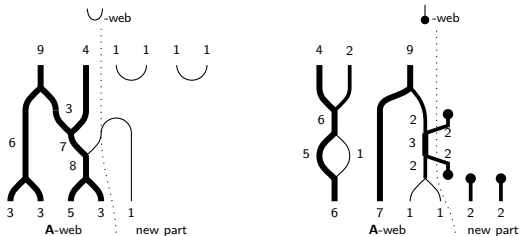


Webs and q -Howe dualities in types BCD

Or: A story about “howe” I failed

Daniel Tubbenhauer



Joint work with Antonio Sartori

April 2017

- 1 The type **A** story
 - Classical Schur-Weyl duality
 - Howe's dualities in type **A**
- 2 The type **BCD** story
 - Classical Schur-Weyl-Brauer duality
 - Howe's dualities in types **BCD**
- 3 The quantum story
 - Various quantizations
 - Concluding remarks

A pioneer of representation theory

▶ Schur's remarkable relationship between \mathfrak{gl}_n and the symmetric group S_k :

Schur ~1901. Let $V = V^{\mathfrak{gl}} = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{gl}_n) \curvearrowright \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \curvearrowright \mathbb{C}[S_k]$$

generating each other's centralizer. The $\mathbf{U}(\mathfrak{gl}_n)$ - $\mathbb{C}[S_k]$ -bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} L(\mathfrak{gl}_n, \lambda) \otimes L(S_k, \lambda^T).$$

The λ 's are partitions (Young diagrams) of k with at most n rows.

A pioneer of representation theory

▶ Schur's remarkable relationship between \mathfrak{gl}_n and the symmetric group S_k :

First statement

Schur ~1901. Let $V = V^{\mathfrak{gl}} = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{gl}_n) \curvearrowright \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \curvearrowright \mathbb{C}[S_k]$$

generating each other's centralizer. The $\mathbf{U}(\mathfrak{gl}_n)$ - $\mathbb{C}[S_k]$ -bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} L(\mathfrak{gl}_n, \lambda) \otimes L(S_k, \lambda^T).$$

The λ 's are partitions (Young diagrams) of k with at most n rows.

A pioneer of representation theory

▶ Schur's remarkable relationship between \mathfrak{gl}_n and the symmetric group S_k :

First statement

Schur ~1901. Let $V = V^{\mathfrak{gl}} = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{gl}_n) \curvearrowright \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \curvearrowright \mathbb{C}[S_k]$$

Second statement

generating each other's centralizer. The $\mathbf{U}(\mathfrak{gl}_n)$ - $\mathbb{C}[S_k]$ -bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} L(\mathfrak{gl}_n, \lambda) \otimes L(S_k, \lambda^T).$$

The λ 's are partitions (Young diagrams) of k with at most n rows.

A pioneer of representation theory

▶ Schur's remarkable relationship between \mathfrak{gl}_n and the symmetric group S_k :

Schur ~1901. Let $V = V^{\mathfrak{gl}} = \mathbb{C}^n$. **First statement**
There are commuting actions

$$\mathbf{U}(\mathfrak{gl}_n) \curvearrowright \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \curvearrowright \mathbb{C}[S_k]$$

Second statement

Third statement

generating each other's centralizer. The $\mathbf{U}(\mathfrak{gl}_n)$ - $\mathbb{C}[S_k]$ -bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} L(\mathfrak{gl}_n, \lambda) \otimes L(S_k, \lambda^T).$$

The λ 's are partitions (Young diagrams) of k with at most n rows.

A pioneer of representation theory

▶ Schur's remarkable relationship between \mathfrak{gl}_n and the symmetric group S_k :

Schur \sim **1901**. Let $V = V^{\mathfrak{gl}} = \mathbb{C}^n$. **First statement**
There are commuting actions

$$\mathbf{U}(\mathfrak{gl}_n) \curvearrowright \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \curvearrowright \mathbb{C}[S_k]$$

Second statement

Third statement

generating each other's centralizer. The $\mathbf{U}(\mathfrak{gl}_n)$ - $\mathbb{C}[S_k]$ -bimodule decomposes as

The precise form does not matter for today. It is only important that one can make it explicit.

$$\bigoplus_{\lambda \in \mathfrak{P}} \dots$$

The λ 's are partitions (Young diagrams) of k with at most n rows.

The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \rightleftarrows \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \rightleftarrows \overset{\text{use}}{\mathbb{C}[S_k]}$$

Schur's [first statement](#) gives a functor

Categorical version of
the symmetric group

$$\mathcal{S} \xrightarrow{\quad \phi \quad} \mathcal{Rep}(\mathfrak{gl}_n)$$

The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowleft \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \circlearrowright \overset{\text{use}}{\mathbb{C}[S_k]}$$

Schur's [second statement](#) gives a full functor

$$\mathcal{S} \xrightarrow[\text{full}]{\phi} \mathcal{R}\text{ep}(\mathfrak{gl}_n)$$

The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowleft \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \circlearrowright \overset{\text{use}}{\mathbb{C}[S_k]}$$

Schur's [third statement](#) gives a full functor

$$\mathcal{S} \xrightarrow[\text{full}]{\Phi} \mathcal{R}\text{ep}(\mathfrak{gl}_n)$$

$$\mathcal{S}/\text{"ker}(\Phi)\text{"} \xrightarrow[\text{fully faithful}]{\Phi} \mathcal{R}\text{ep}(\mathfrak{gl}_n)$$

whose "kernel $\text{ker}(\Phi)$ " can be calculated.

Hence, up to taking duals and Karoubi closures, Schur gave us a presentation of the representation category $\mathcal{R}\text{ep}(\mathfrak{gl}_n)$ of \mathfrak{gl}_n .

▶ diagrammatic

“Thick” Schur-Weyl duality

▶ One of Howe’s remarkable relationships between \mathfrak{gl}_n and \mathfrak{gl}_k :

Howe ~1975. Let $V = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{gl}_n) \curvearrowright \underbrace{\bigwedge^{a_1} V \otimes \cdots \otimes \bigwedge^{a_k} V}_{k \text{ times}} \curvearrowright \mathbf{U}(\mathfrak{gl}_k)$$

generating each other’s centralizer, and $\bigwedge^{a_1} V \otimes \cdots \otimes \bigwedge^{a_k} V$ is the (a_1, \dots, a_k) th weight space as regards $\mathbf{U}(\mathfrak{gl}_k)$. The $\mathbf{U}(\mathfrak{gl}_n)$ - $\mathbf{U}(\mathfrak{gl}_k)$ -bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} L(\mathfrak{gl}_n, \lambda) \otimes L(\mathfrak{gl}_k, \lambda^T).$$

The λ ’s are partitions with at most k columns and n rows.

“Thick” Schur-Weyl duality

▶ One of Howe’s remarkable relationships between \mathfrak{gl}_n and \mathfrak{gl}_k :

Howe ~1975. Let $V = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{gl}_n) \curvearrowright \underbrace{\bigwedge^{a_1} V \otimes \cdots \otimes \bigwedge^{a_k} V}_{k \text{ times}} \curvearrowright \mathbf{U}(\mathfrak{gl}_k)$$

1^{1/2}th statement

generating each other’s centralizer, and $\bigwedge^{a_1} V \otimes \cdots \otimes \bigwedge^{a_k} V$ is the (a_1, \dots, a_k) th weight space as regards $\mathbf{U}(\mathfrak{gl}_k)$. The $\mathbf{U}(\mathfrak{gl}_n)$ - $\mathbf{U}(\mathfrak{gl}_k)$ -bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} L(\mathfrak{gl}_n, \lambda) \otimes L(\mathfrak{gl}_k, \lambda^T).$$

The λ ’s are partitions with at most k columns and n rows.

Again: The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowleft \underbrace{\bigwedge^{\bullet} V \otimes \cdots \otimes \bigwedge^{\bullet} V}_{k \text{ times}} \circlearrowright \overset{\text{use}}{\mathbf{U}(\mathfrak{gl}_k)}$$

Howe's **first statement** gives a functor

Dot version generated by
weight space idempotents 1_{λ} ,
and E_i and F_i

$$\dot{\mathbf{U}}(\mathfrak{gl}_k) \xrightarrow{\Phi_{\mathbf{A}}^{\text{ext}}} \mathcal{R}\text{ep}(\mathfrak{gl}_n)$$

Again: The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowleft \underbrace{\Lambda^\bullet V \otimes \cdots \otimes \Lambda^\bullet V}_{k \text{ times}} \circlearrowright \overset{\text{use}}{\mathbf{U}(\mathfrak{gl}_k)}$$

Howe's [second statement](#) gives a full functor

$$\dot{\mathbf{U}}(\mathfrak{gl}_k) \xrightarrow[\text{full}]{\Phi_A^{\text{ext}}} \mathcal{R}\text{ep}(\mathfrak{gl}_n)$$

Again: The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowleft \underbrace{\bigwedge^{\bullet} \mathbf{V} \otimes \cdots \otimes \bigwedge^{\bullet} \mathbf{V}}_{k \text{ times}} \circlearrowright \overset{\text{use}}{\mathbf{U}(\mathfrak{gl}_k)}$$

Howe's [third statement](#) gives a full functor

$$\dot{\mathbf{U}}(\mathfrak{gl}_k) \xrightarrow[\text{full}]{\Phi_{\mathbf{A}}^{\text{ext}}} \mathcal{R}\text{ep}(\mathfrak{gl}_n)$$

$$\dot{\mathbf{U}}(\mathfrak{gl}_k) / \text{"ker}(\Phi_{\mathbf{A}}^{\text{ext}})\text{"} \xrightarrow[\text{fully faithful}]{\Phi_{\mathbf{A}}^{\text{ext}}} \mathcal{R}\text{ep}(\mathfrak{gl}_n)$$

whose "kernel $\ker(\Phi_{\mathbf{A}}^{\text{ext}})$ " we can calculate.

