Link invariants and $\mathbb{Z}/2\mathbb{Z}$-orbifolds

Or: What makes types ABCD special?

Daniel Tubbenhauer

Joint work in progress (take it with a grain of salt) with Catharina Stroppel and Arik Wilbert (Based on an idea of Mikhail Khovanov)

January 2018
Khovanov style homologies

"Homology easy, topology hard"

"Homology hard, topology easy"

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Khovanov style homologies

Hecke algebras

Lie theory

More...

Commutative algebra

Quantum groups

(Singular) TQFTs

Physics

Geometry

"Homology easy, topology hard"

"Homology hard, topology easy"
A quantum group of type $E_7$ is type $A$-braided!?
Outside of type A

The type A world

- Lie theory
- More...
- Commutative algebra
- Hecke algebras
- Khovanov style homologies
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- (Singular) TQFTs
- Physics
- Geometry

Weyl group side <-> Quantum group side

A quantum group of type $E_7$ is type A-braided!?
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Homologies!! for links??

Weyl group side

Weyl group ♥ Quantum group

Quantum group side

Homologies?? for links!!
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- Commutative algebra
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Homologies!! for links??

Weyl group side

Weyl group $\heartsuit$ Quantum group

Quantum group side

"Homology easy, topology hard"

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A quantum group of type $E_7$ is type $A$-braided!?
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Weyl group side

Weyl group \heartsuit Quantum group

Quantum group side

Yes

Yes

Yes

Homologies!! for links??

Homologies?? for links!!

"Homology easy, topology hard"

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Outside of type A

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Weyl group \xor Quantum group

Quantum group side

Homologies! for links??

Hecke algebras

Yes

"Homology easy, topology hard"

Outside of type A

"Homology hard, topology easy"

"Homology easy, topology hard"

"Homology hard, topology easy"

Yes

Yes

Yes

"Homology easy, topology hard"

"Homology hard, topology easy"

Outside of type A
1. **Tangle diagrams of $\mathbb{Z}/2\mathbb{Z}$-orbifold tangles**
   - Diagrams
   - Tangles in $\mathbb{Z}/2\mathbb{Z}$-orbifolds

2. **Topology of Artin braid groups**
   - The Artin braid groups: algebra
   - Hyperplanes vs. configuration spaces

3. **Invariants**
   - Reshetikhin–Turaev-like theory for some coideals
   - Polynomials and homologies for $\mathbb{Z}/2\mathbb{Z}$-orbifold tangles
Tangle diagrams with cone strands

Let $c\mathcal{T}an$ be the monoidal category defined as follows.

**Generators.** Object generators $\{+,-,c\}$, morphism generators

![Diagrams of usual crossings, usual cups and caps, and cone crossings.]

**Relations.** Reidemeister type relations, and the $\mathbb{Z}/2\mathbb{Z}$-relations:

![Diagrams of relation equations.]

Daniel Tubbenhauer

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January 2018
Tangle diagrams with cone strands

Let $cTan$ be the monoidal category defined as follows.

**Generators.** Object generators $\{+, -, c\}$, morphism generators $\{+, -, c\}$ usual crossings, $\{+, -, c\}$ usual cups and caps, $\{+, -, c\}$ cone crossings

**Relations.** Reidemeister type relations, and the $\mathbb{Z}/2\mathbb{Z}$-relations: $= \equiv$ and $= \equiv$

**Examples.**

Unknot

Essential unknot

Hopf link

Essential Hopf link

Exercise. The relations are actually equivalent.
Let $cTan$ be the monoidal category defined as follows.

**Generators.** Object generators

\[ {\{+, -, c\}} \]

Morphism generators

\[ \text{usual crossings}, \quad \text{usual cups and caps}, \quad \text{cone crossings} \]

**Relations.** 

Reidemeister type relations, and the $\mathbb{Z}/2\mathbb{Z}$-relations:

\[ \begin{align*}
\Uparrow & = \Uparrow \\
\Downarrow & = \Downarrow 
\end{align*} \]
Let $cTan$ be the monoidal category defined as follows.

**Generators.** Object generators \{+, −, $c$\}, morphism generators usual crossings, usual cups and caps, cone crossings.

**Relations.** Reidemeister type relations, and the $\mathbb{Z}/2\mathbb{Z}$-relations:
Let $c\mathcal{T}an$ be the monoidal category defined as follows.

**Generators.** Object generators $\{+,-,c\}$, morphism generators $\{+,-,\text{usual crossings}, \text{usual cups and caps}, \text{cone crossings}\}$

**Relations.** Reidemeister type relations, and the $\mathbb{Z}/2\mathbb{Z}$-relations:

\[\begin{align*}
\xymatrix{+ & + & + & + \\
\ar@{.}@/^/[rr] & \ar@{.}@/^/[rr] & \ar@{.}@/^/[rr] & \ar@{.}@/^/[rr]}
\end{align*}\]

\[\begin{align*}
\xymatrix{+ & + & + & + \\
\ar@{.}@/^/[rr] & \ar@{.}@/^/[rr] & \ar@{.}@/^/[rr] & \ar@{.}@/^/[rr]}
\end{align*}\]
Tangle diagrams with cone strands

Let $cTan$ be the monoidal category defined as follows.

**Generators.** Object generators

$$\{ +, -, c \}$$

Morphism generators

$$+ + + + , - + + + , \text{usual crossings}$$

$$+ + + + , - + + + , \text{usual cups and caps}$$

$$c + c + c + c , c + c + c + c , c + c + c + c , c + c + c + c , \text{cone crossings}$$

**Relations.** Reidemeister type relations, and the $\mathbb{Z}/2\mathbb{Z}$-relations:

$$= \quad \text{and} \quad =$$
Let $cT an$ be the monoidal category defined as follows.

**Generators.** Object generators

- $+, -, c$

Morphism generators

- usual crossings
- usual cups and caps
- cone crossings

**Relations.** Reidemeister type relations, and the $\mathbb{Z}/2\mathbb{Z}$-relations:

$$\begin{align*}
\begin{tikzpicture}[baseline=(current  bounding  box.center)]
  \draw (-1,0) to [out=0,in=180] (1,0);
  \draw (-1,-1) to [out=0,in=180] (1,-1);
\end{tikzpicture}
\end{align*}
\quad =
\begin{align*}
\begin{tikzpicture}[baseline=(current  bounding  box.center)]
  \draw (-1,0) to [out=0,in=180] (1,0);
  \draw (-1,-1) to [out=0,in=180] (1,-1);
\end{tikzpicture}
\end{align*}
\quad \text{and} \quad
\begin{align*}
\begin{tikzpicture}[baseline=(current  bounding  box.center)]
  \draw (-1,0) to [out=0,in=180] (1,0);
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  \draw (-1,-1) to [out=0,in=180] (1,-1);
\end{tikzpicture}
\end{align*}$$
Tangle diagrams with cone strands

Let $cT an$ be the monoidal category defined as follows.

Generators. Object generators $\{+, -, c\}$, morphism generators $\{+\}$, $\{+\}$, $\{-\}$, $\{\}$, usual crossings $\{\}$, $\{\}$, $\{\}$, $\{\}$, usual cups and caps $\{\}$, $\{\}$, $\{\}$, $\{\}$, cone crossings $\{\}$, $\{\}$, $\{\}$, $\{\}$.

Relations. Reidemeister type relations $\{\}$, and the $\mathbb{Z}/2\mathbb{Z}$-relations:

$\{\} = \{\}$ and $\{\} = \{\}$.
Let $cTan$ be the monoidal category defined as follows.

**Generators.** Object generators

$$+$, $-$, $c$$

Morphism generators

usual crossings

usual cups and caps

cone crossings

**Relations.** Reidemeister type relations, and the $\mathbb{Z}/2\mathbb{Z}$-relations:

$$= \quad \text{and} \quad =$$
Tangle diagrams with cone strands

Let $cTan$ be the monoidal category defined as follows.

**Generators.** Object generators\{+\}, \{-\}, \{c\}\}, morphism generators\{+, \(+, \(-, \(+\), \(-, \)-\}, \{+, \(+, \(-, \)+\}, \{-, \}-\}, \{+, \.plus\}, \{-, \}-\}, \{+, \(c\), \(c\), \(c\), \(c\))\}, \{c, \(-, \)+\}, usual crossings, cone crossings.

