] & & \mathcal{W}\mathbf{eb}^{\cup} \arrow[u, "\text{full}", "\Gamma^{\mathrm{ext}}_{\mathbf{BD}}"'] \end{tikzcd}$$

commutes. In particular, $\mathcal{W}\mathbf{eb}^{\cup}$ is a ‣ thick version of the Brauer algebra.

Still alive: The diagrammatic presentation machine

Restricting the action on one side	fix $\mathbf{U}(\mathfrak{gl}_n) \cup \mathbf{U}(\mathfrak{so}_n)$	Hence, we get "old diagram generators" and "new diagram generators" k times	use $\mathbf{U}(\mathfrak{gl}_k) \cap \mathbf{U}(\mathfrak{so}_{2k})$	Increases the centralizer on the other

Howe's $1^1/_2$th statement defines a diagrammatic category $\boldsymbol{\mathcal{W}\mathrm{eb}}^{\cup}$ such that

$$
\begin{tikzcd}
\dot{\mathbf{U}}(\mathfrak{so}_{2k}) \arrow[rr, "\Phi^{\mathrm{ext}}_{\mathbf{BD}}", "\text{full}"'] \arrow[drr] & & \boldsymbol{\mathcal{R}\mathrm{ep}}(\mathfrak{so}_n) \\
\boldsymbol{\mathcal{B}\mathrm{r}}_n \arrow[rr, "\beta_{\cup}", "\text{fully faithful}"'] & & \boldsymbol{\mathcal{W}\mathrm{eb}}^{\cup} \arrow[u, "\text{full}", "\Gamma^{\mathrm{ext}}_{\mathbf{BD}}"']
\end{tikzcd}
$$

commutes. In particular, $\boldsymbol{\mathcal{W}\mathrm{eb}}^{\cup}$ is a thick version of the Brauer algebra.

Some delicate quantizations

$$
\begin{array}{ccccc}
\mathbf{U}(\mathfrak{gl}_n) & \circlearrowright & \bigwedge^{\bullet}\mathrm{V} \otimes \cdots \otimes \bigwedge^{\bullet}\mathrm{V} & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_k) \\
\cup & & \| & & \cap \\
\mathbf{U}(\mathfrak{so}_n) & \circlearrowright & \underbrace{\bigwedge^{\bullet}\mathrm{V} \otimes \cdots \otimes \bigwedge^{\bullet}\mathrm{V}}_{k \text{ times}} & \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k})
\end{array}
$$

Some delicate quantizations

$$\begin{array}{ccc} \mathbf{U}(\mathfrak{gl}_n) & {\textstyle\bigwedge_q^\bullet}\mathrm{V}_q \otimes \cdots \otimes {\textstyle\bigwedge_q^\bullet}\mathrm{V}_q & \mathbf{U}(\mathfrak{gl}_k) \\ \cup & \| & \cap \\ \mathbf{U}(\mathfrak{so}_n) & \underbrace{{\textstyle\bigwedge_q^\bullet}\mathrm{V}_q \otimes \cdots \otimes {\textstyle\bigwedge_q^\bullet}\mathrm{V}_q}_{k \text{ times}} & \mathbf{U}(\mathfrak{so}_{2k}) \end{array}$$

Some delicate quantizations

$$\begin{array}{ccccc}
\mathbf{U}_q(\mathfrak{gl}_n) & \circlearrowright & {\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes {\bigwedge}_q^{\bullet}\mathrm{V}_q & \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) \\
 & & \| & & \\
\mathbf{U}(\mathfrak{so}_n) & & \underbrace{{\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes {\bigwedge}_q^{\bullet}\mathrm{V}_q}_{k \text{ times}} & & \mathbf{U}(\mathfrak{so}_{2k})
\end{array}$$

Quantum skew Howe duality:
Lehrer–Zhang–Zhang $\sim$2009.
(But its quite easy and not their main point.)