Again: The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowleft \underbrace{\bigwedge^{\bullet} V \otimes \cdots \otimes \bigwedge^{\bullet} V}_{k \text{ times}} \circlearrowright \overset{\text{use}}{\mathbf{U}(\mathfrak{gl}_k)}$$

Howe's $1\frac{1}{2}$ th statement ▶ defines a ▶ diagrammatic category \mathbf{Web}^{λ} such that

$$\begin{array}{ccc} \dot{\mathbf{U}}(\mathfrak{gl}_k) & \xrightarrow[\text{full}]{\Phi_{\mathbf{A}}^{\text{ext}}} & \mathcal{R}\text{ep}(\mathfrak{gl}_n) \\ & \searrow & \uparrow \text{full } \Gamma_{\mathbf{A}}^{\text{ext}} \\ \mathcal{S} & \xrightarrow[\text{fully faithful}]{\beta_{\mathbf{A}}} & \mathbf{Web}^{\lambda} \end{array}$$

commutes. In particular, \mathbf{Web}^{λ} is a ▶ thick version of the symmetric group.

The presentation functor

Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{gl}_n)$ -intertwiners

$$\mathcal{Y}_{a,b}^{a+b} : \Lambda^a V \otimes \Lambda^b V \twoheadrightarrow \Lambda^{a+b} V, \quad \mathcal{Y}_{a+b}^{a,b} : \Lambda^{a+b} V \hookrightarrow \Lambda^a V \otimes \Lambda^b V$$

given by projection and inclusion.

The presentation functor is

$$\Gamma_{\mathbf{A}}^{\text{ext}} : \text{Web}^{\mathbf{A}} \rightarrow \mathcal{R}\text{ep}(\mathfrak{gl}_n), \quad a \mapsto \Lambda^a V,$$

$$\begin{array}{c} a+b \\ | \\ \text{Y} \\ | \\ a \quad b \end{array} \mapsto \mathcal{Y}_{a,b}^{a+b}$$

$$\begin{array}{c} a \quad b \\ \text{Y} \\ | \\ a+b \end{array} \mapsto \mathcal{Y}_{a+b}^{a,b}$$

The presentation functor

Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{gl}_n)$ -intertwiners

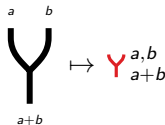
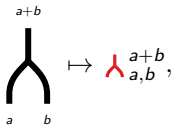
$$\mathcal{A}_{a,b}^{a+b} : \Lambda^a V \otimes \Lambda^b V \twoheadrightarrow \Lambda^{a+b} V, \quad \mathcal{Y}_{a+b}^{a,b} : \Lambda^{a+b} V \hookrightarrow \Lambda^a V \otimes \Lambda^b V$$

given by projection a

The (co)associativity relations say that $\Lambda^\bullet V$ is a (co)algebra with (co)multiplication $\mathcal{A}_{a,b}^{a+b}$ ($\mathcal{Y}_{a+b}^{a,b}$).

The presentation functor is

$$\Gamma_{\mathbf{A}}^{\text{ext}} : \mathbf{Web}^{\wedge} \rightarrow \mathcal{R}\text{ep}(\mathfrak{gl}_n), \quad a \mapsto \Lambda^a V,$$



The presentation functor

Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{gl}_n)$ -intertwiners

$$\mathcal{R}^{a+b} : \Lambda^a V \otimes \Lambda^b V \rightarrow \Lambda^{a+b} V, \quad \mathcal{Y}^{a,b} : \Lambda^{a+b} V \hookrightarrow \Lambda^a V \otimes \Lambda^b V$$

We can play the game the other way around as well by defining Howe's action via:

etc.

$$\Lambda^a V \otimes \Lambda^b V \rightarrow \Lambda^a V \otimes V \otimes \Lambda^{b-1} V \rightarrow \Lambda^{a+1} V \otimes \Lambda^{b-1} V$$

$$\text{cup diagram} \mapsto \mathcal{R}^{a+b}_{a,b}$$

$$\text{cap diagram} \mapsto \mathcal{Y}^{a,b}_{a+b}$$

Another pioneer of representation theory

▶ Brauer's remarkable relationship between $\mathfrak{g}_n = \mathfrak{so}_n, \mathfrak{sp}_n$ and the Brauer algebra Br_n^k :

Brauer \sim **1937**. Let $V = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{g}_n) \curvearrowright \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \curvearrowleft \text{Br}_n^k$$

generating each other's centralizer. The $\mathbf{U}(\mathfrak{g}_n)$ - Br_n^k -bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} L(\mathfrak{g}_n, \lambda) \otimes L(\text{Br}_n^k, \lambda^T).$$

The λ 's are partitions of $k, k-2, k-4, \dots$ whose precise form depend on \mathfrak{g}_n .

Another pioneer of representation theory

Be careful: One needs to work with \mathfrak{o}_n in type **D**.
Today, I silently stay with \mathfrak{so}_n , and thus, in type **B**.

▶ Brauer's remarkable relationship between $\mathfrak{g}_n = \mathfrak{so}_n, \mathfrak{sp}_n$ and the Brauer algebra Br_n^k :

Brauer \sim **1937**. Let $V = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{g}_n) \curvearrowright \underbrace{V \otimes \dots \otimes V}_{k \text{ times}} \curvearrowleft \text{Br}_n^k$$

generating each other's centralizer. The $\mathbf{U}(\mathfrak{g}_n)$ - Br_n^k -bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} L(\mathfrak{g}_n, \lambda) \otimes L(\text{Br}_n^k, \lambda^T).$$

The λ 's are partitions of $k, k-2, k-4, \dots$ whose precise form depend on \mathfrak{g}_n .

The diagrammatic presentation machine – it still works fine

$$\mathbf{U}(\mathfrak{g}_n) \overset{\text{fix}}{\hookrightarrow} \underbrace{V \otimes \dots \otimes V}_{k \text{ times}} \overset{\text{use}}{\hookrightarrow} \mathbf{Br}_n^k$$

As usual, Brauer's insights give a full functor

Categorical version of
the Brauer algebra

$$\mathbf{Br}_n \xrightarrow[\text{full}]{\Phi} \mathcal{R}\text{ep}(\mathfrak{g}_n)$$

$$\mathbf{Br}_n / \text{"ker}(\Phi) \xrightarrow[\text{fully faithful}]{\Phi} \mathcal{R}\text{ep}(\mathfrak{g}_n)$$

whose "kernel $\text{ker}(\Phi)$ " can be calculated.

Hence, up to Spin's and Karoubi closures, Brauer gave us a diagrammatic presentation of the representation category $\mathcal{R}\text{ep}(\mathfrak{g}_n)$ of \mathfrak{g}_n .

“Thick” Schur-Weyl-Brauer duality

Another one of Howe’s remarkable relationships:

Howe \sim **1975**. Let $V = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{so}_n) \curvearrowright \underbrace{\bigwedge^{\bullet} V \otimes \cdots \otimes \bigwedge^{\bullet} V}_{k \text{ times}} \curvearrowleft \mathbf{U}(\mathfrak{so}_{2k})$$

generating each other’s centralizer, and $\bigwedge^{a_1} V \otimes \cdots \otimes \bigwedge^{a_k} V$ is the $(\bar{a}_1, \dots, \bar{a}_k)$ th weight space of $\mathbf{U}(\mathfrak{so}_{2k})$. The $\mathbf{U}(\mathfrak{so}_n)$ - $\mathbf{U}(\mathfrak{so}_{2k})$ -bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} \mathbf{L}(\mathfrak{so}_n, \lambda) \otimes \mathbf{L}(\mathfrak{so}_{2k}, \sum_{j=1}^k (\lambda_j^T - n/2) \varepsilon_j).$$

The λ ’s again satisfy certain explicit conditions and $\bar{a}_i = a_i + n/2$.

“Thick” Schur-Weyl-Brauer duality

Another one of Howe’s remarkable relationships:

Note that the action of $\mathbf{U}(\mathfrak{so}_{2k})$ is not as clear as it was for $\mathbf{U}(\mathfrak{gl}_k)$!

Howe ~1975. Let $V = \mathbb{C}^n$. There are com-

$$\mathbf{U}(\mathfrak{so}_n) \curvearrowright \underbrace{\bigwedge^{\bullet} V \otimes \cdots \otimes \bigwedge^{\bullet} V}_{k \text{ times}} \curvearrowright \mathbf{U}(\mathfrak{so}_{2k})$$

generating each other’s centralizer, and $\bigwedge^{a_1} V \otimes \cdots \otimes \bigwedge^{a_k} V$ is the $(\bar{a}_1, \dots, \bar{a}_k)$ th weight space of $\mathbf{U}(\mathfrak{so}_{2k})$. The $\mathbf{U}(\mathfrak{so}_n)$ - $\mathbf{U}(\mathfrak{so}_{2k})$ -bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} \mathbf{L}(\mathfrak{so}_n, \lambda) \otimes \mathbf{L}(\mathfrak{so}_{2k}, \sum_{j=1}^k (\lambda_j^T - n/2) \varepsilon_j).$$

The λ ’s again satisfy certain explicit conditions and $\bar{a}_i = a_i + n/2$.