**Relations.** Reidemeister type relations, and the $\mathbb{Z}/2\mathbb{Z}$-relations:

```
\begin{align*}
\text{Example.} & \quad \text{Exercise. The relations are actually equivalent.}
\end{align*}
```

**Exercise.** The relations are actually equivalent.

**Relations.** Reidemeister type relations, and the $\mathbb{Z}/2\mathbb{Z}$-relations:

\begin{align*}
\text{Example.}
\end{align*}
“Definition”. An orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

Main example. $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}^2$ by rotation by $\pi$ around a fixed point $c$:

$$c_{\text{1 orb}} = \mathbb{R}^2/\mathbb{Z}/2\mathbb{Z} \overset{\text{$\mathbb{Z}/2\mathbb{Z}$ action}}{\sim} X_{c_{\text{1 orb}}} \approx \mathbb{R}^2/\mathbb{Z}/2\mathbb{Z}$$

Philosophy. $c$ is half-way in between a regular point and a puncture:
“Definition”. An orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

Main example. $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}^2$ by rotation by $\pi$ around a fixed point $c$:

$$c_{10rb} = \mathbb{R}^2/\mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi} X_{c_{10rb}} \cong$$

Philosophy. $c$ is half-way in between a regular point and a puncture:
“Definition”. An orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

Main example. $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}^2$ by rotation by $\pi$ around a fixed point $c$:

$$c_{10rb} = \mathbb{R}^2/\mathbb{Z}/2\mathbb{Z} \rightsquigarrow X_{c_{10rb}} \approx$$

Philosophy. $c$ is half-way in between a regular point and a puncture:
An orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

$\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}^2$ by rotation by $\pi$ around a fixed point $c$:

$$c_{1Orb} = \mathbb{R}^2/\mathbb{Z}/2\mathbb{Z} \xrightarrow{\mathbb{Z}/2\mathbb{Z} \text{ action}} X_{c_{1Orb}} \approx c$$

$c$ is half-way in between a regular point and a puncture:
“Definition”. An orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

Main example. \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathbb{R}^2 \) by rotation by \( \pi \) around a fixed point \( c \):

\[
\text{c Orb} = \mathbb{R}^2 / \mathbb{Z}/2\mathbb{Z} \quad \sim \quad X_{\text{c Orb}} \approx \text{cone point}
\]

Philosophy. \( c \) is half-way in between a regular point and a puncture:
**Definition**. An orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

**Main example.** \(\mathbb{Z}/2\mathbb{Z}\) acts on \(\mathbb{R}^2\) by rotation by \(\pi\) around a fixed point \(c\):

\[
\text{c}_1\text{0}rb = \mathbb{R}^2/\mathbb{Z}/2\mathbb{Z} \quad \xrightarrow{\mathbb{Z}/2\mathbb{Z}\text{ action}} \quad X_{\text{c}_1\text{0}rb} \approx \mathbb{R}^2/\mathbb{Z} = -z
\]

**Philosophy.** \(c\) is half-way in between a regular point and a puncture:
\(\mathbb{Z}/2\mathbb{Z}\)-orbifolds

“Definition”. An orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

Main example. \(\mathbb{Z}/2\mathbb{Z}\) acts on \(\mathbb{R}^2\) by rotation by \(\pi\) around a fixed point \(c\):

\[
c_1\text{Orb} = \mathbb{R}^2/\mathbb{Z}/2\mathbb{Z} \quad \overset{\mathbb{Z}/2\mathbb{Z} \text{ action}}{\longrightarrow} \quad X_{c_1\text{Orb}} \approx \c
\]

Philosophy. \(c\) is half-way in between a regular point and a puncture:
"Definition". An orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

Main example. $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}^2$ by rotation by $\pi$ around a fixed point $c$:

$$c_{10rb} = \mathbb{R}^2/\mathbb{Z}/2\mathbb{Z} \simeq X_{c_{10rb}}$$

Philosophy. $c$ is half-way in between a regular point and a puncture:
An orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

Main example. \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathbb{R}^2 \) by rotation by \( \pi \) around a fixed point \( c \):

\[
\mathbb{C}_1 orb = \mathbb{R}^2 / \mathbb{Z}/2\mathbb{Z} \quad \Leftrightarrow \quad \mathbb{X}_c \approx \quad \text{cone point}
\]

Philosophy. \( c \) is half-way in between a regular point and a puncture:

- Regular point: \( \pi^0 orb = 1 \)
- Cone point: \( \pi^0 orb = \mathbb{Z} \)
- Puncture: never trivial
**$\mathbb{Z}/2\mathbb{Z}$-orbifolds**

“Definition”. An orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

**Main example.** $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}^2$ by rotation by $\pi$ around a fixed point $c$:

$$\mathbb{C}_{Orb} = \mathbb{R}^2/\mathbb{Z}/2\mathbb{Z} \sim \mathbf{X}_{C_{Orb}} \approx \mathbb{R}^2/_{z=-z}$$

**Philosophy.** $c$ is half-way in between a regular point and a puncture:
"Definition". An orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

Main example. \(\mathbb{Z}/2\mathbb{Z}\) acts on \(\mathbb{R}^2\) by rotation by \(\pi\) around a fixed point \(c\):

\[
c_{10rb} = \mathbb{R}^2/\mathbb{Z}/2\mathbb{Z}
\]

Philosophy. \(c\) is half-way in between a regular point and a puncture:
"Definition". An orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

Main example. \( \mathbb{Z}_2 \) acts on \( \mathbb{R}^2 \) by rotation by \( \pi \) around a fixed point \( c \):

\[
\mathcal{C}_{10rb} = \mathbb{R}^2 / \mathbb{Z}_2
\]

\( \mathbb{Z}_2 \) action \( \mathbb{R}^2 \mathcal{C}_{10rb} \approx \) cone point

Philosophy. \( c \) is half-way in between a regular point and a puncture:

\[
\pi^0_{1rb} = 1
\]

\[
\pi^0_{1rb} = \mathbb{Z}_2
\]

\[
\pi^0_{1rb} = \mathbb{Z}
\]
**$\mathbb{Z}/2\mathbb{Z}$-orbifolds**

“Definition”. An orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

**Main example.** $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}^2$ by rotation by $\pi$ around a fixed point $c$: $c_{1\text{orb}} = \mathbb{R}^2/\mathbb{Z}/2\mathbb{Z} \rightarrow X_c$.

**Philosophy.** $c$ is half-way in between a regular point and a puncture:

- Regular point: $\pi^0_{\text{orb}} = 1$
- Cone point: $\pi^0_{\text{orb}} = \mathbb{Z}/2\mathbb{Z}$
- Puncture: $\pi^0_{\text{orb}} = \mathbb{Z}$
Pioneers of algebra

Let $\Gamma$ be a Coxeter graph.

Artin $\sim 1925$, Tits $\sim 1961$. The Artin braid groups and its Coxeter group quotients are given by generators-relations:

$$\mathcal{A}_\Gamma = \langle b_i \mid \cdots b_i b_j b_i = \cdots b_j b_i b_j \rangle$$

$$W_\Gamma = \langle s_i \mid s_i^2 = 1, \cdots s_i s_j s_i = \cdots s_j s_i s_j \rangle$$

Artin braid groups generalize classical braid groups, Coxeter groups Weyl groups.

We want to understand these better.
Let Γ be a Coxeter graph.

**Artin ∼1925, Tits ∼1961**. The Artin braid groups and its Coxeter group quotients are given by generators-relations:

\[
\mathcal{A}_Γ = \langle b_i \mid \cdots b_j b_i b_j = \cdots b_j b_i b_j \rangle
\]

\[
\mathcal{W}_Γ = \langle s_i \mid s_i^2 = 1, \cdots s_j s_i s_j = \cdots s_j s_i s_j \rangle
\]

Artin braid groups generalize classical braid groups, Coxeter groups Weyl groups.

We want to understand these better.
$\mathcal{W}_{A_2} = \langle s, t \rangle$ acts faithfully on $\mathbb{R}^2$ by reflecting in hyperplanes (for each reflection):

$\mathcal{W}_{A_2}$ acts freely on $M_{A_2} = \mathbb{R}^2 \setminus$ hyperplanes. Set $N_{A_2} = M_{A_2} / \mathcal{W}_{A_2}$. 

Coxeter $\sim$ 1934, Tits $\sim$ 1961.

This works in ridiculous generality. (Up to some minor technicalities in the infinite case.)

Brieskorn $\sim$ 1971, van der Lek $\sim$ 1983.

This works in ridiculous generality. (Up to some minor technicalities in the infinite case.)
I follow hyperplanes

\( \mathcal{W}_{A_2} = \langle s, t \rangle \) acts faithfully on \( \mathbb{R}^2 \) by reflecting in hyperplanes (for each reflection):

\[ t \]

\( \mathcal{W}_{A_2} \) acts freely on \( M_{A_2} = \mathbb{R}^2 \setminus \text{hyperplanes} \). Set \( N_{A_2} = M_{A_2} / \mathcal{W}_{A_2} \).
\[ W_{A_2} = \langle s, t \rangle \] acts faithfully on \( \mathbb{R}^2 \) by reflecting in hyperplanes (for each reflection):

\[ W_{A_2} \text{ acts freely on } M_{A_2} = \mathbb{R}^2 \setminus \text{hyperplanes. Set } N_{A_2} = M_{A_2} / W_{A_2}. \]
I follow hyperplanes

\( \mathcal{W}_{A_2} = \langle s, t \rangle \) acts faithfully on \( \mathbb{R}^2 \) by reflecting in hyperplanes (for each reflection):

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I follow hyperplanes

\[ W_{A_2} = \langle s, t \rangle \] acts faithfully on \( \mathbb{R}^2 \) by reflecting in hyperplanes (for each reflection):

\[
\begin{align*}
\text{Coxeter } &\sim 1934, \quad \text{Tits } \sim 1961. \\
&\text{This works in ridiculous generality.} \\
&\text{(Up to some minor technicalities in the infinite case.)}
\end{align*}
\]

\[ W_{A_2} \] acts freely on \( M_{A_2} = \mathbb{R}^2 \setminus \text{hyperplanes} \). Set \( N_{A_2} = M_{A_2} / W_{A_2} \).
I follow hyperplanes

$W_{A_2} = \langle s, t \rangle$ acts faithfully on $\mathbb{R}^2$ by reflecting in hyperplanes (for each reflection):

$W_{A_2}$ acts freely on $M_{A_2} = \mathbb{R}^2 \setminus \text{hyperplanes}$. Set $N_{A_2} = M_{A_2} / W_{A_2}$.