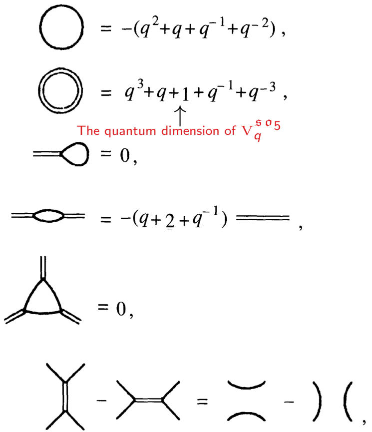
Some delicate quantizations

$$\begin{array}{ccccc} & \mathbf{U}_q(\mathfrak{gl}_n) & \circlearrowright \quad {\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes {\bigwedge}_q^{\bullet}\mathrm{V}_q \quad \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) & \\ \text{Does not quantize!} & \not\cup & \| & \cap & \text{Quantizes easily} \\ & \mathbf{U}_q(\mathfrak{so}_n) & \underbrace{{\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes {\bigwedge}_q^{\bullet}\mathrm{V}_q}_{k \text{ times}} & \mathbf{U}_q(\mathfrak{so}_{2k}) & \end{array}$$

Some delicate quantizations

$$\begin{array}{ccccc} & \mathbf{U}_q(\mathfrak{gl}_n) & \circlearrowright & {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q & \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) & \\ & \not\cup & & \| & & \cap & \\ \text{No action at all.} & \mathbf{U}_q(\mathfrak{so}_n) & \not\circlearrowright & \underbrace{{\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q}_{k \text{ times}} & ??? & \mathbf{U}_q(\mathfrak{so}_{2k}) & \text{Action unclear.} \end{array}$$

The quantum dimension of $\mathrm{V}_q^{\mathfrak{gl}}$ is $[n]$.
The quantum dimension of $\mathrm{V}_q^{\mathfrak{so}}$ is $[n-1]+1$.
Hence, $\mathrm{V}_q^{\mathfrak{so}}$ does not come from $\mathrm{V}_q^{\mathfrak{gl}}$!
This "flaw" propagates all the way through:
$\bigwedge_q^a \mathrm{V}_q^{\mathfrak{so}}$ have "weird" quantum dimensions.



Above: Kuperberg's $\mathbf{B}_2$ web relations ~1995.

Some delicate

The quantum dimension of $\mathrm{V}_q^{\mathfrak{gl}}$ is $[n]$.
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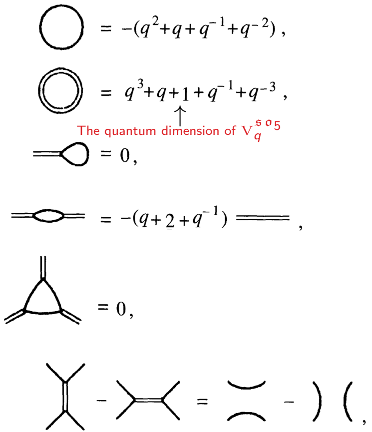


Above: Kuperberg's $\mathbf{B}_2$ web relations $\sim$1995.

We wanted to generalize Kuperberg's results. We failed because quantization is hard outside of type **A**.

But let me explain what we can do.

Some delicate quantizations

Using a coideal subalgebra does the trick.

$$\begin{array}{ccccc}
\mathbf{U}_q(\mathfrak{gl}_n) & \circlearrowright & \bigwedge_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^{\bullet}\mathrm{V}_q & \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) \\
\cup & & \| & & \cap \\
\mathbf{U}_q'(\mathfrak{so}_n) & \circlearrowright & \underbrace{\bigwedge_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^{\bullet}\mathrm{V}_q}_{k \text{ times}} & \circlearrowleft & \mathbf{U}_q(\mathfrak{so}_{2k})
\end{array}$$

The action is constructed using the unquantized diagrammatics.

Some delicate quantizations

$$\begin{array}{ccccc} \mathbf{U}_q(\mathfrak{gl}_n) & \circlearrowright & \bigwedge_q^\bullet \mathrm{V}_q \otimes\cdots\otimes \bigwedge_q^\bullet \mathrm{V}_q & \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k)\\ \cup & & \| & & \cap \\ \mathbf{U}_q'(\mathfrak{so}_n) & \circlearrowright & \underbrace{\bigwedge_q^\bullet \mathrm{V}_q \otimes\cdots\otimes \bigwedge_q^\bullet \mathrm{V}_q}_{k\text{ times}} & \circlearrowleft & \mathbf{U}_q(\mathfrak{so}_{2k})\end{array}$$

Using a q-monoidal ‣ diagrammatic category $\mathcal{W}\mathbf{eb}^{\cup}_{q,q^n}$ we can ‣ define a full Howe functor $\Phi^{\mathrm{ext}}_{\mathbf{BD}}$ such that we get a commuting diagram

$$\begin{tikzcd}
\dot{\mathbf{U}}_q(\mathfrak{so}_{2k}) \arrow[r, red, "\Phi^{\mathrm{ext}}_{\mathbf{BD}}"] \arrow[dr, blue] \arrow[dr, phantom, "\text{define}"] & \mathcal{R}\mathbf{ep}'_q(\mathfrak{so}_n) \\
\mathcal{B}\mathbf{r}_{q,q^n} \arrow[r, "\beta_{\cup}", "\text{fully faithful}"'] & \mathcal{W}\mathbf{eb}^{\cup}_{q,q^n} \arrow[u, red, "\Gamma^{\mathrm{ext}}_{\mathbf{BD}}"']
\end{tikzcd}$$