Still alive: The diagrammatic presentation machine

$$\begin{array}{ccc}
 \text{fix} & & \text{use} \\
 \mathbf{U}(\mathfrak{gl}_n) \circlearrowleft \bigwedge^{\bullet} V \otimes \cdots \otimes \bigwedge^{\bullet} V \circlearrowright & & \mathbf{U}(\mathfrak{gl}_k) \\
 \cup & \parallel & \cap \\
 \mathbf{U}(\mathfrak{so}_n) \circlearrowleft \underbrace{\bigwedge^{\bullet} V \otimes \cdots \otimes \bigwedge^{\bullet} V}_{k \text{ times}} \circlearrowright & & \mathbf{U}(\mathfrak{so}_{2k})
 \end{array}$$

Howe's $1^{1/2}$ th statement ▶ defines a ▶ diagrammatic category \mathbf{Web}^{\cup} such that

$$\begin{array}{ccc}
 \mathbf{U}(\mathfrak{so}_{2k}) & \xrightarrow[\text{full}]{\Phi_{\text{BD}}^{\text{ext}}} & \mathcal{R}\text{ep}(\mathfrak{so}_n) \\
 & \searrow \beta_{\cup} & \uparrow \Gamma_{\text{BD}}^{\text{ext}} \\
 \mathcal{B}\mathbf{r}_n & \xrightarrow[\text{fully faithful}]{} & \mathbf{Web}^{\cup}
 \end{array}$$

commutes. In particular, \mathbf{Web}^{\cup} is a ▶ thick version of the Brauer algebra.

Still alive: The diagrammatic presentation machine

Restricting the
action
on one side

$$\begin{array}{ccc}
 \text{fix} & & \text{use} \\
 \mathbf{U}(\mathfrak{gl}_n) \circlearrowleft \bigwedge^{\bullet} V \otimes \cdots \otimes \bigwedge^{\bullet} V \circlearrowright & & \mathbf{U}(\mathfrak{gl}_k) \\
 \cup & \parallel & \cap \\
 \mathbf{U}(\mathfrak{so}_n) \circlearrowleft \underbrace{\bigwedge^{\bullet} V \otimes \cdots \otimes \bigwedge^{\bullet} V}_{k \text{ times}} \circlearrowright & & \mathbf{U}(\mathfrak{so}_{2k})
 \end{array}$$

Howe's $1^{1/2}$ th statement ▶ defines a ▶ diagrammatic category \mathbf{Web}^{\cup} such that

$$\begin{array}{ccc}
 \mathbf{U}(\mathfrak{so}_{2k}) & \xrightarrow[\text{full}]{\Phi_{\text{BD}}^{\text{ext}}} & \mathcal{R}\text{ep}(\mathfrak{so}_n) \\
 & \searrow \beta_{\cup} & \uparrow \Gamma_{\text{BD}}^{\text{ext}} \\
 \mathcal{B}\mathbf{r}_n & \xrightarrow[\text{fully faithful}]{} & \mathbf{Web}^{\cup}
 \end{array}$$

commutes. In particular, \mathbf{Web}^{\cup} is a ▶ thick version of the Brauer algebra.

Still alive: The diagrammatic presentation machine

Restricting the action on one side

$$\begin{array}{ccc}
 \text{fix} & & \text{use} \\
 \mathbf{U}(\mathfrak{gl}_n) \circlearrowleft \bigwedge^{\bullet} V \otimes \cdots \otimes \bigwedge^{\bullet} V \circlearrowright & & \mathbf{U}(\mathfrak{gl}_k) \\
 \cup & \parallel & \cap \\
 \mathbf{U}(\mathfrak{so}_n) \circlearrowleft \underbrace{\bigwedge^{\bullet} V \otimes \cdots \otimes \bigwedge^{\bullet} V}_{k \text{ times}} \circlearrowright & & \mathbf{U}(\mathfrak{so}_{2k})
 \end{array}$$

Increases the centralizer on the other

Howe's $1\frac{1}{2}$ th statement ▶ defines a ▶ diagrammatic category \mathbf{Web}^{\cup} such that

$$\begin{array}{ccc}
 \mathbf{U}(\mathfrak{so}_{2k}) & \xrightarrow[\text{full}]{\Phi_{\text{BD}}^{\text{ext}}} & \mathcal{R}\text{ep}(\mathfrak{so}_n) \\
 & \searrow \beta_{\cup} & \uparrow \Gamma_{\text{BD}}^{\text{ext}} \\
 \mathcal{B}\mathfrak{r}_n & \xrightarrow[\text{fully faithful}]{} & \mathbf{Web}^{\cup}
 \end{array}$$

commutes. In particular, \mathbf{Web}^{\cup} is a ▶ thick version of the Brauer algebra.

Still alive: The diagrammatic presentation machine

Restricting the action on one side

$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \cup \mathbf{U}(\mathfrak{so}_n)$

 Hence, we get
 “old diagram generators”
 and
 “new diagram generators”
 k times

 $\overset{\text{use}}{\mathbf{U}(\mathfrak{gl}_k)} \cap \mathbf{J}(\mathfrak{so}_{2k})$

Increases the centralizer on the other

Howe's $1\frac{1}{2}$ th statement ▶ defines a ▶ diagrammatic category \mathbf{Web}^\cup such that

$$\begin{array}{ccc}
 \mathbf{U}(\mathfrak{so}_{2k}) & \xrightarrow[\text{full}]{\Phi_{\text{BD}}^{\text{ext}}} & \mathbf{Rep}(\mathfrak{so}_n) \\
 & \searrow^{\beta_\cup} & \uparrow^{\text{full } \Gamma_{\text{BD}}^{\text{ext}}} \\
 \mathbf{Br}_n & \xrightarrow[\text{fully faithful}]{} & \mathbf{Web}^\cup
 \end{array}$$

commutes. In particular, \mathbf{Web}^\cup is a ▶ thick version of the Brauer algebra.

Some delicate quantizations

$$\begin{array}{ccccc} \mathbf{U}(\mathfrak{gl}_n) & \hookrightarrow & \Lambda^\bullet V & \otimes \cdots \otimes & \Lambda^\bullet V & \hookrightarrow & \mathbf{U}(\mathfrak{gl}_k) \\ \cup & & & \parallel & & & \cap \\ \mathbf{U}(\mathfrak{so}_n) & \hookrightarrow & \underbrace{\Lambda^\bullet V \otimes \cdots \otimes \Lambda^\bullet V}_{k \text{ times}} & \hookrightarrow & \mathbf{U}(\mathfrak{so}_{2k}) \end{array}$$

Some delicate quantizations

$$\begin{array}{ccc} \mathbf{U}(\mathfrak{gl}_n) & \Lambda_q^\bullet V_q \otimes \cdots \otimes \Lambda_q^\bullet V_q & \mathbf{U}(\mathfrak{gl}_k) \\ \cup & \parallel & \cap \\ \mathbf{U}(\mathfrak{so}_n) & \underbrace{\Lambda_q^\bullet V_q \otimes \cdots \otimes \Lambda_q^\bullet V_q}_{k \text{ times}} & \mathbf{U}(\mathfrak{so}_{2k}) \end{array}$$

Some delicate quantizations

$$\begin{array}{ccc} \mathbf{U}_q(\mathfrak{gl}_n) \circlearrowleft \Lambda_q^\bullet V_q \otimes \cdots \otimes \Lambda_q^\bullet V_q \circlearrowright & & \mathbf{U}_q(\mathfrak{gl}_k) \\ & \parallel & \\ \mathbf{U}(\mathfrak{so}_n) & \underbrace{\Lambda_q^\bullet V_q \otimes \cdots \otimes \Lambda_q^\bullet V_q}_{k \text{ times}} & \mathbf{U}(\mathfrak{so}_{2k}) \end{array}$$

Quantum skew Howe duality:
Lehrer–Zhang–Zhang ~2009.
(But its quite easy and not their main point.)

Some delicate quantizations

Does not quantize!

$$\begin{array}{ccc}
 \mathbf{U}_q(\mathfrak{gl}_n) & \overset{\circlearrowleft}{\cong} & \Lambda_q^\bullet V_q \otimes \cdots \otimes \Lambda_q^\bullet V_q & \overset{\circlearrowright}{\cong} & \mathbf{U}_q(\mathfrak{gl}_k) \\
 \Downarrow & & \parallel & & \Downarrow \\
 \mathbf{U}_q(\mathfrak{so}_n) & & \underbrace{\Lambda_q^\bullet V_q \otimes \cdots \otimes \Lambda_q^\bullet V_q}_{k \text{ times}} & & \mathbf{U}_q(\mathfrak{so}_{2k})
 \end{array}$$

Quantizes easily

Some delicate quantizations

$$\mathbf{U}_q(\mathfrak{gl}_n) \overset{\circlearrowleft}{\simeq} \underbrace{\Lambda_q^\bullet V_q \otimes \cdots \otimes \Lambda_q^\bullet V_q}_{\parallel} \overset{\circlearrowright}{\simeq} \mathbf{U}_q(\mathfrak{gl}_k)$$

No action at all.

$$\mathbf{U}_q(\mathfrak{so}_n) \overset{\circlearrowleft}{\simeq} \underbrace{\Lambda_q^\bullet V_q \otimes \cdots \otimes \Lambda_q^\bullet V_q}_{k \text{ times}} \overset{\circlearrowright}{\simeq} \mathbf{U}_q(\mathfrak{so}_{2k})$$

Action unclear.