Complexifying the action: $\mathbb{R}^2 \rightsquigarrow \mathbb{C}^2$, $M_{A_2} \rightsquigarrow M_{A_2}^\mathbb{C}$, $N_{A_2} \rightsquigarrow N_{A_2}^\mathbb{C}$. Then:

$$\pi_1(N_{A_2}^\mathbb{C}) \cong \mathcal{A}r_{A_2} = \langle b_s, b_t \mid b_s b_t b_s = b_t b_s b_t \rangle$$
\( \mathcal{W}_{A_2} = \langle s, t \rangle \) acts faithfully on \( \mathbb{R}^2 \) by reflecting in hyperplanes (for each reflection):

\( \mathcal{W}_{A_2} \) acts freely on \( M_{A_2} = \mathbb{R}^2 \setminus \) hyperplanes. Set \( N_{A_2} = M_{A_2} / \mathcal{W}_{A_2} \).

Complexifying the action: \( \mathbb{R}^2 \leadsto \mathbb{C}^2, M_{A_2} \leadsto M_{A_2}^\mathbb{C}, N_{A_2} \leadsto N_{A_2}^\mathbb{C} \). Then:

\[
\pi_1(N_{A_2}^\mathbb{C}) \cong \mathcal{A}r_{A_2} = \langle b_s, b_t \mid b_s b_t b_s = b_t b_s b_t \rangle
\]
\( \mathcal{W}_{A_2} = \langle s, t \rangle \) acts faithfully on \( \mathbb{R}^2 \) by reflecting in hyperplanes (for each reflection):

\[ \mathcal{W}_{A_2} \text{ acts freely on } M_{A_2} = \mathbb{R}^2 \setminus \text{hyperplanes}. \text{ Set } N_{A_2} = M_{A_2} / \mathcal{W}_{A_2}. \]

Complexifying the action: \( \mathbb{R}^2 \leadsto \mathbb{C}^2, M_{A_2} \leadsto M_{A_2}^\mathbb{C}, N_{A_2} \leadsto N_{A_2}^\mathbb{C} \). Then:

\[
\pi_1(N_{A_2}^\mathbb{C}) \cong \mathcal{A}r_{A_2} = \langle b_s, b_t \mid b_s b_t b_s = b_t b_s b_t \rangle
\]
\( \mathcal{W}_{A_2} = \langle s, t \rangle \) acts faithfully on \( \mathbb{R}^2 \) by reflecting in hyperplanes (for each reflection):

\[ \begin{array}{ccc}
    \mathcal{W}_{A_2} \text{ acts freely on } M_{A_2} = \mathbb{R}^2 \setminus \text{hyperplanes. Set } N_{A_2} = M_{A_2} / \mathcal{W}_{A_2}. \\
\end{array} \]

Complexifying the action: \( \mathbb{R}^2 \leadsto \mathbb{C}^2, M_{A_2} \leadsto M_{A_2}^C, N_{A_2} \leadsto N_{A_2}^C. \) Then:

\[ \pi_1(N_{A_2}^C) \cong \mathcal{A}_{r_{A_2}} = \langle b_s, b_t \mid b_s b_t b_s = b_t b_s b_t \rangle \]

\[ \text{Coxeter } \sim 1934, \text{ Tits } \sim 1961. \text{ This works in ridiculous generality.} \]

\[ \text{Brieskorn } \sim 1971, \text{ van der Lek } \sim 1983. \text{ This works in ridiculous generality.} \]

(Up to some minor technicalities in the infinite case.)
I follow hyperplanes

\( \mathcal{W}_{A_2} = \langle s, t \rangle \) acts faithfully on \( \mathbb{R}^2 \) by reflecting in hyperplanes (for each reflection):

\[ \mathcal{B}_t \]

\textbf{Brieskorn} \( \sim 1971 \), \textbf{van der Lek} \( \sim 1983 \). This works in ridiculous generality.

(Up to some minor technicalities in the infinite case.)

\( \mathcal{W}_{A_2} \) acts freely on \( M_{A_2} = \mathbb{R}^2 \setminus \text{hyperplanes} \). Set \( N_{A_2} = M_{A_2} / \mathcal{W}_{A_2} \).

Complexifying the action: \( \mathbb{R}^2 \hookrightarrow \mathbb{C}^2 \), \( M_{A_2} \hookrightarrow M_{A_2}^\mathbb{C} \), \( N_{A_2} \hookrightarrow N_{A_2}^\mathbb{C} \). Then:

\[ \pi_1(N_{A_2}^\mathbb{C}) \cong \mathcal{A}_r_{A_2} = \langle b_s, b_t \mid b_s b_t b_s = b_t b_s b_t \rangle \]
Configuration spaces

Artin \(\sim 1925\). There is a topological model of \(\mathcal{A}r_A\) via configuration spaces.

Example. Take \(Conf_{A_2} = (\mathbb{R}^2)^3 \setminus \text{fat diagonal}/S_3\). Then \(\pi_1(Conf_{A_2}) \cong \mathcal{A}r_{A_2}\).

Philosophy. Having a configuration spaces is the same as having braid diagrams:

![Diagram](image)

Crucial. Note that – by explicitly calculating the equations defining the hyperplanes – one can directly check that:

“Hyperplane picture equals configuration space picture.”
**Configuration spaces**

**Artin ~1925.** There is a topological model of $Ar_A$ via configuration spaces.

**Example.** Take $Conf_{\mathbb{A}^2} = (\mathbb{R}^2)^3 \setminus \text{fat diagonal}/\text{fat diagonal}$. Then $\pi_1(Conf_{\mathbb{A}^2}) \cong Ar_{\mathbb{A}^2}$.

**Philosophy.** Having a configuration spaces is the same as having braid diagrams:

```
\begin{align*}
\sigma &= (13) \\
\text{a usual braid}
\end{align*}
```

**Critical.** Note that – by explicitly calculating the equations defining the hyperplanes – one can directly check that:

“Hyperplane picture equals configuration space picture.”
Configuration spaces

**Artin ∼1925.** There is a topological model of $\mathcal{A}r_A$ via configuration spaces.

**Example.** Take $\text{Conf}_A^2 = (\mathbb{R}^2)^3 \setminus \text{fat diagonal} / S_3$. Then $\pi_1(\text{Conf}_A^2) \cong A_r A^2$.

**Philosophy.** Having a configuration spaces is the same as having braid diagrams:

But we can’t compute the hyperplanes...

**Crucial.** Note that – by explicitly calculating the equations defining the hyperplanes – one can directly check that:

“Hyperplane picture equals configuration space picture.”
Configuration spaces

Artin $\sim 1925$. There is a topological model of $Ar_A$ via configuration spaces.

Example. Take $Conf_{A^2} = (\mathbb{R}^2)^3 \setminus \text{fat diagonal} / S_3$. Then $\pi_1(Conf_{A^2}) \sim A_{A^2}$.

Philosophy. Having a configuration spaces is the same as having braid diagrams:

In words: The $\mathbb{Z}/2\mathbb{Z}$-orbifolds provide the framework to study Artin braid groups of classical (affine) type - one can directly compute the hyperplanes...

Crucial. "Hyperplane picture equals configuration space picture."

In those cases one can compute the hyperplanes! This is very special for (affine) types $ABCD$.

Hope. The same works for Coxeter diagrams which are “locally type ABCD”, e.g.:

In words: The $\mathbb{Z}/2\mathbb{Z}$-orbifolds provide the framework to study Artin braid groups of classical (affine) type - one can directly compute the hyperplanes...

"Hyperplane picture equals configuration space picture."

Link invariants and $\mathbb{Z}/2\mathbb{Z}$-orbifolds
Configuration spaces

**Artin ~1925.** There is a topological model of $\mathcal{A}r_A$ via configuration spaces.

**Example.** Take $\text{Conf}_{A_2} = (\mathbb{R}^2)^3 \setminus \text{fat diagonal}/S_3$. Then $\pi_1(\text{Conf}_{A_2}) \cong \mathcal{A}r_{A_2}$.

**Philosophy.** Having a configuration spaces is the same as having braid diagrams:

![Example.](image)

**Crucial.** Note that – by explicitly calculating the equations defining the hyperplanes – one can directly check that:

“Hyperplane picture equals configuration space picture.”
Configuration spaces

Artin \(\sim 1925\). There is a topological model of \(Ar_A\) via configuration spaces.

Example. Take \(Conf_{\mathbb{A}^2} = (\mathbb{R}^2)^3 \setminus \text{fat diagonal}/S_3\). Then \(\pi_1(Conf_{\mathbb{A}^2}) \cong Ar_{\mathbb{A}^2}\).