Hereby, $\mathcal{R}\mathbf{ep}'_q(\mathfrak{so}_n)$ is the q-monoidal representation category of ‣ $\mathbf{U}'_q(\mathfrak{so}_n)$, and $\mathcal{B}\mathbf{r}_{q,q^n}$ is Molev's q-Brauer category ($\sim$ 2002).

Further directions

$$\begin{array}{ccccc}
\mathbf{U}_q(\mathfrak{gl}_n) & \circlearrowright & \textstyle\bigwedge_q^{\bullet}\mathrm{V}_q^{\mathfrak{gl}} \otimes \cdots \otimes \textstyle\bigwedge_q^{\bullet}\mathrm{V}_q^{\mathfrak{gl}} & \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) \\
\cup & & \| & & \cap \\
\mathbf{U}_q'(\mathfrak{so}_n) & \circlearrowright & \underbrace{\textstyle\bigwedge_q^{\bullet}\mathrm{V}_q^{\mathfrak{gl}} \otimes \cdots \otimes \textstyle\bigwedge_q^{\bullet}\mathrm{V}_q^{\mathfrak{gl}}}_{k \text{ times}} & \circlearrowleft & \mathbf{U}_q(\mathfrak{so}_{2k})
\end{array}$$

- Use a similar approach to get the quantum group to work. (Needs probably some mixed Howe duality à la Queffelec–Sartori.)

Further directions

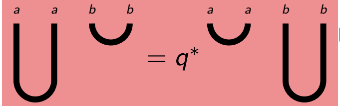
$* = {}^{n-1}/_2$ for type **B**,
$* = {}^{n}/_2$ for type **D**.

$$\begin{array}{ccccc}
\mathbf{U}_q(\mathfrak{gl}_*) & \circlearrowright & \bigwedge_q^\bullet \mathrm{V}_q^{\mathfrak{so}} \otimes \cdots \otimes \bigwedge_q^\bullet \mathrm{V}_q^{\mathfrak{so}} & \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) \\
\cap & & \| & & \cup \\
\mathbf{U}_q(\mathfrak{so}_n) & \circlearrowright & \underbrace{\bigwedge_q^\bullet \mathrm{V}_q^{\mathfrak{so}} \otimes \cdots \otimes \bigwedge_q^\bullet \mathrm{V}_q^{\mathfrak{so}}}_{k \text{ times}} & \circlearrowleft & ?\mathbf{U}_q'(\mathfrak{so}_{2k})?
\end{array}$$

- Use a similar approach to get the quantum group to work. (Needs probably some mixed Howe duality à la Queffelec–Sartori.)

This should give the quantum group story,
but it is much trickier since e.g.
$\mathrm{V}_q^{\mathfrak{so}} \cong \mathrm{V}_q^{\mathfrak{gl}} \oplus (\mathrm{V}_q^{\mathfrak{gl}})^* \oplus \mathbb{C}$
as $\mathbf{U}_q(\mathfrak{gl}_*)$-modules in type **B**.
Thus, the above is not the usual $\mathbf{U}(\mathfrak{gl}_*)$-$\mathbf{U}(\mathfrak{gl}_k)$ duality.

$$\begin{array}{ccc} \mathbf{U}_q(\mathfrak{gl}_n) & \circlearrowright \; {\textstyle\bigwedge_q^{\bullet}} \mathrm{V}_q^{\mathfrak{gl}} \otimes \cdots \otimes {\textstyle\bigwedge_q^{\bullet}} \mathrm{V}_q^{\mathfrak{gl}} \; \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) \\ \cup & & \cap \\ \mathbf{U}'_q(\mathfrak{so}_n) & & \mathbf{U}_q(\mathfrak{so}_{2k}) \end{array}$$



$= q^*$

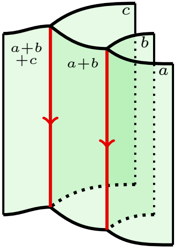
q-interchange law

$*$ =some power depending on a, b

- Use a similar approac … o work. (Needs probably some mixed Howe du …)
- q-monoidal categories are probably very useful to study representation categories of coideal subalgebras. An abstract formulation à la Brundan–Ellis ("super monoidal") should be useful.