The quantum dimension of V_q^{gl} is $[n]$.
 The quantum dimension of V_q^{so} is $[n-1]+1$.

Hence, V_q^{so} does not come from V_q^{gl} !

This "flaw" propagates all the way through:

$\bigwedge_q^a V_q^{\text{so}}$ have "weird" quantum dimensions.

$$\bigcirc = -(q^2 + q + q^{-1} + q^{-2}),$$

$$\bigcirc\bigcirc = q^3 + q + 1 + q^{-1} + q^{-3},$$

$$\text{⌢} = 0,$$

↑
The quantum dimension of V_q^{so}

$$\text{⌢⌢} = -(q + 2 + q^{-1}) \text{⌢⌢},$$

$$\text{⌢⌢⌢} = 0,$$

$$\text{⌢⌢⌢} - \text{⌢⌢⌢} = \text{⌢} - \text{⌢},$$

Above: Kuperberg's \mathbf{B}_2 web relations ~ 1995 .

The quantum dimension of V_q^{gl} is $[n]$.
 The quantum dimension of V_q^{so} is $[n-1]+1$.

Hence, V_q^{so} does not come from V_q^{gl} !

This "flaw" propagates all the way through:
 $\bigwedge_q^a V_q^{so}$ have "weird" quantum dimensions.

$$\bigcirc = -(q^2 + q + q^{-1} + q^{-2}),$$

$$\bigcirc\bigcirc = q^3 + q + 1 + q^{-1} + q^{-3},$$

↑
The quantum dimension of V_q^{so5}

$$\text{loop} = 0,$$

$$\text{link} = -(q + 2 + q^{-1}) = \text{link},$$

$$\text{triangle} = 0,$$

$$\text{web} - \text{web} = \text{web} - \text{web},$$

Above: Kuperberg's B_2 web relations ~ 1995 .

We wanted to generalize Kuperberg's results. We failed because quantization is hard outside of type **A**.

But let me explain what we can do.

Some delicate quantizations

Using a coideal subalgebra does the trick.

$$\begin{array}{ccc} \mathbf{U}_q(\mathfrak{gl}_n) \circlearrowleft \underbrace{\Lambda_q^\bullet V_q \otimes \cdots \otimes \Lambda_q^\bullet V_q}_{k \text{ times}} \circlearrowright \mathbf{U}_q(\mathfrak{gl}_k) & & \\ \cup & \parallel & \cap \\ \mathbf{U}'_q(\mathfrak{so}_n) \circlearrowleft \underbrace{\Lambda_q^\bullet V_q \otimes \cdots \otimes \Lambda_q^\bullet V_q}_{k \text{ times}} \circlearrowright \mathbf{U}_q(\mathfrak{so}_{2k}) & & \end{array}$$

The action is constructed using the unquantized diagrammatics.

Some delicate quantizations

$$\begin{array}{ccc}
 \mathbf{U}_q(\mathfrak{gl}_n) \circlearrowleft \underbrace{\Lambda_q^\bullet V_q \otimes \cdots \otimes \Lambda_q^\bullet V_q}_{k \text{ times}} \circlearrowright \mathbf{U}_q(\mathfrak{gl}_k) \\
 \cup \qquad \qquad \qquad \parallel \qquad \qquad \qquad \cap \\
 \mathbf{U}'_q(\mathfrak{so}_n) \circlearrowleft \underbrace{\Lambda_q^\bullet V_q \otimes \cdots \otimes \Lambda_q^\bullet V_q}_{k \text{ times}} \circlearrowright \mathbf{U}_q(\mathfrak{so}_{2k})
 \end{array}$$

Using a q -monoidal [diagrammatic](#) category $\mathbf{Web}_{q,q^n}^\cup$ we can [define](#) a full Howe functor $\Phi_{\mathbf{BD}}^{\text{ext}}$ such that we get a commuting diagram

$$\begin{array}{ccc}
 \mathbf{U}_q(\mathfrak{so}_{2k}) & \xrightarrow{\Phi_{\mathbf{BD}}^{\text{ext}}} & \mathbf{Rep}'_q(\mathfrak{so}_n) \\
 & \swarrow \text{define} & \uparrow \Gamma_{\mathbf{BD}}^{\text{ext}} \\
 & & \mathbf{Web}_{q,q^n}^\cup \\
 \mathbf{Br}_{q,q^n} & \xrightarrow{\beta_\cup \text{ fully faithful}} & \mathbf{Web}_{q,q^n}^\cup
 \end{array}$$

Hereby, $\mathbf{Rep}'_q(\mathfrak{so}_n)$ is the q -monoidal representation category of [\$\mathbf{U}'_q\(\mathfrak{so}_n\)\$](#) , and \mathbf{Br}_{q,q^n} is Molev's q -Brauer category (~ 2002).

Further directions

$$\begin{array}{ccc}
 \mathbf{U}_q(\mathfrak{gl}_n) \circlearrowleft \Lambda_q^\bullet V_q^{\mathfrak{gl}} \otimes \cdots \otimes \Lambda_q^\bullet V_q^{\mathfrak{gl}} \circlearrowright \mathbf{U}_q(\mathfrak{gl}_k) & & \\
 \cup & \parallel & \cap \\
 \mathbf{U}'_q(\mathfrak{so}_n) \circlearrowleft \underbrace{\Lambda_q^\bullet V_q^{\mathfrak{gl}} \otimes \cdots \otimes \Lambda_q^\bullet V_q^{\mathfrak{gl}}}_{k \text{ times}} \circlearrowright \mathbf{U}_q(\mathfrak{so}_{2k}) & &
 \end{array}$$

- Use a similar approach to get the quantum group to work. (Needs probably some mixed Howe duality à la Queffelec–Sartori.)

Further directions

* = $n-1/2$ for type **B**,
 * = $n/2$ for type **D**.

$$\begin{array}{ccc}
 \mathbf{U}_q(\mathfrak{gl}_*) \circlearrowleft \Lambda_q^\bullet V_q^{s_0} \otimes \cdots \otimes \Lambda_q^\bullet V_q^{s_0} \circlearrowright \mathbf{U}_q(\mathfrak{gl}_k) & & \\
 \cap & \parallel & \cup \\
 \mathbf{U}_q(\mathfrak{so}_n) \circlearrowleft \underbrace{\Lambda_q^\bullet V_q^{s_0} \otimes \cdots \otimes \Lambda_q^\bullet V_q^{s_0}}_{k \text{ times}} \circlearrowright ?\mathbf{U}'_q(\mathfrak{so}_{2k})? & &
 \end{array}$$

- Use a similar approach to get the quantum group to work. (Needs probably some mixed Howe duality à la Queffelec–Sartori.)

This should give the quantum group story,
 but it is much trickier since e.g.

$$V_q^{s_0} \cong V_q^{\mathfrak{gl}} \oplus (V_q^{\mathfrak{gl}})^* \oplus \mathbb{C}$$

as $\mathbf{U}_q(\mathfrak{gl}_*)$ -modules in type **B**.

Thus, the above is not the usual $\mathbf{U}(\mathfrak{gl}_*)$ - $\mathbf{U}(\mathfrak{gl}_k)$ duality.

Further directions

$$\begin{array}{ccc} \mathbf{U}_q(\mathfrak{gl}_n) \circlearrowleft \Lambda_q^\bullet \mathbf{V}_q^{\mathfrak{gl}} \otimes \dots \otimes \Lambda_q^\bullet \mathbf{V}_q^{\mathfrak{gl}} \circlearrowright \mathbf{U}_q(\mathfrak{gl}_k) \\ \cup & & \cap \\ \mathbf{U}'_q(\mathfrak{so}_n) & \begin{array}{c} \begin{array}{cccc} a & a & b & b \end{array} \\ \text{U-shaped diagram} \\ \end{array} & = q^* & \begin{array}{c} \begin{array}{cccc} a & a & b & b \end{array} \\ \text{U-shaped diagram} \\ \end{array} \\ & \text{\textit{q-interchange law}} & & \\ & * = \text{some power depending on } a, b & & \end{array}$$

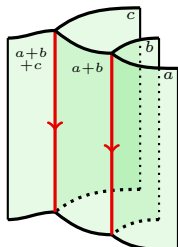
- ▶ Use a similar approach to [reference] to work. (Needs probably some mixed Howe duality à la Queneau–Saiton.)
- ▶ q -monoidal categories are probably very useful to study representation categories of coideal subalgebras. An abstract formulation à la Brundan–Ellis (“super monoidal”) should be useful.

Further directions

$$\mathbf{U}_q(\mathfrak{gl}_n)$$

$$\cup$$

$$\mathbf{U}'_q(\mathfrak{so}_n)$$



Singular cobordisms ("foams",
à la Khovanov–Rozansky and Mackaay–Stošić–Vaz)
categorify webs.