Philosophy. Having a configuration spaces is the same as having braid diagrams:

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Configuration spaces

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Philosophy. Having a configuration spaces is the same as having braid diagrams:

\[ b_i b'_i = b'_i b_i, \text{ if } b_i \]

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Configuration spaces

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Daniel Tubbenhauer  
Link invariants and \(\mathbb{Z}/2\mathbb{Z}\)-orbifolds  
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Configuration spaces

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Example. Take \(\text{Conf}_{A_2} = (\mathbb{R}^2)^3 \setminus \text{fat diagonal}/S_3\). Then \(\pi_1(\text{Conf}_{A_2}) \cong A_r\).

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Example.

\[
\begin{align*}
&\bullet \ b'_i \quad \uparrow \quad \uparrow \quad \uparrow \\
&\bullet \ b_i
\end{align*}
\]

\(b_i b'_i = b'_i b_i\), if

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Reshetikhin–Turaev theory half-way in between

Reshetikhin–Turaev \(\sim 1991\). Construct link and tangle invariants as functors

\[ uRT : uTan \to \text{well-behaved target category}. \]

Today: Target categories = \( \mathcal{R}ep(U_v(sl_2)) \) and friends.

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Question. What could the \( \mathbb{Z}/2\mathbb{Z} \)-analog be?

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Daniel Tubbenhauer

Link invariants and \( \mathbb{Z}/2\mathbb{Z} \)-orbifolds

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Reshetikhin–Turaev \(\sim 1991\). Construct link and tangle invariants as functors

\[ uRT : uTan \to \text{well-behaved target category}. \]

Today: Target categories = \( \text{Rep}(\mathcal{U}_v(\mathfrak{sl}_2)) \) and friends.

**Question.** What could the \( \mathbb{Z}/2\mathbb{Z} \)-analog be?

\[ \text{C}(v) = \text{ground field}, \quad \text{V}_v = \text{vector representation of} \quad \mathcal{U}_v(\mathfrak{sl}_2). \]

\[ \text{?? : V}_v \to V_v \text{should be non-trivial.} \]

But \( V_v \) is irreducible for \( \mathcal{U}_v \)...

Same issue...

Orbifold-philosophy. We need something half-way in between \( \text{C}(v) \) and \( \mathcal{U}_v \).
Reshetikhin–Turaev theory half-way in between

Reshetikhin–Turaev $\sim 1991$. Construct link and tangle invariants as functors

$$u\mathcal{RT} : u\mathcal{T}an \rightarrow \text{well-behaved target category}.$$ 

Today: Target categories $= \mathcal{R}ep(\mathcal{U}_v(sl_2))$ and friends.

Question. What could the $\mathbb{Z}/2\mathbb{Z}$-analog be?

$\mathbb{C}(v)$

$\mathbb{C}(v)$
Reshetikhin–Turaev theory half-way in between

Reshetikhin–Turaev \(\sim 1991\). Construct link and tangle invariants as functors

\[
u\mathcal{RT} : u\mathcal{T}an \rightarrow \text{well-behaved target category.}
\]

Today: Target categories = \(\text{Rep}(u_v(\mathfrak{sl}_2))\) and friends.

---

**Question.** What could the \(\mathbb{Z}/2\mathbb{Z}\)-analog be?

![Diagram](image)

\(\mathbb{C}(v)\) = ground field, 
\(V_v = \text{vector representation of } u_v = u_v(\mathfrak{sl}_2)\).

\(V_v \otimes V_v \overset{\text{ev}^*}{\rightarrow} \mathbb{C}(v)\)
Reshetikhin–Turaev theory half-way in between

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\[ C(v) = \text{ground field}, \quad V_v = \text{vector representation of } \mathcal{U}_v(\mathfrak{sl}_2). \]

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But $V_v$ is irreducible for $\mathcal{U}_v$...?

Same issue...

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Daniel Tubbenhauer

Link invariants and $\mathbb{Z}/2\mathbb{Z}$-orbifolds
Reshetikhin–Turaev theory half-way in between

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Today: Target categories = \( \mathcal{R}ep(u_v(\mathfrak{sl}_2)) \) and friends.

**Question.** What could the \( \mathbb{Z}/2\mathbb{Z} \)-analog be?

\[ \mathbb{C}(v) \]

\[ \mathbb{C}(v) \]

??

??: \( V_v \to V_v \) should be non-trivial.

But \( V_v \) is irreducible for \( u_v \)...

\[ V_v \otimes V_v \otimes V_v \otimes V_v \]

\[ V_v \otimes V_v \otimes V_v \otimes V_v \]

\[ \uparrow \text{id} \otimes \text{id} \otimes \text{id} \]

\[ \text{id} \otimes \text{id} \otimes \text{ev}^* \]

\[ \text{ev}^* \]

\[ \mathbb{C}(v) \]
Reshetikhin–Turaev theory half-way in between

Reshetikhin–Turaev $\sim 1991$. Construct link and tangle invariants as functors

$$u\mathcal{RT} : u\mathcal{T}an \rightarrow \text{well-behaved target category}.$$ Today: Target categories $= \mathcal{R}ep(\mathcal{U}_v(sl_2))$ and friends.

Question. What could the $\mathbb{Z}/2\mathbb{Z}$-analog be?

$$\mathbb{C}(v)$$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$$\uparrow \text{id} \otimes \mathcal{R} \otimes \text{id}$$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$$\uparrow ?? \otimes \text{id} \otimes \text{id} \otimes \text{id}$$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$$\uparrow \text{id} \otimes \text{id} \otimes \text{ev}^*$$

$$V_v \otimes V_v$$

$$\uparrow \text{ev}^*$$

$$\mathbb{C}(v)$$
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Today: Target categories $= \mathcal{R} ep(\mathcal{U}_v(\mathfrak{sl}_2))$ and friends.

---

**Question.** What could the $\mathbb{Z}/2\mathbb{Z}$-analog be?

---

$\mathbb{C}(v)$

$\mathbb{V}_v \otimes \mathbb{V}_v \otimes \mathbb{V}_v \otimes \mathbb{V}_v$

$\uparrow \text{id} \otimes \text{R} \otimes \text{id}$

$\mathbb{V}_v \otimes \mathbb{V}_v \otimes \mathbb{V}_v \otimes \mathbb{V}_v$

$\uparrow \text{id} \otimes \text{R} \otimes \text{id}$

$\mathbb{V}_v \otimes \mathbb{V}_v \otimes \mathbb{V}_v \otimes \mathbb{V}_v$

$\uparrow ?? \otimes \text{id} \otimes \text{id} \otimes \text{id}$

$\mathbb{V}_v \otimes \mathbb{V}_v \otimes \mathbb{V}_v \otimes \mathbb{V}_v$

$\uparrow \text{id} \otimes \text{id} \otimes \text{ev}^*$

$\mathbb{V}_v \otimes \mathbb{V}_v$

$\uparrow \text{ev}^*$

$\mathbb{C}(v)$

??
Reshetikhin–Turaev theory half-way in between

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Today: Target categories = $\mathcal{R}ep(\mathcal{U}_v(sl_2))$ and friends.

---

Question. What could the $\mathbb{Z}/2\mathbb{Z}$-analog be?

$$\mathbb{C}(v)$$

Same issue...

$$\mathbb{C}(v)$$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$$\uparrow \quad ?? \otimes \text{id} \otimes \text{id} \otimes \text{id}$$

$$V_v \otimes V_v \otimes V_v$$

$$\uparrow \quad \text{id} \otimes R \otimes \text{id}$$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$$\uparrow \quad \text{id} \otimes R \otimes \text{id}$$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$$\uparrow \quad ?? \otimes \text{id} \otimes \text{id} \otimes \text{id}$$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$$\uparrow \quad \text{id} \otimes \text{id} \otimes \text{ev}^*$$

$$V_v \otimes V_v$$

$$\uparrow \quad \text{ev}^*$$
**Reshetikhin–Turaev theory half-way in between**

**Reshetikhin–Turaev $\sim 1991$.** Construct link and tangle invariants as functors $u\mathcal{R}\mathcal{T} : u\mathcal{T}an \to$ well-behaved target category.

Today: Target categories $= \mathcal{R}ep(\mathcal{U}_v(sl_2))$ and friends.

---

**Question.** What could the $\mathbb{Z}/2\mathbb{Z}$-analog be?

$$C(v)$$

$$V_v \otimes V_v$$

$\uparrow \text{id} \otimes \text{id} \otimes \text{ev}$

$$V_v \otimes V_v \otimes V_v$$

$\uparrow \text{id} \otimes \text{id} \otimes \text{id}$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$\uparrow \text{id} \otimes \text{id} \otimes \text{id}$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$\uparrow \text{id} \otimes \text{id} \otimes \text{id}$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$\uparrow \text{id} \otimes \text{id} \otimes \text{id}$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$\uparrow \text{id} \otimes \text{id} \otimes \text{id}$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$\uparrow \text{id} \otimes \text{id} \otimes \text{id}$

$$\mathbb{C}(v)$$
Reshetikhin–Turaev theory half-way in between

Reshetikhin–Turaev $\sim 1991$. Construct link and tangle invariants as functors

$$u\mathcal{R}\mathcal{T} : u\mathcal{T}an \to \text{well-behaved target category}.$$ 

Today: Target categories $= \mathcal{R}ep(\mathcal{U}_v(sl_2))$ and friends.