Further directions

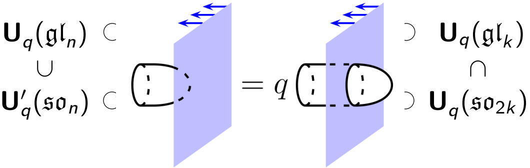
$$\mathbf{U}_q(\mathfrak{gl}_n) \qquad\qquad \mathbf{U}_q(\mathfrak{gl}_k)$$
$$\cup \qquad\qquad\qquad \cap$$
$$\mathbf{U}_q'(\mathfrak{so}_n) \qquad\qquad \mathbf{U}_q(\mathfrak{so}_{2k})$$



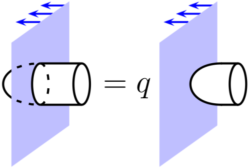
Singular cobordisms ("foams", à la Khovanov–Rozansky and Mackaay–Stošić–Vaz) categorify webs. The q-monoidal property has to be smartly encoded.

- Use a similar approac … to work. (Needs probably some mixed Howe du…
- q-monoidal categories … study representation categories of coideal s… …ulation à la Brundan–Ellis ("super monoidal") sl…
- Coideal subalgebras are amenable to categorification, cf. Ehrig–Stroppel or Bao–Shan–Wang–Webster. Similarly, their representation categories should be amenable to categorification.

Further directions

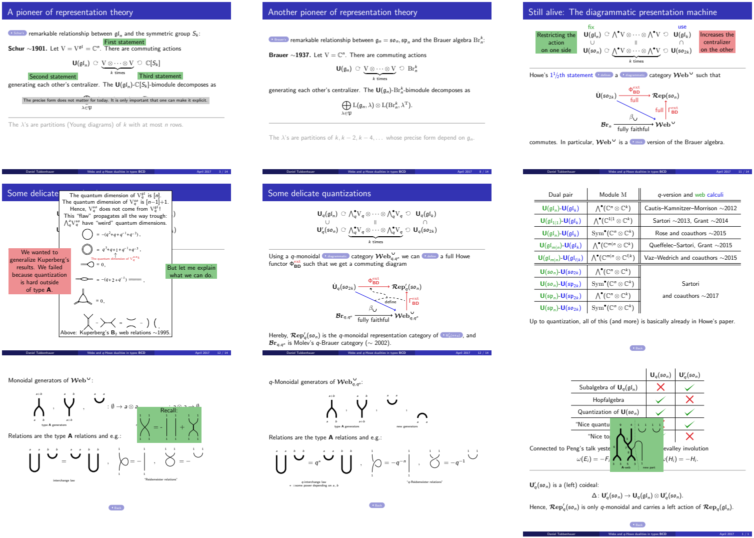


- Use a similar approach ... to work. (Needs probably some mixed Howe dual ...
- q-monoidal categories ... study representation categories of coideal su... ...mulation à la Brundan–Ellis ("super monoidal") sho...
- Coideal subalgebras are ... ion, cf. Ehrig–Stroppel or Bao–Shan–Wang–Webs... ...entation categories should be amenable to categorification.

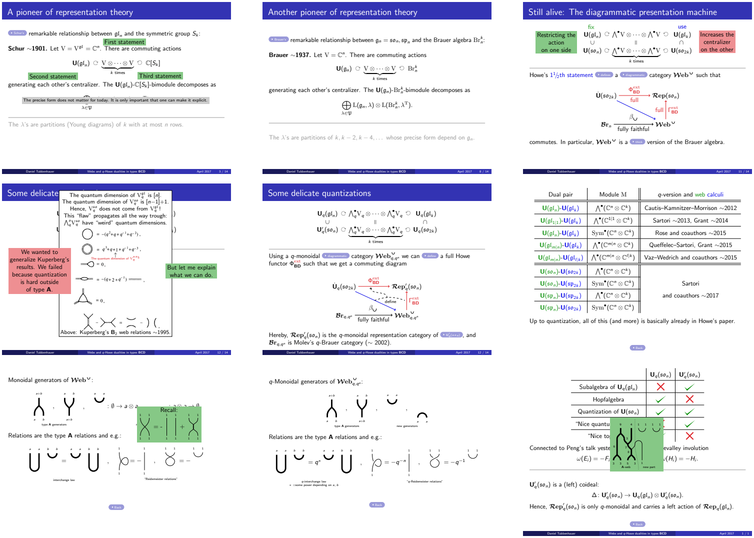


2-q-monoidal foams.
(Maybe connected to Beliakova–Putyra–Wehrli whose pictures I shamelessly stole.)