The q -monoidal property has to be smartly encoded.

$$\mathbf{U}_q(\mathfrak{gl}_k)$$

$$\cap$$

$$\mathbf{U}_q(\mathfrak{so}_{2k})$$

- Use a similar approach to some mixed Howe dualities

- q -monoidal categories of coideal subalgebras ("super monoidal")

- Coideal subalgebras are amenable to categorification, cf. Ehrig–Stroppel or Bao–Shan–Wang–Webster. Similarly, their representation categories should be amenable to categorification.

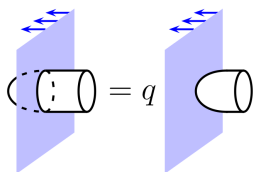
to work. (Needs probably

study representation
theory à la Brundan–Ellis

Further directions

$$\begin{array}{c}
 \mathbf{U}_q(\mathfrak{gl}_n) \subset \\
 \cup \\
 \mathbf{U}'_q(\mathfrak{so}_n) \subset
 \end{array}
 \begin{array}{c}
 \text{Diagram 1: A cylinder with a vertical dashed line and a blue plane with arrows} \\
 = q \cdot \\
 \text{Diagram 2: A cylinder with two vertical dashed lines and a blue plane with arrows}
 \end{array}
 \begin{array}{c}
 \supset \mathbf{U}_q(\mathfrak{gl}_k) \\
 \cap \\
 \supset \mathbf{U}_q(\mathfrak{so}_{2k})
 \end{array}$$

- ▶ Use a similar approach to work. (Needs probably some mixed Howe dual categories)
- ▶ q -monoidal categories : study representation categories of coideal su- algebras (“super monoidal”) should be amenable to categorification. (Needs probably some mixed Howe dual categories)
- ▶ Coideal subalgebras are Bao–Shan–Wang–Webster’s “2- q -monoidal foams.” (Maybe connected to Beliakova–Putyra–Wehrli whose pictures I shamelessly stole.)
- ▶ Formulate everything in a “2- q -monoidal language”. (Again, à la Brundan–Ellis.)



2- q -monoidal foams.

(Maybe connected to Beliakova–Putyra–Wehrli whose pictures I shamelessly stole.)

to work. (Needs probably some mixed Howe dual categories)

study representation categories of coideal subalgebras (“super monoidal”) should be amenable to categorification.

ion, cf. Ehrig–Stroppel or categorification of representations of Lie algebras should be amenable to categorification.

A pioneer of representation theory

remarkable relationship between gl_n and the symmetric group S_n :
Scher – 1901. Let $V = V^{\oplus n} = \mathbb{C}^n$. There are commuting actions

$$U(gl_n) \otimes V \cong \bigoplus_{\lambda \vdash n} V_{\lambda} \otimes \mathbb{C}[S_n]$$
 This statement
 generating each other's centralizer. The $U(gl_n) \otimes \mathbb{C}[S_n]$ bimodule decomposes as
 The precise form does not matter for today. It is only important that one can make it explicit!

$$\bigoplus_{\lambda \vdash n} U(gl_n, \lambda) \otimes L(B_n^{\vee}, \lambda^T)$$

The λ 's are partitions (Young diagrams) of k with at most n rows.

Another pioneer of representation theory

remarkable relationship between $gl_n = \mathfrak{sl}_n, \mathfrak{sp}_n$ and the Brauer algebra B_n^{\pm} :
Brauer – 1937. Let $V = \mathbb{C}^n$. There are commuting actions

$$U(\mathfrak{g}_n) \otimes V \cong \bigoplus_{\lambda \vdash n} V_{\lambda} \otimes B_n^{\pm}$$
 generating each other's centralizer. The $U(\mathfrak{g}_n) \otimes B_n^{\pm}$ bimodule decomposes as

$$\bigoplus_{\lambda \vdash n} U(\mathfrak{g}_n, \lambda) \otimes L(B_n^{\pm}, \lambda^T)$$
 The λ 's are partitions of $k, k-2, k-4, \dots$, whose precise form depend on \mathfrak{g}_n .

Still alive: The diagrammatic presentation machine

Minimizing the action

$$U(\mathfrak{g}_n) \otimes \wedge^k V \otimes \dots \otimes \wedge^k V \otimes U(\mathfrak{g}_n) \cap U(\mathfrak{so}_n)$$
 Increases the centralizer
 on the other

$$U(\mathfrak{so}_n) \otimes \wedge^k V \otimes \dots \otimes \wedge^k V \otimes U(\mathfrak{so}_n)$$
 How's $\mathbb{1}$'s job statement? $\mathbb{1}$ is a \mathbb{C} -category $\mathbb{W}eb^{\vee}$ such that

$$\begin{array}{ccc} U(\mathfrak{so}_n) & \xrightarrow{\text{left}} & \mathbb{R}ep(\mathfrak{so}_n) \\ & \searrow \text{left} & \downarrow \text{left} \\ B_n^{\vee} & \xrightarrow{\text{fully faithful}} & \mathbb{W}eb^{\vee} \end{array}$$
 commutes. In particular, $\mathbb{W}eb^{\vee}$ is a \mathbb{C} -version of the Brauer algebra.

Some delicate

The quantum dimension of $V^{\otimes n}$ is $[n]!$
 The quantum dimension of $V^{\otimes n}$ is $[n-1]!$.
 Hence $V^{\otimes n}$ does not come from $V^{\otimes n}$.
 This "bar" propagates all the way through.
 $\Lambda_n^{\otimes n}$'s have "large" quantum dimension.

$$\bigcirc = -[n]! \text{tr}(e^{\otimes n})$$

$$\bigcirc = e^{\otimes n} \text{tr}(e^{\otimes n})$$

$$\bigcirc = [n]!$$

 But let me explain what we can do.

$$\bigcirc = -[n-2]! \text{tr}(e^{\otimes n})$$

 Above: Kapranov's \mathbb{B} , web relations – 2005.

Some delicate quantizations

$$U_q(\mathfrak{g}_n) \otimes \wedge^k V_q \otimes \dots \otimes \wedge^k V_q \otimes U_q(\mathfrak{g}_n)$$

$$U_q(\mathfrak{so}_n) \otimes \wedge^k V_q \otimes \dots \otimes \wedge^k V_q \otimes U_q(\mathfrak{so}_n)$$
 Using a q -monoidal \mathbb{C} -category $\mathbb{W}eb_{q, \mathbb{C}}^{\vee}$, we can \mathbb{C} -ify a full Howe functor $\Phi_{\mathbb{C}, q}$ such that we get a commuting diagram

$$\begin{array}{ccc} U_q(\mathfrak{so}_n) & \xrightarrow{\text{left}} & \mathbb{R}ep_q(\mathfrak{so}_n) \\ & \searrow \text{left} & \downarrow \text{left} \\ B_n^{\vee} & \xrightarrow{\text{fully faithful}} & \mathbb{W}eb_{q, \mathbb{C}}^{\vee} \end{array}$$
 Hence $\mathbb{R}ep_q(\mathfrak{so}_n)$ is the q -monoidal representation category of \mathbb{C} -ified, and B_n^{\vee} is Mal'cev's q -Brauer category (~ 2002).

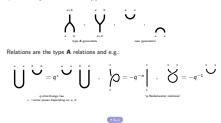
Dual pair	Module M	q-version and web calculi
$U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_n)$	$\Lambda_n^{\pm}(\mathbb{C}^n \otimes \mathbb{C}^n)$	Coxeter-Kamnitzer-Matsumoto – 2012
$U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_n)$	$\Lambda_n^{\pm}(\mathbb{C}^n \otimes \mathbb{C}^n)$	Sartori – 2013, Grant – 2014
$U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_n)$	$\text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n)$	Rowe and coauthors – 2015
$U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_n)$	$\Lambda_n^{\pm}(\mathbb{C}^n \otimes \mathbb{C}^n)$	Quaffele-Sartori, Grant – 2015
$U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_n)$	$\Lambda_n^{\pm}(\mathbb{C}^{2n} \otimes \mathbb{C}^n)$	Vaz-Webbich and coauthors – 2015
$U(\mathfrak{so}_n) \otimes U(\mathfrak{so}_n)$	$\Lambda_n^{\pm}(\mathbb{C}^n \otimes \mathbb{C}^n)$	Sartori and coauthors – 2017
$U(\mathfrak{so}_n) \otimes U(\mathfrak{so}_n)$	$\text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n)$	
$U(\mathfrak{sp}_n) \otimes U(\mathfrak{sp}_n)$	$\Lambda_n^{\pm}(\mathbb{C}^n \otimes \mathbb{C}^n)$	
$U(\mathfrak{sp}_n) \otimes U(\mathfrak{sp}_n)$	$\text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n)$	

Up to quantization, all of this (and more) is basically already in Howe's paper.

Monoidal generators of $\mathbb{W}eb^{\vee}$:



q-Monoidal generators of $\mathbb{W}eb_{q, \mathbb{C}}^{\vee}$:



There is still much to do...

	$U_q(\mathfrak{so}_n)$	$U_q(\mathfrak{sp}_n)$
Subalgebra of $U_q(\mathfrak{gl}_n)$	✗	✗
Hopf algebra	✓	✓
Quantization of $U(\mathfrak{so}_n)$	✓	✓
"Nice quantum" $\mathbb{1}$	✓	✓
Connected to Peng's talk years ago $\omega(E_i) = -F_i$	✓	✗

$U_q(\mathfrak{so}_n)$ is a (left) coideal:
 $\Delta: U_q(\mathfrak{so}_n) \rightarrow U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{so}_n)$.
 Hence, $\mathbb{R}ep_q(\mathfrak{so}_n)$ is only q -monoidal and carries a left action of $\mathbb{R}ep_q(\mathfrak{gl}_n)$.