**Question.** What could the $\mathbb{Z}/2\mathbb{Z}$-analog be?

\[ C(v) \quad \xrightarrow{ev} \quad V_v \otimes V_v \quad \xrightarrow{id \otimes id \otimes ev} \quad V_v \otimes V_v \otimes V_v \otimes V_v \quad \xrightarrow{id \otimes id \otimes id} \quad V_v \otimes V_v \otimes V_v \otimes V_v \quad \xrightarrow{id \otimes R \otimes id} \quad V_v \otimes V_v \otimes V_v \otimes V_v \quad \xrightarrow{id \otimes R \otimes id} \quad V_v \otimes V_v \otimes V_v \otimes V_v \quad \xrightarrow{?? \otimes id \otimes id \otimes id} \quad V_v \otimes V_v \otimes V_v \otimes V_v \quad \xrightarrow{id \otimes id \otimes ev^*} \quad V_v \otimes V_v \quad \xrightarrow{ev^*} \quad C(v) \]
Reshetikhin–Turaev theory half-way in between

Reshetikhin–Turaev $\sim 1991$. Construct link and tangle invariants as functors

$$uRT : u\mathcal{T}an \rightarrow \text{well-behaved target category}.$$ 

Today: Target categories $=$ $\mathcal{R}ep(\mathcal{U}_v(\mathfrak{sl}_2))$ and friends.

**Question.** What could the $\mathbb{Z}/2\mathbb{Z}$-analog be?

**Orbifold-philosophy.** We need something half-way in between $\mathbb{C}(v)$ and $\mathcal{U}_v$. 

$$\mathbb{C}(v)$$

$$\uparrow$$

$$\text{ev}$$

$$V_v \otimes V_v$$

$$\uparrow$$

$$\text{id} \otimes \text{id} \otimes \text{ev}$$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$$\uparrow$$

$$?? \otimes \text{id} \otimes \text{id} \otimes \text{id}$$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$$\uparrow$$

$$\text{id} \otimes R \otimes \text{id}$$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$$\uparrow$$

$$?? \otimes \text{id} \otimes \text{id} \otimes \text{id}$$

$$V_v \otimes V_v \otimes V_v \otimes V_v$$

$$\uparrow$$

$$\text{id} \otimes \text{id} \otimes \text{ev}^*$$

$$V_v \otimes V_v$$

$$\uparrow$$

$$\text{ev}^*$$

$$\mathbb{C}(v)$$
Half-way in between trivial $\subset \mathbb{U}_v \subset \mathbb{U}_v$ – part I

**Kulish–Reshetikhin ~1981.** $\mathbb{U}_v$ is the associative, unital $\mathbb{C}(v)$-algebra generated by $E, F, K^{\pm 1}$ subject to the usual relations.

$V_v:
\begin{align*}
E v_+ &= 0, & F v_+ &= v_-, & K v_+ &= v v_+, \\
E v_- &= v_+, & F v_- &= 0, & K v_- &= v^{-1} v_-.
\end{align*}$

Define $\mathbb{U}_v$-intertwiners:

$\mathcal{C} : \mathbb{C}(v) \to V_v \otimes V_v, \quad 1 \mapsto v_- \otimes v_+ - v^{-1} v_+ \otimes v_-,$

$\mathcal{C} : V_v \otimes V_v \to \mathbb{C}(v), \quad \begin{cases} v_+ \otimes v_+ \mapsto 0, & v_+ \otimes v_- \mapsto 1, \\
v_- \otimes v_+ \mapsto -v, & v_- \otimes v_- \mapsto 0,\end{cases}$

$\mathcal{Y} : V_v \otimes V_v \to V_v \otimes V_v, \quad \mathcal{Y} = v | | + v^2 \mathcal{C}.$

Not really important...
Half-way in between trivial $\subset ?? \subset \mathcal{U}_v$ – part I

Kulish–Reshetikhin $\sim 1981$. $\mathcal{U}_v$ is the associative, unital $\mathbb{C}(v)$-algebra generated by $E, F, K^{\pm 1}$ subject to the usual relations.

$V_v$: 
- $E_v^+ = 0$, $F_v^+ = v^-$, $K_v^+ = v v^+$,
- $E_v^- = v^+$, $F_v^- = 0$, $K_v^- = v^{-1} v^{-}$.

**Fact.** $\mathcal{U}_v$ is a Hopf algebra $\Rightarrow$ We can tensor representations.

Define $\mathcal{U}_v$-intertwiners:

- $\mathcal{O}: \mathbb{C}(v) \rightarrow V_v \otimes V_v$, $\quad 1 \mapsto v^- \otimes v^+ - v^{-1} v^+ \otimes v^-,$
- $\mathcal{C}: V_v \otimes V_v \rightarrow \mathbb{C}(v)$, 
  \[ \begin{cases} 
  v^+ \otimes v^+ & \mapsto 0, \\
  v^- \otimes v^+ & \mapsto -v, \\
  v^- \otimes v^- & \mapsto 0, \\
  v^+ \otimes v^- & \mapsto 1, 
  \end{cases} \]
- $\mathcal{Y}: V_v \otimes V_v \rightarrow V_v \otimes V_v$, $\quad \mathcal{Y} = v|l| + v^2 \mathcal{O}.$
Half-way in between trivial $\subset \mathcal{U}_v - $ part I

Kulish–Reshetikhin $\sim 1981$. $\mathcal{U}_v$ is the associative, unital $\mathbb{C}(v)$-algebra generated by $E, F, K^{\pm 1}$ subject to the usual relations.

\[ V_v: \quad E v_+ = 0, \quad F v_+ = v_, \quad K v_+ = v v_+; \]

\[ V_v: \quad E v_- = v_+, \quad F v_- = 0, \quad K v_- = v^{-1} v_. \]

Define $\mathcal{U}_v$-intertwiners:

\[ \bigcirc: \mathbb{C}(v) \to V_v \otimes V_v, \quad 1 \mapsto v_+ \otimes v_- - v^{-1} v_+ \otimes v_-; \]

\[ \bigcirc: V_v \otimes V_v \to \mathbb{C}(v), \quad \begin{cases} v_+ \otimes v_+ \mapsto 0, & v_+ \otimes v_- \mapsto 1, \\ v_- \otimes v_+ \mapsto -v, & v_- \otimes v_- \mapsto 0, \end{cases} \]

\[ \chi: V_v \otimes V_v \to V_v \otimes V_v, \quad \chi = v_+ \bigcirc + v^2 \bigcirc. \]
Half-way in between trivial $\subset \mathbb{R} \subset \mathcal{U}_v$ – part I

Kulish–Reshetikhin $\sim 1981$. $\mathcal{U}_v$ is the associative, unital $\mathbb{C}(v)$-algebra generated by $E, F, K^{\pm 1}$ subject to the usual relations.

$V_v :$

Define $\mathcal{U}_v$-intertwiners:

$\mathcal{C}(v) \rightarrow V_v \otimes V_v, \quad 1 \mapsto v \otimes v + v \otimes 1$.

Example. We can not see the cone strands.
Half-way in between trivial \( \subset \mathcal{U}_v \subset \mathcal{U}_v - \text{part I} \)

**Kulish–Reshetikhin \( \sim 1981 \).** \( \mathcal{U}_v \) is the associative, unital \( \mathbb{C}(v) \)-algebra generated by \( E, F, K^{\pm 1} \) subject to the usual relations.

\[
\begin{align*}
V_v: & \quad E_v + v = 0, \\
& \quad E_v - v = v + 1, \\
& \quad F_v + v = v - 1, \\
& \quad F_v - v = 0, \\
& \quad K_v + v = v v + 1, \\
& \quad K_v - v = v - 1 v - 1.
\end{align*}
\]

Define \( \mathcal{U}_v \)-intertwiners:

- \( \mathcal{U}_v \rightarrow V_v \otimes V_v, \quad 1 \mapsto v \}

Not really important...

**Example.** We can not see the cone strands.
Half-way in between trivial $\subset \mathcal{U}_v$ – part II

Let $c\mathcal{U}_v$ be the coideal subalgebra of $\mathcal{U}_v$ generated by $B = v^{-1}EK^{-1} + F$.

$$V_v: Bv_+ = v_-, \quad Bv_- = v_+.$$  

Define $c\mathcal{U}_v$-intertwiners:

$\dagger: V_v \to V_v$, \quad $v_+ \mapsto v_-, \quad v_- \mapsto v_+$,

$\Upsilon: \mathbb{C}(v) \to V_v \otimes V_v$, \quad $1 \mapsto v_+ \otimes v_+ - v^{-1}v_- \otimes v_-,$

$\Lambda: V_v \otimes V_v \to \mathbb{C}(v)$, \quad \begin{align*}
    v_+ \otimes v_+ & \mapsto -v, \quad v_+ \otimes v_- \mapsto 0, \\
    v_- \otimes v_+ & \mapsto 0, \quad v_- \otimes v_- \mapsto 1,
\end{align*}

$\Upsilon = \dagger = \Lambda$ and $\Upsilon = \mid = \Lambda$.