- Formulate everything in a "2-q-monoidal language". (Again, à la Brundan–Ellis.)



There is still much to do...



Thanks for your attention!

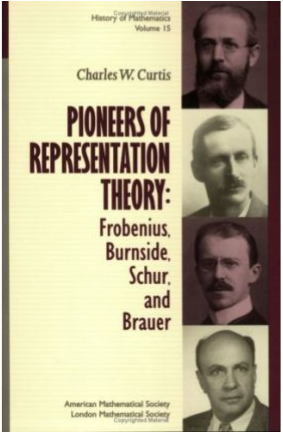


Figure: Two of the main players for today: Schur and Brauer.

Curtis, C.W. *Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer*.

Back

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

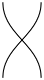
VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

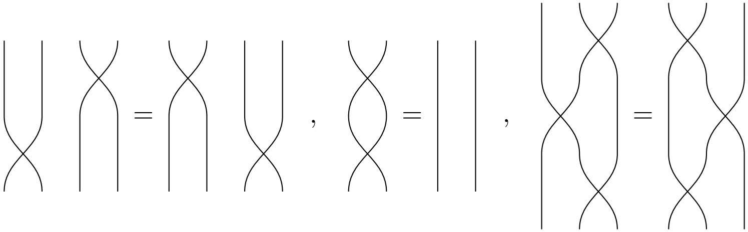
Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

Back

Monoidal generator of $\mathcal{S}$:

 $: 2 \to 2.$

Relations e.g.:



interchange law

"Reidemeister relations"

◂ Back

Dual pair	Module M	q-version and web calculi
$\mathbf{U}(\mathfrak{gl}_n)$-$\mathbf{U}(\mathfrak{gl}_k)$	$\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	Cautis–Kamnitzer–Morrison $\sim$2012
$\mathbf{U}(\mathfrak{gl}_{1\vert 1})$-$\mathbf{U}(\mathfrak{gl}_k)$	$\bigwedge^\bullet(\mathbb{C}^{1\vert 1} \otimes \mathbb{C}^k)$	Sartori $\sim$2013, Grant $\sim$2014
$\mathbf{U}(\mathfrak{gl}_n)$-$\mathbf{U}(\mathfrak{gl}_k)$	$\mathrm{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	Rose and coauthors $\sim$2015
$\mathbf{U}(\mathfrak{gl}_{m\vert n})$-$\mathbf{U}(\mathfrak{gl}_k)$	$\bigwedge^\bullet(\mathbb{C}^{m\vert n} \otimes \mathbb{C}^k)$	Queffelec–Sartori, Grant $\sim$2015
$\mathbf{U}(\mathfrak{gl}_{m\vert n})$-$\mathbf{U}(\mathfrak{gl}_{l\vert k})$	$\bigwedge^\bullet(\mathbb{C}^{m\vert n} \otimes \mathbb{C}^{l\vert k})$	Vaz–Wedrich and coauthors $\sim$2015
$\mathbf{U}(\mathfrak{so}_n)$-$\mathbf{U}(\mathfrak{so}_{2k})$	$\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	Sartori and coauthors $\sim$2017
$\mathbf{U}(\mathfrak{so}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$	$\mathrm{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	
$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$	$\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	
$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{so}_{2k})$	$\mathrm{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	

Up to quantization, all of this (and more) is basically already in Howe's paper.

Back

Type **A** is in a fairly good shape:
The story partially works "integrally" (Elias ~2015).
Applications to link polynomials (e.g. Wedrich–Vaz and coauthors ~2015).
Partially categorified (e.g. Huerfano–Khovanov ~2002, Mackaay ~2009).
Applications to link homologies (e.g. Lauda–Queffelec–Rose ~2012).
Applications to canonical bases and geometry (e.g. Cautis–Kamnitzer ~2016).
...

Dual pair	Module M	q-version and web calculi
$\mathbf{U}(\mathfrak{so}_n)$-$\mathbf{U}(\mathfrak{so}_{2k})$	$\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	
$\mathbf{U}(\mathfrak{so}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$	$\mathrm{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	Sartori
$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$	$\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	and coauthors ~2017
$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{so}_{2k})$	$\mathrm{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	

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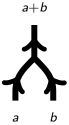
...