A pioneer of representation theory

remarkable relationship between gl_n and the symmetric group S_n :

Schur – 1901. Let $V = V^{\lambda} \in \mathbb{C}^n$. There are commuting actions

$$U(gl_n) \otimes V \cong \bigoplus_{\lambda \vdash n} U(S_n) \otimes \mathbb{C}[S_n]$$

Second statement: This statement generating each other's centralizer. The $U(gl_n) \otimes \mathbb{C}[S_n]$ bimodule decomposes as

$$\bigoplus_{\lambda \vdash n} U(gl_n, \lambda) \otimes L(\mathbb{C}[S_n]^\lambda)$$

The λ 's are partitions (Young diagrams) of k with at most n rows.

Another pioneer of representation theory

remarkable relationship between $gl_n = \mathfrak{sl}_n, \mathfrak{sp}_n$ and the Brauer algebra B_n^2 :

Brauer – 1937. Let $V \in \mathbb{C}^n$. There are commuting actions

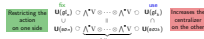
$$U(\mathfrak{g}_n) \otimes V \cong \bigoplus_{\lambda \vdash n} U(B_n^2) \otimes \mathbb{C}[B_n^2]$$

generating each other's centralizer. The $U(\mathfrak{g}_n) \otimes \mathbb{C}[B_n^2]$ bimodule decomposes as

$$\bigoplus_{\lambda \vdash n} U(\mathfrak{g}_n, \lambda) \otimes L(\mathbb{C}[B_n^2]^\lambda)$$

The λ 's are partitions of $k, k-2, k-4, \dots$, whose precise form depend on \mathfrak{g}_n .

Still alive: The diagrammatic presentation machine



How's $\mathbb{1}$'s statement: $\mathbb{1}$ is a q -monoidal category $\mathbb{W}eb^{\mathbb{1}}$ such that



commutes. In particular, $\mathbb{W}eb^{\mathbb{1}}$ is a q -version of the Brauer algebra.

Some delicate

The quantum dimension of V^{λ} is $q^{|\lambda|}$. The quantum dimension of V^{μ} is $[n-|\mu|+1]$. Hence V^{λ} does not come from V^{μ} . This "bar" propagates all the way through. Λ_n^{\pm} 's have "large" quantum dimensions.

$$\bigcirc = -q^2 + q + q^{-1}$$

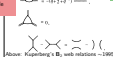
$$\bigcirc = q^2 + q + q^{-1}$$

$$\bigcirc = q$$

$$\bigcirc = -q^{-1} + q + q^{-1}$$

We wanted to generalize Kapranov's results. We failed because quantization is hard outside of type A.

But let me explain what we can do.



Some delicate quantizations

$$U_q(\mathfrak{gl}_n) \otimes \Lambda_n^+ \cong \bigoplus_{\lambda \vdash n} \Lambda_n^+ \otimes U_q(\mathfrak{gl}_n)$$

$$U_q(\mathfrak{sl}_n) \otimes \Lambda_n^+ \cong \bigoplus_{\lambda \vdash n} \Lambda_n^+ \otimes U_q(\mathfrak{sl}_n)$$

Using a q -monoidal category $\mathbb{W}eb_{\mathbb{1}}^{\mathbb{1}}$, we can get a full Howe functor $\Phi_{\mathbb{1}}^{\mathbb{1}}$ such that we get a commuting diagram

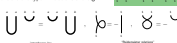


Hence $\mathbb{R}ep_{\mathbb{1}}^{\mathbb{1}}(\mathfrak{sl}_n)$ is the q -monoidal representation category of $\mathbb{1}$, and $\mathbb{B}r_{\mathbb{1}}^{\mathbb{1}}$ is Mal'cev's q -Brauer category (~2002).

Monoidal generators of $\mathbb{W}eb^{\mathbb{1}}$:



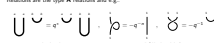
Relations are the type A relations and e.g.:



q -Monoidal generators of $\mathbb{W}eb_{\mathbb{1}}^{\mathbb{1}}$:



Relations are the type A relations and e.g.:



Dual pair	Module M	q-version and web calculi
$U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_n)$	$\Lambda_n^{\pm}(\mathbb{C}^n \otimes \mathbb{C}^n)$	Coxeter-Kamnitzer-Matsumoto – 2012
$U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_n)$	$\Lambda_n^{\pm}(\mathbb{C}^n \otimes \mathbb{C}^n)$	Sartori – 2013, Grant – 2014
$U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_n)$	$\text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n)$	Rowe and coauthors – 2015
$U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_n)$	$\Lambda_n^{\pm}(\mathbb{C}^n \otimes \mathbb{C}^n)$	Quaffele-Sartori, Grant – 2015
$U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_n)$	$\Lambda_n^{\pm}(\mathbb{C}^{2n} \otimes \mathbb{C}^n)$	Vaz-Webbich and coauthors – 2015
$U(\mathfrak{sl}_n) \otimes U(\mathfrak{sl}_n)$	$\Lambda_n^{\pm}(\mathbb{C}^n \otimes \mathbb{C}^n)$	Sartori and coauthors – 2017
$U(\mathfrak{sl}_n) \otimes U(\mathfrak{sp}_n)$	$\text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n)$	
$U(\mathfrak{sp}_n) \otimes U(\mathfrak{gl}_n)$	$\Lambda_n^{\pm}(\mathbb{C}^n \otimes \mathbb{C}^n)$	
$U(\mathfrak{sp}_n) \otimes U(\mathfrak{sp}_n)$	$\text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n)$	

Up to quantization, all of this (and more) is basically already in Howe's paper.

	$U_q(\mathfrak{sl}_n)$	$U_q(\mathfrak{sp}_n)$
Subalgebra of $U_q(\mathfrak{gl}_n)$	✗	✗
Hopf algebra	✓	✓
Quantization of $U(\mathfrak{sl}_n)$	✓	✓
"Nice quantum" $\mathbb{1}$ -invariant	✓	✗

Connected to Peng's talk years ago: $(H) = -F$ (Mal'cev involution)

$U_q(\mathfrak{sl}_n)$ is a (left) coideal: $\Delta: U_q(\mathfrak{sl}_n) \rightarrow U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{sl}_n)$. Hence, $\mathbb{R}ep_{\mathbb{1}}^{\mathbb{1}}(\mathfrak{sl}_n)$ is only q -monoidal and carries a left action of $\mathbb{R}ep_{\mathbb{1}}^{\mathbb{1}}(\mathfrak{gl}_n)$.

Thanks for your attention!

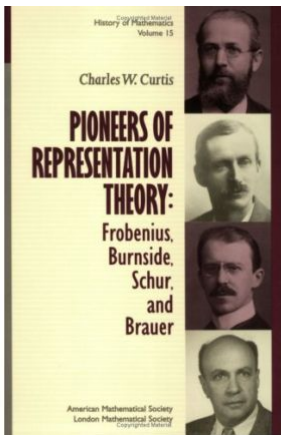


Figure: Two of the main players for today: Schur and Brauer.

Curtis, C.W. *Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer.*

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

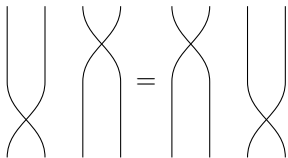
In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from “Theory of Groups of Finite Order” by Burnside. Top: first edition (1897); bottom: second edition (1911).

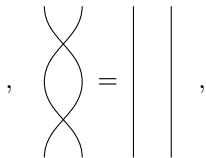
Monoidal generator of \mathcal{S} :

$$\begin{array}{c} \diagup \\ \diagdown \\ \diagdown \\ \diagup \end{array} : 2 \rightarrow 2.$$

Relations e.g.:



interchange law



"Reidemeister relations"

Dual pair	Module M	q -version and web calculi
$U(\mathfrak{gl}_n) - U(\mathfrak{gl}_k)$	$\Lambda^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	Cautis–Kamnitzer–Morrison ~ 2012
$U(\mathfrak{gl}_{1 1}) - U(\mathfrak{gl}_k)$	$\Lambda^\bullet(\mathbb{C}^{1 1} \otimes \mathbb{C}^k)$	Sartori ~ 2013 , Grant ~ 2014
$U(\mathfrak{gl}_n) - U(\mathfrak{gl}_k)$	$\text{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	Rose and coauthors ~ 2015
$U(\mathfrak{gl}_{m n}) - U(\mathfrak{gl}_k)$	$\Lambda^\bullet(\mathbb{C}^{m n} \otimes \mathbb{C}^k)$	Queffelec–Sartori, Grant ~ 2015
$U(\mathfrak{gl}_{m n}) - U(\mathfrak{gl}_{l k})$	$\Lambda^\bullet(\mathbb{C}^{m n} \otimes \mathbb{C}^{l k})$	Vaz–Wedrich and coauthors ~ 2015
$U(\mathfrak{so}_n) - U(\mathfrak{so}_{2k})$	$\Lambda^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	Sartori and coauthors ~ 2017
$U(\mathfrak{so}_n) - U(\mathfrak{sp}_{2k})$	$\text{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	
$U(\mathfrak{sp}_n) - U(\mathfrak{sp}_{2k})$	$\Lambda^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	
$U(\mathfrak{sp}_n) - U(\mathfrak{so}_{2k})$	$\text{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	

Up to quantization, all of this (and more) is basically already in Howe's paper.