Aside. This drops out of a coideal version of Schur–Weyl duality.
Let $c U_v$ be the coideal subalgebra of $U_v$ generated by $B = v^{-1}EK^{-1} + F$.

\[ V_v : Bv_+ = v_-, \quad Bv_- = v_+ \]

Define $c U_v$-intertwiners:

- $\Upsilon : \mathbb{C}(v) \to V_v \otimes V_v$, $1 \mapsto v_+ \otimes v_+ - v^{-1}v_- \otimes v_-$,

- $\Lambda : V_v \otimes V_v \to \mathbb{C}(v)$, $\begin{cases} v_+ \otimes v_+ \mapsto -v, & v_+ \otimes v_- \mapsto 0, \\ v_- \otimes v_+ \mapsto 0, & v_- \otimes v_- \mapsto 1, \end{cases}$

\[ \Upsilon = \downarrow = \downarrow \quad \text{and} \quad \Lambda = | = \uparrow. \]

Aside. This drops out of a coideal version of Schur–Weyl duality.
Half-way in between trivial $\subset \mathcal{U}_v - \text{part II}$

Let $c\mathcal{U}_v$ be the coideal subalgebra of $\mathcal{U}_v$ generated by $B = v^{-1}EK^{-1} + F$.

\[ V_v: Bv_+ = v_-, \quad Bv_- = v_+. \]

Define $c\mathcal{U}_v$-intertwiners:

\[ \triangledown: V_v \to V_v, \quad v_+ \mapsto v_-, \quad v_- \mapsto v_+. \]

\[ \triangledown: \mathbb{C}(v) \to V_v \otimes V_v, \quad 1 \mapsto v_+ \otimes v_+ - v_+ \otimes v_- - v_- \otimes v_-. \]

\[ \triangledown: V_v \otimes V_v \to \mathbb{C}(v), \]

\[ \left\{ \begin{array}{c}
    v_+ \otimes v_+ \mapsto -v, & v_+ \otimes v_- \mapsto 0, \\
    v_- \otimes v_+ \mapsto 0, & v_- \otimes v_- \mapsto 1,
\end{array} \right. \]

\[ \triangledown = \triangledown = \triangledown \quad \text{and} \quad \triangledown = \triangledown = \triangledown. \]

Aside. This drops out of a coideal version of Schur–Weyl duality.
Half-way in between trivial ⊂ 𝑈_v ⊂ 𝑈_v − part II

Let $c\mathcal{U}_v$ be the coideal subalgebra of $\mathcal{U}_v$ generated by $B = v^{-1}EK^{-1} + F$.

**Example.** We can see the cone strands.

**Aside.** This drops out of a coideal version of Schur–Weyl duality.
Half-way in between trivial $\subset \square \subset \mathcal{U}_v$ – part II

Let $c\mathcal{U}_v$ be the coideal subalgebra of $\mathcal{U}_v$ generated by $B = v^{-1} E K^{-1} + F$.

**Example.** We can see the cone strands.

**Aside.** This drops out of a coideal version of Schur–Weyl duality.

We have now $\neq \neq \neq$. 

**Define $c\mathcal{U}_v$-inertwiners:**

$$V_v \rightarrow V_v, \quad v \mapsto v - v, \quad v \mapsto v + v - v + B B$$

$$C(v) \rightarrow V_v \otimes V_v,$$

$$\{v \mapsto v + v - v + B B, 1 \mapsto v + v - v + B B\}$$

$$V_v \otimes V_v \rightarrow C(v),$$

$$\{v \mapsto v + v - v + B B, 1 \mapsto v + v - v + B B\}$$
Let $c\mathcal{U}_v$ be the coideal subalgebra of $\mathcal{U}_v$ generated by $B = v^{-1}EK^{-1} + F$.

Define $c\mathcal{U}_v$.

But what is the replacement of $c\mathcal{U}_v$ outside of classical or affine classical type?

(Affine) ABCD are again very special.

Aside. This drops out of a coideal version of Schur–Weyl duality.
Let $\mathsf{mArc}$ be the monoidal category defined as follows.

**Generators.** Object generator $\{\circ\}$, morphism generators

![Diagram of cups and caps, m cups and caps, and markers]

**Relations.** “Coideal” relations:

![Diagram with relations and technicality]

A technicality: $q = -v$. $\circ$ circle removal, $m$ circle removals

marker removal, marker isotopies
Let $\mathcal{Arc}$ be the monoidal category defined as follows.

**Generators.** Object generator $\{o\}$, morphism generators $o \circ o$, $o \circ o \circ \circ$, cups and caps $o \circ o$.

**Relations.** “Coideal” relations:
- $q + q^{-1} - 1 = 0$, circle removal,
- $q + q^{-1} = 0 = m$, circle removals,
- $q + q^{-1} = 0$ and marker isotopies.

**Examples.**
- $(q + q^{-1})^2 = (q + q^{-1})^2$,
- $0 = 0$,
- $= 0$ and $= 0$.

Marker removal,
- marker isotopies.
A polynomial invariant à la Jones & Kauffman

We define a monoidal functor $\langle \_ \rangle_c : \mathcal{T}an \rightarrow \mathcal{A}rc$ as follows. On objects,

$$\langle + \rangle_c = \circ \quad , \quad \langle - \rangle_c = \circ \quad , \quad \langle c \rangle_c = \emptyset$$

and on morphisms by

$$\langle \quad \rangle_c = q^0 \text{reso.} \quad , \quad \langle \quad \rangle_c = q^{-1} \text{reso.}$$

The skein relations.

$$\langle \quad \rangle_c = q \quad , \quad \langle \quad \rangle_c = -q^2 \text{reso.} \quad , \quad \langle \quad \rangle_c = -q^{-2} \text{reso.} \quad + q^{-1}$$

and

$$\langle \quad \rangle_c = \quad \quad \text{and} \quad \langle \quad \rangle_c = \quad$$

adds a marker

$$\langle \quad \rangle_c = \quad \quad \text{and} \quad \langle \quad \rangle_c = \quad$$

does not add a marker

Theorem.

This is a $\mathbb{Z}/2\mathbb{Z}$-tangle invariant.

Proof.

Check relations, e.g.:

$$\langle \quad \rangle_c = \quad = \quad = \quad$$

Example.

Here the Hopf link.

Its cube:

$$q \quad , \quad -q^2 \quad , \quad -q^{-2} \quad + q^{-1}$$

Hence, they are different.

A homological invariant à la Khovanov & Bar-Natan. Works mutatis mutandis. Here is the picture:

cone crossings

usual crossings

$$\langle \quad \rangle_c = q^0 \quad , \quad \langle \quad \rangle_c = q^{-1}$$

$$\langle \quad \rangle_c = \quad \quad \text{and} \quad \langle \quad \rangle_c = \quad$$

adds a marker

$$\langle \quad \rangle_c = \quad \quad \text{and} \quad \langle \quad \rangle_c = \quad$$

does not add a marker
We define a monoidal functor \( \langle - \rangle_c : \mathcal{C}Tan \rightarrow \mathcal{M}Arc \) as follows. On objects,
\[
\langle + \rangle_c = \circ \quad , \quad \langle - \rangle_c = \circ \quad , \quad \langle c \rangle_c = \emptyset
\]
and on morphisms by
\[
\langle \ \rangle_c = q^{0\text{-reso.}} - q^{2\text{-reso.}} \quad , \quad \langle \ \rangle_c = -q^{-2\text{-reso.}} + q^{-1\text{-reso.}}
\]

The skein relations.

The \( \mathbb{Z}/2\mathbb{Z} \)-skein relations.

adds a marker

does not add a marker

A homological invariant à la Khovanov & Bar-Natan.

Works mutatis mutandis.

A homological invariant à la Khovanov & Rozansky.

Everything generalizes to higher ranks.

一名代数描述
A polynomial invariant à la Jones & Kauffman

We define a monoidal functor \( \langle - \rangle_c : c\mathcal{T}an \to m\mathcal{A}rc \) as follows. On objects,

\[
\langle + \rangle_c = o , \quad \langle - \rangle_c = o , \quad \langle c \rangle_c = \emptyset
\]

and on morphisms by

\[
\langle \begin{array}{c}
\text{usual crossings} \\
\text{cone crossings}
\end{array} \rangle_c = q + \text{resolvant terms} + q^{-1}
\]

**Theorem.** This is a \( \mathbb{Z}/2\mathbb{Z} \)-tangle invariant.

**Proof.** Check relations, e.g.:

\[
\langle \text{usual crossings} \rangle_c = \text{resolvant terms} = \langle \text{cone crossings} \rangle_c \\
\text{and}
\]

does not add a marker.
A polynomial invariant à la Jones & Kauffman

We define a monoidal functor \( \langle \_ \rangle_c : \mathcal{T}an \to \mathcal{M}Arc \) as follows. On objects,

\[
\langle + \rangle_c = \circ , \quad \langle - \rangle_c = \circ , \quad \langle c \rangle_c = \emptyset
\]

**Example.** Here the Hopf link.

\[
\langle h \rangle_c = q^2(q + q^{-1})^2 - 2q^3(q + q^{-1}) + q^4(q + q^{-1})^2
\]

does not add a marker
A polynomial invariant à la Jones & Kauffman

We define a monoidal functor \( \langle - \rangle_c : cTan \to mArc \) as follows. On objects,

\[
\langle + \rangle_c = \emptyset, \quad \langle - \rangle_c = \emptyset, \quad \langle c \rangle_c = \emptyset
\]

**Example.** Here the essential Hopf link.

\[
\langle eh \rangle_c = q^2(q + q^{-1})^2 - 2q^3(q + q^{-1}) + 0
\]

does not add a marker
A polynomial invariant à la Jones & Kauffman

We define a monoidal functor $\langle - \rangle_c : \mathcal{C}Tan \to \mathcal{M}Arc$ as follows. On objects,

$\langle + \rangle_c = o, \quad \langle - \rangle_c = o, \quad \langle c \rangle_c = \emptyset$

Example. Here the essential Hopf link.