Dual pair	Module M	q-version and web calculi
$\mathbf{U}(\mathfrak{so}_n)$-$\mathbf{U}(\mathfrak{so}_{2k})$	$\bigwedge^{\bullet}(\mathbb{C}^n \otimes \mathbb{C}^k)$	Sartori and coauthors ∼2017
$\mathbf{U}(\mathfrak{so}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$	$\mathrm{Sym}^{\bullet}(\mathbb{C}^n \otimes \mathbb{C}^k)$	
$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$	$\bigwedge^{\bullet}(\mathbb{C}^n \otimes \mathbb{C}^k)$	
$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{so}_{2k})$	$\mathrm{Sym}^{\bullet}(\mathbb{C}^n \otimes \mathbb{C}^k)$	

Types **BCD** are not really understood.

Up to quantization, all of this (and more) is basically already in Howe's paper.

Monoidal generators of $\mathcal{W}\mathrm{eb}^{\curlywedge}$:

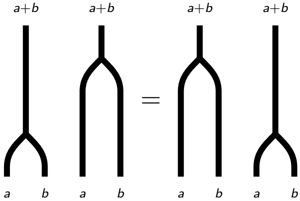


$: a \otimes b \to a + b$ and


$: a + b \to a \otimes b.$

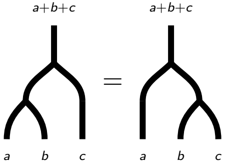
One needs orientations in type **A**, but I am going to ignore them.

Relations e.g.:



interchange law

,



Associativity

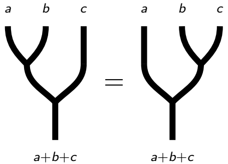
◂ Back

Monoidal generators of $\mathcal{W}\mathrm{eb}^{\curlywedge}$:

$: a \otimes b \to a + b$ and

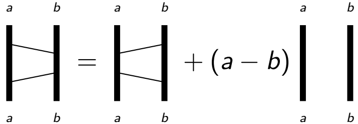

$: a + b \to a \otimes b.$

Relations e.g.:



Coassociativity

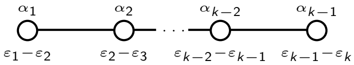
,



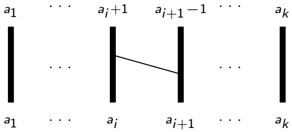
square switch

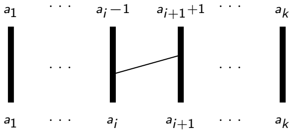
Back

Root conventions is type **A**:



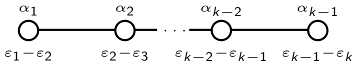
Thus, because of statement $1\frac{1}{2}$, we should set

$\mathrm{E}_i 1_\lambda \longmapsto$  , for all $i = 1, \dots, k-1$,

$\mathrm{F}_i 1_\lambda \longmapsto$  , for all $i = 1, \dots, k-1$.

Back

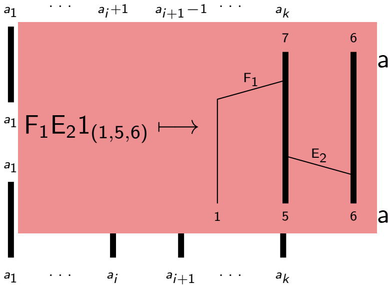
Root conventions is type **A**:



Thus, because of statement $1\frac{1}{2}$, we should set

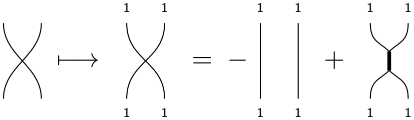
$E_i 1_\lambda \longmapsto$ all $i = 1, \ldots, k-1,$

$F_i 1_\lambda \longmapsto$ all $i = 1, \ldots, k-1.$



Back

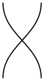
$$\beta_{\mathsf{A}} \colon \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{W}}\mathbf{eb}^{\curlywedge}$$



$$\mathbb{C}[S_k] \xrightarrow{\cong} \mathrm{End}_{\boldsymbol{\mathcal{W}}\mathbf{eb}^{\curlywedge}}(1^{\otimes k})$$

Back

Monoidal generators of $\mathcal{B}\mathbf{r}_n$:

 ,  $: \emptyset \to 2$,  $: 2 \to \emptyset.$

Relations e.g.:

 $=$  ,  $= \pm n.$

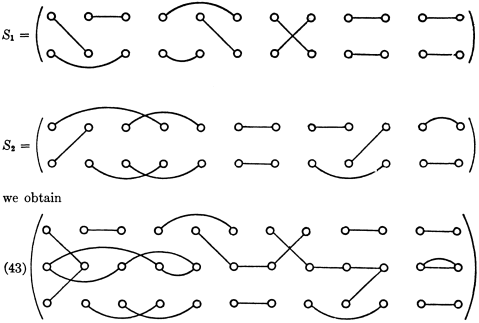
interchange law

circle removal

Back

Monoidal generators of $\mathcal{B}\mathbf{r}_n$:

Relati

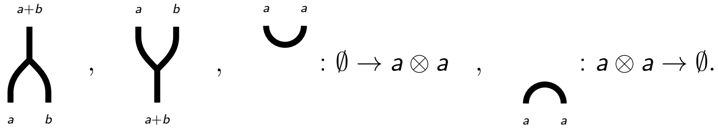


From “Brauer, R. *On algebras which are connected with the semisimple continuous groups*. Ann. of Math. (2) 38 (1937), no. 4, 857–872.”

Back

Monoidal generators of $\mathcal{W}\mathbf{eb}^{\cup}$:

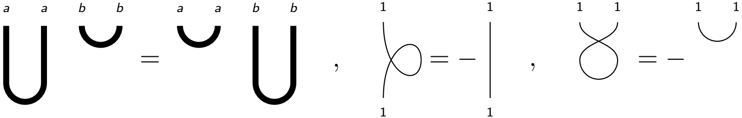
No orientations needed in types **BCD**.



type **A** generators

new generators

Relations are the type **A** relations and e.g.:

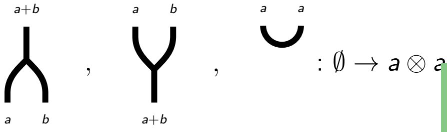


interchange law

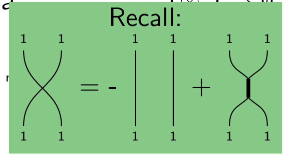
"Reidemeister relations"

◂ Back

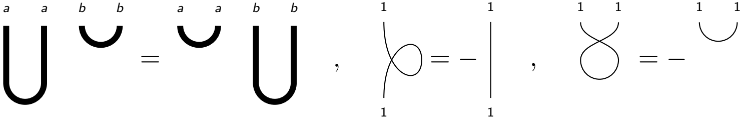
Monoidal generators of $\mathcal{W}\mathbf{eb}^{\cup}$:



type **A** generators



Relations are the type **A** relations and e.g.:

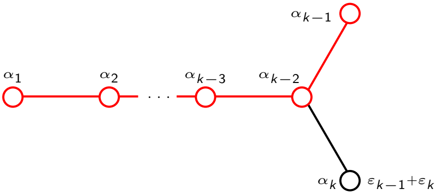


interchange law

“Reidemeister relations”

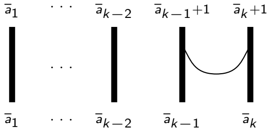
Back

Root conventions is type **D**:

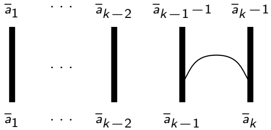


Thus, because of statement $1\frac{1}{2}$, we should set

$E_k 1_\lambda \longmapsto$

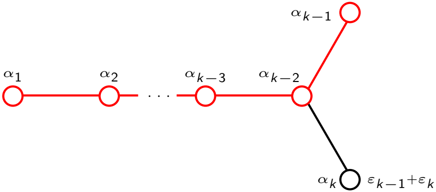


$F_k 1_\lambda \longmapsto$



Back

Root conventions is type **D**:



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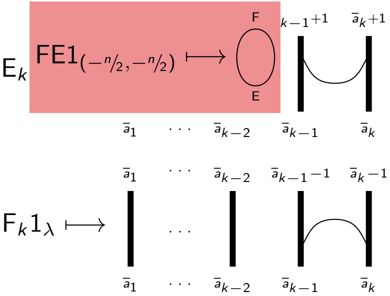
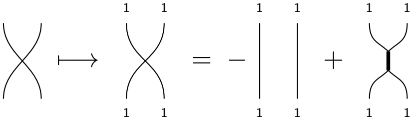
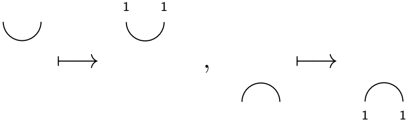


Back

$$\beta_{\cup} \colon \boldsymbol{\mathcal{B}}\mathbf{r}_n \to \boldsymbol{\mathcal{W}}\mathbf{eb}^{\cup}$$