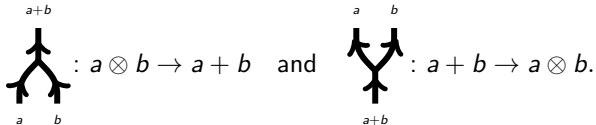
Dual pair	Module M	q -version and web calculi
<p>Type A is in a fairly good shape:</p> <p>The story partially works “integrally” (Elias ~2015).</p> <p>Applications to link polynomials (e.g. Wedrich–Vaz and coauthors ~2015).</p> <p>Partially categorified (e.g. Huerfano–Khovanov ~2002, Mackaay ~2009).</p> <p>Applications to link homologies (e.g. Lauda–Queffelec–Rose ~2012).</p> <p>Applications to canonical bases and geometry (e.g. Cautis–Kamnitzer ~2016).</p>		
$\mathcal{U}(\mathfrak{so}_m \mathfrak{n})$ - $\mathcal{U}(\mathfrak{sp}_l \mathfrak{k})$	$\Lambda^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	Vaz, Wedrich and coauthors ~2015
$\mathcal{U}(\mathfrak{so}_n)$ - $\mathcal{U}(\mathfrak{so}_{2k})$	$\Lambda^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	Sartori and coauthors ~2017
$\mathcal{U}(\mathfrak{so}_n)$ - $\mathcal{U}(\mathfrak{sp}_{2k})$	$\text{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	
$\mathcal{U}(\mathfrak{sp}_n)$ - $\mathcal{U}(\mathfrak{sp}_{2k})$	$\Lambda^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	
$\mathcal{U}(\mathfrak{sp}_n)$ - $\mathcal{U}(\mathfrak{so}_{2k})$	$\text{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	

Up to quantization, all of this (and more) is basically already in Howe’s paper.

Dual pair	Module M	q -version and web calculi
<p>Type A is in a fairly good shape:</p> <p>The story partially works “integrally” (Elias ~2015).</p> <p>Applications to link polynomials (e.g. Wedrich–Vaz and coauthors ~2015).</p> <p>Partially categorified (e.g. Huerfano–Khovanov ~2002, Mackaay ~2009).</p> <p>Applications to link homologies (e.g. Lauda–Queffelec–Rose ~2012).</p> <p>Applications to canonical bases and geometry (e.g. Cautis–Kamnitzer ~2016).</p>		
$U(\mathfrak{so}_m n)$ – $U(\mathfrak{sp}_l k)$	$\Lambda^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	Vaz, Wedrich and coauthors ~2015
$U(\mathfrak{so}_n)$ – $U(\mathfrak{so}_{2k})$	$\Lambda^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	
$U(\mathfrak{so}_n)$ – $U(\mathfrak{sp}_{2k})$	$\text{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	Sartori
<p>Types BCD are not really understood.</p>		
$U(\mathfrak{sp}_n)$ – $U(\mathfrak{sp}_{2k})$	$\Lambda^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	and coauthors ~2017
$U(\mathfrak{sp}_n)$ – $U(\mathfrak{so}_{2k})$	$\text{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	

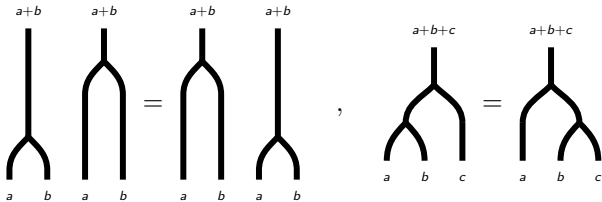
Up to quantization, all of this (and more) is basically already in Howe’s paper.

Monoidal generators of \mathcal{Web}^{\wedge} :



Relations e.g.:

One needs orientations in type **A**,
but I am going to ignore them.

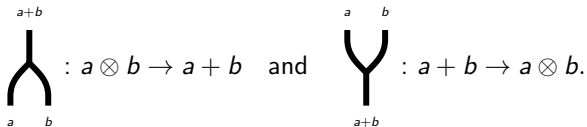


interchange law

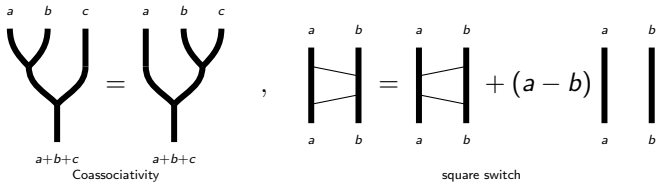
Associativity

◀ Back

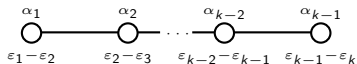
Monoidal generators of \mathcal{Web}^\wedge :



Relations e.g.:



Root conventions is type **A**:

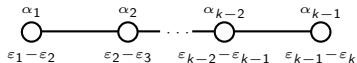


Thus, because of statement 1^{1/2}, we should set

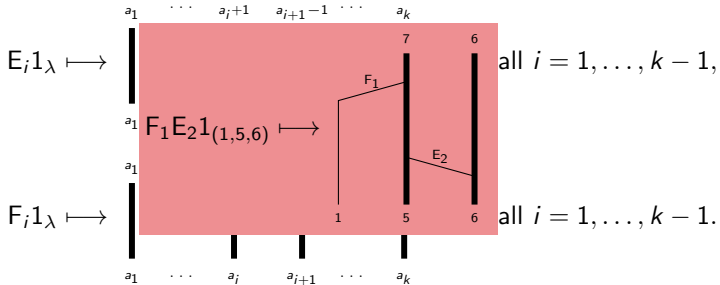
$$E_i 1_\lambda \mapsto \begin{array}{cccccc} a_1 & \cdots & a_{i+1} & a_{i+1}-1 & \cdots & a_k \\ | & & | & | & & | \\ \vdots & & \diagdown & \diagup & & \vdots \\ | & & | & | & & | \\ a_1 & \cdots & a_i & a_{i+1} & \cdots & a_k \end{array}, \quad \text{for all } i = 1, \dots, k-1,$$

$$F_i 1_\lambda \mapsto \begin{array}{cccccc} a_1 & \cdots & a_{i-1} & a_{i+1}+1 & \cdots & a_k \\ | & & | & | & & | \\ \vdots & & \diagdown & \diagup & & \vdots \\ | & & | & | & & | \\ a_1 & \cdots & a_i & a_{i+1} & \cdots & a_k \end{array}, \quad \text{for all } i = 1, \dots, k-1.$$

Root conventions is type **A**:



Thus, because of statement 1^{1/2}, we should set



$$\beta_{\mathbf{A}}: \mathcal{S} \rightarrow \mathbf{Web}^{\mathbf{A}}$$

The diagram shows an equality between three configurations of strands. On the left is a crossing of two strands. An arrow points to the first configuration on the right, which consists of two strands crossing, with a '1' label above and below each strand. This is equal to the second configuration, which consists of two parallel vertical strands, each with a '1' label above and below. This is then added to the third configuration, which consists of two strands crossing, with a '1' label above and below each strand, and a thick vertical line segment connecting the two strands at the crossing point.

$$\mathbb{C}[S_k] \xrightarrow{\cong} \text{End}_{\mathbf{Web}^{\mathbf{A}}}(1^{\otimes k})$$

◀ Back

Monoidal generators of $\mathcal{B}r_n$:

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}, \quad \cup : \emptyset \rightarrow 2, \quad \cap : 2 \rightarrow \emptyset.$$

Relations e.g.:

$$\begin{array}{c} \cup \\ \cup \end{array} \cup = \cup \begin{array}{c} \cup \\ \cup \end{array}, \quad \bigcirc = \pm n.$$

interchange law circle removal

◀ Back

Monoidal generators of $\mathcal{B}r_n$:

Relati

$$S_1 = \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right)$$

$$S_2 = \left(\begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \\ \text{Diagram 9} \\ \text{Diagram 10} \\ \text{Diagram 11} \end{array} \right)$$

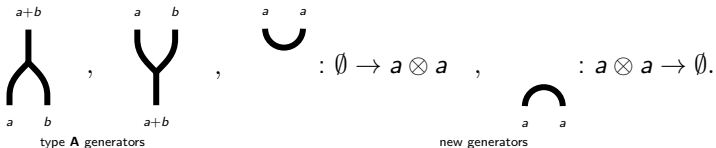
we obtain

$$(43) \left(\begin{array}{c} \text{Diagram 12} \\ \text{Diagram 13} \\ \text{Diagram 14} \\ \text{Diagram 15} \\ \text{Diagram 16} \\ \text{Diagram 17} \\ \text{Diagram 18} \\ \text{Diagram 19} \\ \text{Diagram 20} \end{array} \right)$$

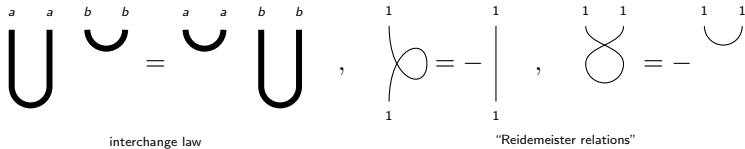
From "Brauer, R. *On algebras which are connected with the semisimple continuous groups.*
Ann. of Math. (2) 38 (1937), no. 4, 857-872."