Example. Here the Hopf link.

Hence, they are different.

$\langle e h \rangle_c = q^2(q + q^{-1})^2 - 2q^3(q + q^{-1}) + 0$

does not add a marker
We define a monoidal functor $\langle - \rangle_c : c\mathcal{T}an \to m\mathcal{A}rc$ as follows. On objects, 

A homological invariant à la Khovanov & Bar-Natan.
Works mutatis mutandis. Here is the picture:

\[
m\mathcal{Z} \left( \begin{array}{c} \text{cone crossings} \\ \text{usual crossings} \end{array} \right) = \begin{cases} \mathbb{Z}[X]/(X^2), & \text{if } m \text{ is even}, \\ 0, & \text{if } m \text{ is odd}, \end{cases}
\]

does not add a marker
A polynomial invariant à la Jones & Kauffman

We define a monoidal functor \( \langle - \rangle_c : \mathcal{C} \text{Tan} \to \mathcal{M} \text{Arc} \) as follows. On objects,

\[
\langle + \rangle_c = o, \quad \langle - \rangle_c = o, \quad \langle C \rangle_c = \emptyset
\]

and on morphisms by

\[
\langle \rangle_c = q_0\text{-reso.} - q_2 q_1, \quad \langle h \rangle_c = - q_{-2} q_{-1}, \quad \langle h \rangle_c = \text{adds a marker,} \quad \langle \rangle_c = \text{does not add a marker}
\]

The skein relations.

The \( \mathbb{Z}/2\mathbb{Z} \)-skein relations. 

Theorem. This is a \( \mathbb{Z}/2\mathbb{Z} \)-tangle invariant.

Proof. Check relations, e.g.:

\[
\langle h \rangle_c = \text{and } \langle \rangle_c
\]

Example. Here the Hopf link. Its \( \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \) cube

\[
\begin{align*}
q_3 & 10 \\
q_2 & 00 \\
q_1 & 11 \\
q_0 & \quad (q_0 + q_0 - 1)^2 \\
q_2 & 2 \quad q_3 \quad q_4 \quad (q_0 + q_0 - 1)^2
\end{align*}
\]

\[
\langle h \rangle_c = - + \quad \langle \rangle_c
\]

Example. Here the essential Hopf link. Its \( \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \) cube

\[
\begin{align*}
q_3 & 10 \\
q_2 & 00 \\
q_1 & 11 \\
q_0 & \quad (q_0 + q_0 - 1)^2 \\
q_2 & 2 \quad q_3 \quad q_4 \quad (q_0 + q_0 - 1)^2
\end{align*}
\]

\[
\langle h \rangle_c = - + \quad \langle \rangle_c
\]

Hence, they are different.

A homological invariant à la Khovanov & Bar-Natan.

Works mutatis mutandis. Here is the picture:

In case of type ABCD this comes from a categorification of the Schur–Weyl-coideal duality. (“Web and arc algebras of type D”.)

\[
\mathcal{M} \mathcal{Z} \begin{pmatrix} m \\ m \end{pmatrix} = \begin{cases} \mathbb{Z}[X]/(X^2), & \text{if } m \text{ is even}, \\ 0, & \text{if } m \text{ is odd}, \end{cases}
\]

does not add a marker
We define a monoidal functor \( \langle - \rangle_c : \mathcal{C}Tan \to \mathcal{M}Arc \) as follows. On objects,

\[
\langle + \rangle_c = \circ, \quad \langle - \rangle_c = \circ, \quad \langle c \rangle_c = \emptyset
\]

and on morphisms by

\[
\begin{align*}
\langle \begin{array}{c}
\text{usual crossings} \\
\end{array} \rangle_c &= q \begin{array}{c} \text{cone crossings} \end{array} - q^2 \begin{array}{c} \text{cone crossings} \end{array}, \\
\langle \begin{array}{c}
\text{usual crossings} \\
\end{array} \rangle_c &= -q^{-2} \begin{array}{c} \text{cone crossings} \end{array} + q^{-1} \begin{array}{c} \text{cone crossings} \end{array}
\end{align*}
\]

A homological invariant à la Khovanov & Rozansky.
Everything generalizes to higher ranks.
(“Webs”, “foams”, etc.)
Tangle diagrams with cone strands

Let $\mathcal{C}$ be the monoidal category defined as follows.

Generators. Object generators $\{x, y\}$ and the $\mathbb{Z}/2\mathbb{Z}$ relations:

$$x \sim y \quad \text{and} \quad x \sim x^{-1}$$

Relations. To $\mathcal{C}$ add the $\mathbb{Z}/2\mathbb{Z}$ relations:

$$x \sim y \quad \text{and} \quad x \sim x^{-1}$$

I follow hyperplanes

$W_m = \langle x, y \rangle$ acts faithfully on $\mathbb{R}^2$ by reflecting in hyperplanes (for each reflection):

$$W_m = \langle x, y \rangle$$

Complicating the action: $\mathbb{R}^2 \rightarrow \mathbb{C}$, $\mathcal{H}_m \rightarrow \mathcal{H}_m$, $\mathcal{H}_m \rightarrow \mathcal{H}_m$. Then:

$$\mathcal{H}_m(W_m) \cong \mathcal{H}_m = (k, k \mid A A A = A)$$

Configuration spaces

Artin -- 1925. There is a topological model of $\mathcal{H}_m$ via configuration spaces.

Example. Take $\text{Conf}_m = \{\mathbb{R}^2, \text{fat diagram}\}$. Then $\tau_x(\text{Conf}_m) \cong \mathcal{H}_m$.

Philosophy. Having a configuration space is the same as having braid diagrams:

$$\mathcal{H}_m \cong \text{Braids}$$

Crucial. Note that -- by explicitly calculating the configuration space -- one can directly check that:

"Hyperplane picture equals configuration space picture."

A polynomial invariant à la Jones & Kauffman

We define a monoidal functor $(-)_m : \mathcal{C} \rightarrow \text{Vec}$ as follows. On objects,

$$x \mapsto x_m \quad \text{and} \quad y \mapsto y_m$$

Example. Here the essential Hopf tie:

$$h \circ \alpha$$

A version of Schur's remarkable duality.

$$\mathcal{U}(\mathcal{C}) \otimes \mathcal{U}(\mathcal{C}) \cong \mathcal{U}(\mathcal{C})$$

Ehrig-Stroppel, Bau-Wang -- 2013. The actions of $\mathcal{U}(\mathcal{C})$ and $\mathcal{U}(\mathcal{C})$ on $\mathbb{R}^2$ commute and generate each other's centralizer.

A polynomial invariant à la Jones & Kauffman

We define a monoidal functor $(-)_m : \mathcal{C} \rightarrow \text{Vec}$ as follows. On objects, and on:

$$m \mapsto \mathcal{U}(\mathcal{C})$$

A homological invariant à la Khovanov & Bar-Natan.

Works mutatis mutandis. Here is the picture:

$$\mathcal{U}(\mathcal{C})$$

There is still much to do...
**Configuration spaces**

Artin – 1925. There is a topological model of $TX$ via configuration spaces.

**Example.** Take $\text{Conf}_n = (\mathbb{R}^3)^n \setminus \text{fat diagonal}$, then $\tau_0(\text{Conf}_n) = TX$.

**Philosophy.** Having a configuration space is the same as having braid diagrams:

![Braid diagrams](image)

A polynomial invariant à la Jones & Kauffman

We define a monoidal functor $(\cdot)_! : \mathcal{O} \to \operatorname{Ob} \mathsf{Gr}$ as follows. On objects,

$$\mathcal{O}(c) \mapsto \bigoplus_n c^n \otimes \mathcal{O}(c^n)$$

**Example.** Here the essential Hopf map:

![Essential Hopf map](image)

A version of Schur’s remarkable duality.

$$\mathcal{O}(c) \otimes \mathcal{O}(d) \ni x \otimes y \mapsto (x \otimes y)(c)$$

**Ehrig–Stroppel, Bau–Wang – 2013.** The actions of $\mathcal{O}(c^n)$ and $\mathcal{O}(d^n)$ commute and generate each other’s centralizer.

A homological invariant à la Khovanov & Bar-Natan.

Works mutates mutations. Here is the picture:

![Mutation](image)

**I follow hyperplanes**

$W_0 = (x/1)$ acts faithfully on $\mathbb{R}^2$ by reflecting in hyperplanes (for each reflection):

$W_0$ acts freely on $\mathbb{R}^2 \setminus \text{hyperplanes}$. Set $\mathbb{R}_0 = \mathbb{R}_0 / W_0$:

Compositifying the action: $\mathbb{R}^2 \to \mathbb{C}^2$, $\mathbb{R}_0 \to \mathbb{C}^2$, $\mathbb{R}_0 \to \mathbb{C}^2$. Then:

$$n_1(B^3_2) \cong \mathbb{R}_0 = (k,k \mid \text{AAA} = \text{AAA})$$

**Half-way in between trivial $\subset \mathbb{F}_v$ – part II**

Let $\mathbb{F}_v$ be the $\mathbb{C}$-subalgebra of $\mathcal{O}_v$ generated by $v = e^{i \pi/2} - i$.