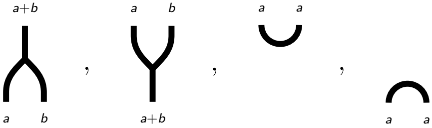




$$\mathrm{Br}_n^k \xrightarrow{\cong} \mathsf{End}_{\boldsymbol{\mathcal{W}}\mathbf{eb}^{\cup}}(1^{\otimes k})$$

Back

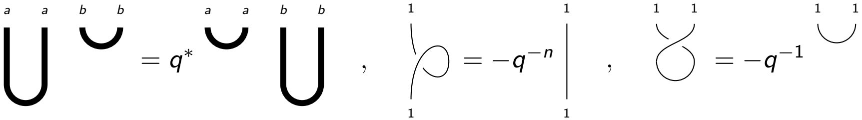
q-Monoidal generators of $\mathcal{W}\mathbf{eb}^{\cup}_{q,q^n}$:



type **A** generators

new generators

Relations are the type **A** relations and e.g.:

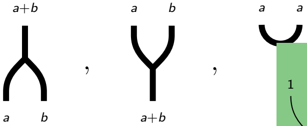


q-interchange law
$*$ =some power depending on a, b

"q-Reidemeister relations"

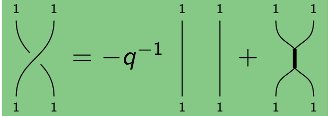
◂ Back

q-Monoidal generators of $\mathcal{W}\mathbf{eb}^{\cup}_{q,q^n}$:



type **A** generators

Type A crossing:



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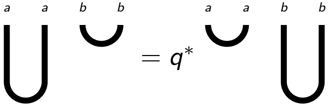

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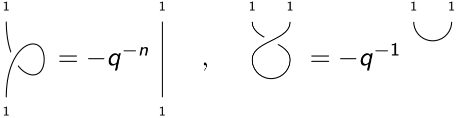


"q-Reidemeister relations"

Back

Via restriction, we see that the $\mathbf{U}_q(\mathfrak{gl}_n)$-intertwiners $\curlywedge^{a+b}_{a,b}$ and $\curlyvee^{a,b}_{a+b}$ are $\mathbf{U}'_q(\mathfrak{so}_n)$-equivariant as well.

Note that $\mathrm{V} \otimes \mathrm{V}$ contains a copy of the trivial $\mathbf{U}(\mathfrak{so}_n)$-module. One shows that the same holds with q and one gets inclusions and projections

$$\cup : \mathbb{C}_q \to \mathrm{V}_q \otimes \mathrm{V}_q, \qquad \cap : \mathrm{V}_q \otimes \mathrm{V}_q \to \mathbb{C}_q.$$

As before, use these to quantize Howe's duality.

Back

	$\mathbf{U}_q(\mathfrak{so}_n)$	$\mathbf{U}'_q(\mathfrak{so}_n)$
Subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$	✗	✓
Hopfalgebra	✓	✗
Quantization of $\mathbf{U}(\mathfrak{so}_n)$	✓	✓
"Nice quantum numbers"	✗	✓
"Nice topology"	✓	✗

Connected to Peng's talk yesterday: $\theta = \omega$, the Chevalley involution

$$\omega(E_i) = -F_i, \quad \omega(F_i) = -E_i, \quad \omega(H_i) = -H_i.$$

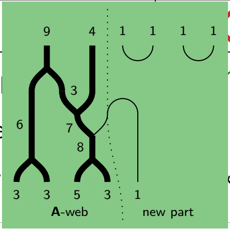
$\mathbf{U}'_q(\mathfrak{so}_n)$ is a (left) coideal:

$$\Delta\colon \mathbf{U}'_q(\mathfrak{so}_n) \to \mathbf{U}_q(\mathfrak{gl}_n) \otimes \mathbf{U}'_q(\mathfrak{so}_n).$$

Hence, $\mathcal{R}\mathbf{ep}'_q(\mathfrak{so}_n)$ is only q-monoidal and carries a left action of $\mathcal{R}\mathbf{ep}_q(\mathfrak{gl}_n)$.

Back

	$\mathbf{U}_q(\mathfrak{so}_n)$	$\mathbf{U}'_q(\mathfrak{so}_n)$
Subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$	✗	✓
Hopfalgebra	✓	✗
Quantization of $\mathbf{U}(\mathfrak{so}_n)$	✓	✓
“Nice quantu	✗	✓
“Nice to	✓	✗



Connected to Peng’s talk yeste … evalley involution

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