Monoidal generators of \mathbf{Web}^\cup :

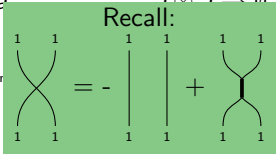
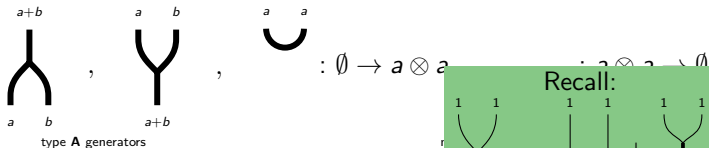
No orientations needed in types **BCD**.



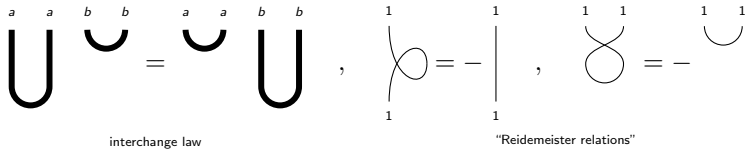
Relations are the type **A** relations and e.g.:



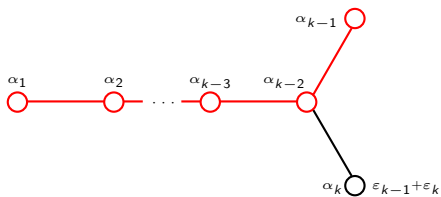
Monoidal generators of $\mathcal{W}eb^{\cup}$:



Relations are the type **A** relations and e.g.:



Root conventions is type **D**:

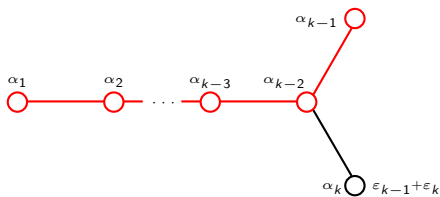


Thus, because of statement 1^{1/2}, we should set

$$E_k 1_\lambda \mapsto \begin{array}{cccccc} \bar{a}_1 & \cdots & \bar{a}_{k-2} & \bar{a}_{k-1}+1 & \bar{a}_k+1 \\ | & & | & \text{---} & | \\ \bar{a}_1 & \cdots & \bar{a}_{k-2} & \bar{a}_{k-1} & \bar{a}_k \end{array}$$

$$F_k 1_\lambda \mapsto \begin{array}{cccccc} \bar{a}_1 & \cdots & \bar{a}_{k-2} & \bar{a}_{k-1}-1 & \bar{a}_k-1 \\ | & & | & \text{---} & | \\ \bar{a}_1 & \cdots & \bar{a}_{k-2} & \bar{a}_{k-1} & \bar{a}_k \end{array}$$

Root conventions is type **D**:



Thus, because of statement 1^{1/2}, we should set

$$E_k \text{ FE1}_{(-n/2, -n/2)} \mapsto \begin{array}{c} \text{F} \\ \text{E} \end{array} \begin{array}{c} k-1+1 \\ \bar{a}_{k-1} \end{array} \begin{array}{c} \bar{a}_{k+1} \\ \bar{a}_k \end{array}$$

$$F_k 1_\lambda \mapsto \begin{array}{c} \bar{a}_1 \quad \cdots \quad \bar{a}_{k-2} \quad \bar{a}_{k-1}-1 \quad \bar{a}_{k-1} \\ \bar{a}_1 \quad \cdots \quad \bar{a}_{k-2} \quad \bar{a}_{k-1} \quad \bar{a}_k \end{array}$$

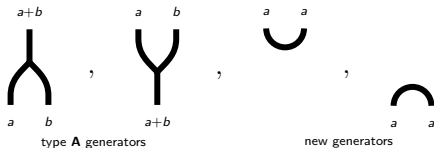
$$\beta_{\cup}: \mathbf{Br}_n \rightarrow \mathbf{Web}^{\cup}$$

$$\begin{array}{c}
 \text{Crossing} \mapsto \begin{array}{c} 1 \quad 1 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ 1 \quad 1 \end{array} = - \begin{array}{c} 1 \quad 1 \\ | \quad | \\ 1 \quad 1 \end{array} + \begin{array}{c} 1 \quad 1 \\ \diagdown \quad \diagup \\ \text{thick line} \\ \diagup \quad \diagdown \\ 1 \quad 1 \end{array} \\
 \\
 \text{Cup} \mapsto \begin{array}{c} 1 \quad 1 \\ \cup \\ \end{array}, \quad \text{Cap} \mapsto \begin{array}{c} \cap \\ 1 \quad 1 \end{array}
 \end{array}$$

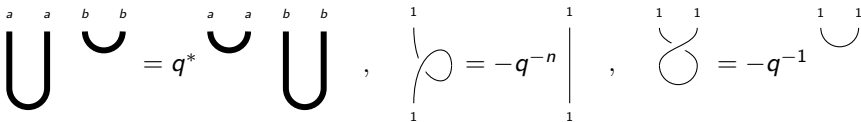
$$\mathbf{Br}_n^k \xrightarrow{\cong} \text{End}_{\mathbf{Web}^{\cup}}(1^{\otimes k})$$

◀ Back

q -Monoidal generators of $\mathcal{Web}_{q,q^n}^\cup$:



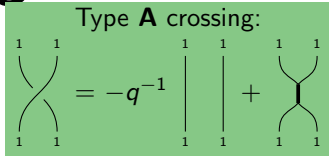
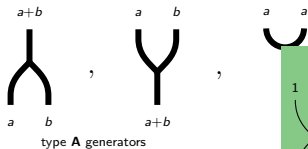
Relations are the type **A** relations and e.g.:



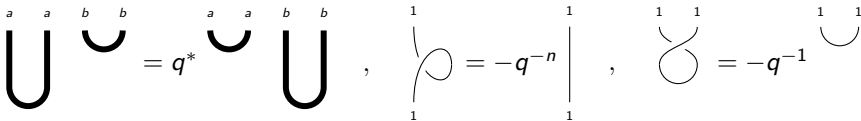
q -interchange law
 * =some power depending on a, b

" q -Reidemeister relations"

q -Monoidal generators of $\mathcal{Web}_{q, q^n}^\cup$:



Relations are the type **A** relations and e.g.:



q -interchange law
 * =some power depending on a, b

" q -Reidemeister relations"

Via restriction, we see that the $\mathbf{U}_q(\mathfrak{gl}_n)$ -intertwiners $\mathbf{R}_{a,b}^{a+b}$ and $\mathbf{Y}_{a+b}^{a,b}$ are $\mathbf{U}'_q(\mathfrak{so}_n)$ -equivariant as well.

Note that $V \otimes V$ contains a copy of the trivial $\mathbf{U}(\mathfrak{so}_n)$ -module. One shows that the same holds with q and one gets inclusions and projections

$$\cup : \mathbb{C}_q \rightarrow V_q \otimes V_q, \quad \cap : V_q \otimes V_q \rightarrow \mathbb{C}_q.$$

As before, use these to quantize Howe's duality.

[◀ Back](#)

	$\mathbf{U}_q(\mathfrak{so}_n)$	$\mathbf{U}'_q(\mathfrak{so}_n)$
Subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$	✗	✓
Hopfalgebra	✓	✗
Quantization of $\mathbf{U}(\mathfrak{so}_n)$	✓	✓
“Nice quantum numbers”	✗	✓
“Nice topology”	✓	✗

Connected to Peng's talk yesterday: $\theta = \omega$, the Chevalley involution

$$\omega(E_i) = -F_i, \quad \omega(F_i) = -E_i, \quad \omega(H_i) = -H_i.$$

$\mathbf{U}'_q(\mathfrak{so}_n)$ is a (left) coideal:

$$\Delta: \mathbf{U}'_q(\mathfrak{so}_n) \rightarrow \mathbf{U}_q(\mathfrak{gl}_n) \otimes \mathbf{U}'_q(\mathfrak{so}_n).$$

Hence, $\mathcal{R}\text{ep}'_q(\mathfrak{so}_n)$ is only q -monoidal and carries a left action of $\mathcal{R}\text{ep}_q(\mathfrak{gl}_n)$.

	$\mathbf{U}_q(\mathfrak{so}_n)$	$\mathbf{U}'_q(\mathfrak{so}_n)$
Subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$	✗	✓
Hopfalgebra	✓	✗
Quantization of $\mathbf{U}(\mathfrak{so}_n)$	✓	✓
“Nice quantum”	✗	✓
“Nice to”	✓	✗

Connected to Peng's talk yesterday

$\omega(E_i) = -F_i$ $\omega(H_i) = -H_i$

$\mathbf{U}'_q(\mathfrak{so}_n)$ is a (left) coideal:

$$\Delta: \mathbf{U}'_q(\mathfrak{so}_n) \rightarrow \mathbf{U}_q(\mathfrak{gl}_n) \otimes \mathbf{U}'_q(\mathfrak{so}_n).$$

Hence, $\mathcal{R}\text{ep}'_q(\mathfrak{so}_n)$ is only q -monoidal and carries a left action of $\mathcal{R}\text{ep}_q(\mathfrak{gl}_n)$.