**Example.** We can see the cone strands.

![Cone strands](image)

Aside. This drops out of a new version of Schur–Weyl duality.

A polynomial invariant à la Jones & Kauffman

We define a monoidal functor $(\cdot)_! : \mathcal{O} \to \operatorname{Ob} \mathsf{Gr}$ as follows. On objects, and on:

![Monoidal functor](image)

**Thanks for your attention!**
These guys and friends come for free.

$\in Hom_{\mathcal{T}an}(c, -c)$
These guys and friends come for free.

I see them as diagrams – no topological interpretation intended at the moment.
Satake ∼1956 ("V-manifold"), Thurston ∼1978, Haefliger ∼1990 ("orbihedron"), etc. A triple $\text{Orb} = (X_{\text{orb}}, \bigcup_i U_i, G_i)$ of a Hausdorff space $X_{\text{orb}}$, a covering $\bigcup_i U_i$ of it (closed under finite intersections) and a collection of finite groups $G_i$ is called an orbifold (of dimension $m$) if for each $U_i$ there exists a open subset $V_i \subset \mathbb{R}^m$ carrying an action of $G_i$, and some compatibility conditions.

Fact. A two-dimensional ("smooth") orbifold is locally modeled on:

- Cone points $\leftrightarrow$ rotation action of $\mathbb{Z}/l\mathbb{Z}$.
- Reflector corners $\leftrightarrow$ reflection action of the dihedral group.
- Mirror points $\leftrightarrow$ reflection action of $\mathbb{Z}/2\mathbb{Z}$.
Satake ~1956 ("V-manifold"), Thurston ~1978, Haefliger ~1990 ("orbihedron"), etc.

A triple $O_{rb} = (X_{rb}, \bigcup_i U_i, G_i)$ of a Hausdorff space $X_{rb}$, a covering $\bigcup_i U_i$ of it (closed under finite intersections) and a collection of finite groups $G_i$ is called an orbifold (of dimension $m$) if for each $U_i$ there exists an open subset $V_i \subset \mathbb{R}^m$ carrying an action of $G_i$, and some compatibility conditions.

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Topologically an orbifold is sometimes the same as its underlying space. So all notions concerning orbifolds have to take this into account.
Satake ∼1956 ("V-manifold"), Thurston ∼1978, Haefliger ∼1990
("orbihedron").

A triple \( O = (X_{orb}, \bigcup_i U_i, G_i) \) of a Hausdorff space \( X_{orb} \), a covering \( \bigcup_i U_i \) of it (closed under finite intersections) and a collection of finite groups \( G_i \) is called an orbifold (of dimension \( m \)) if for each \( U_i \) there exists an open subset \( V_i \subset \mathbb{R}^m \) carrying an action of \( G_i \), and some compatibility conditions.

**Fact.** A two-dimensional ("smooth") orbifold is locally modeled on: \( \exists \) Cone points \( \leftrightarrow \) rotation action of \( \mathbb{Z}/l\mathbb{Z} \). \( \exists \) Reflector corners \( \leftrightarrow \) reflection action of the dihedral group. \( \exists \) Mirror points \( \leftrightarrow \) reflection action of \( \mathbb{Z}/2\mathbb{Z} \).

Topologically an orbifold is sometimes the same as its underlying space. So all notions concerning orbifolds have to take this into account. Below the following of a two-dimensional orbifold (with \( \mathbb{Z}/3\mathbb{Z} \) and \( \mathbb{Z}/2\mathbb{Z} \) actions).

**Quote by Thurston about the name orbifold:**

“This terminology should not be blamed on me. It was obtained by a democratic process in my course of 1976-77. An orbifold is something with many folds; unfortunately, the word ‘manifold’ already has a different definition. I tried ‘foldamani’, which was quickly displaced by the suggestion of ‘manifolded’. After two months of patiently saying ‘no, not a manifold, a manifold dead,’ we held a vote, and ‘orbifold’ won.”
Satake ∼1956 ("V-manifold"), Thurston ∼1978, Haefliger ∼1990 ("orbihedron"), etc. A triple $\text{O}rb = (X_{\text{orb}}, \mathbin{\cup_i\mathcal{U}_i}, G_i)$ of a Hausdorff space $X_{\text{orb}}$, a covering $\mathbin{\cup_i\mathcal{U}_i}$ of it (closed under finite intersections) and a collection of finite groups $G_i$ is called an orbifold (of dimension $m$) if for each $\mathcal{U}_i$ there exists an open subset $V_i \subset \mathbb{R}^m$ carrying an action of $G_i$, and some compatibility conditions.

**Fact.** A two-dimensional ("smooth") orbifold is locally modeled on:

- Cone points $\rightsquigarrow$ rotation action of $\mathbb{Z}/l\mathbb{Z}$.
- Reflector corners $\rightsquigarrow$ reflection action of the dihedral group.
- Mirror points $\rightsquigarrow$ reflection action of $\mathbb{Z}/2\mathbb{Z}$. 

Not super important. Only one thing to stress: Topologically an orbifold is sometimes the same as its underlying space. So all notions concerning orbifolds have to take this into account.

Quote by Thurston about the name orbifold:

"This terminology should not be blamed on me. It was obtained by a democratic process in my course of 1976-77. An orbifold is something with many folds; unfortunately, the word 'manifold' already has a different definition. I tried 'foldamani', which was quickly displaced by the suggestion of 'manifolded'. After two months of patiently saying 'no, not a manifold, a manifoldead', we held a vote, and 'orbifold' won."

Examples. $A \mathbb{Z}/2\mathbb{Z}$-orbifold tangle $= A \mathbb{Z}/3\mathbb{Z}$-orbifold tangle etc. "Puncture = lim$_{l \to \infty} l$-cone point".
Satake ~1956 ("V-manifold"), Thurston ~1978, Haefliger ~1990 ("orbihedron"), etc.

A triple \( \mathcal{O}rb = (X_{\mathcal{O}rb}, \bigcup_i U_i, G_i) \) of a Hausdorff space \( X_{\mathcal{O}rb} \), a covering \( \bigcup_i U_i \) of it (closed under finite intersections) and a collection of finite groups \( G_i \) is called an orbifold (of dimension \( m \)) if for each \( U_i \) there exists an open subset \( V_i \subset \mathbb{R}^m \) carrying an action of \( G_i \), and some compatibility conditions.

**Fact.** A two-dimensional ("smooth") orbifold is locally modeled on:

- **Cone points** \( \cong \) rotation action of \( \mathbb{Z}/l\mathbb{Z} \).
- **Reflector corners** \( \leftrightarrow \) reflection action of the dihedral group.
- **Mirror points** \( \leftrightarrow \) reflection action of \( \mathbb{Z}/2\mathbb{Z} \).

"Puncture = \( \lim_{l \to \infty} l \)-cone point".

Examples.

\begin{align*}
\text{A } \mathbb{Z}/2\mathbb{Z}\text{-orbifold tangle} & \quad \text{A } \mathbb{Z}/3\mathbb{Z}\text{-orbifold tangle} \\
\Rightarrow & \quad \Rightarrow \\
\text{ Examples. } & \quad \\
\text{ "Puncture = } \lim_{l \to \infty} l \text{-cone point". } & \\
\end{align*}
**Figure:** The Coxeter graphs of finite type.

**Example.** The type $A$ family is given by the symmetric groups using the simple transpositions as generators.

(Picture from https://en.wikipedia.org/wiki/Coxeter_group.)
**Example.** The type $A$ family is given by the symmetric groups using the simple transpositions as generators.

(Picture from [https://en.wikipedia.org/wiki/Coxeter_group](https://en.wikipedia.org/wiki/Coxeter_group).)
Example. The type $A$ family is given by the symmetric groups using the simple transpositions as generators.

(Picture from https://en.wikipedia.org/wiki/Coxeter_group.)
Example. The type $A$ family is given by the symmetric groups using the simple transpositions. I want to answer ??? in this case, and partially in general.

(Picture from https://en.wikipedia.org/wiki/Coxeter_group.)
**Figure:** The Coxeter graphs of affine type.

**Example.** The type $\tilde{A}_n$ corresponds to the affine Weyl group for $\mathfrak{sl}_n$.

(Picture from https://en.wikipedia.org/wiki/Coxeter_group.)
### Hyperplane Equations

- **A₂**
  - Positive root: \( \alpha_1 = (1, -1, 0) \), \( \alpha_2 = (0, 1, -1) \), \( \alpha_1 + \alpha_2 = (1, 0, -1) \)
  - Reflection action:
    - \( x_1 \leftrightarrow x_2 \)
    - \( x_2 \leftrightarrow x_3 \)
    - \( x_1 \leftrightarrow x_3 \)
  - \( \perp \)-hyperplane:
    - \( \{ (x, x, 0) \} \)
    - \( \{ (0, y, y) \} \)
    - \( \{ (z, 0, z) \} \)

**Hyperplane equations:** \( \{(x, y, z) \in (\mathbb{R}^2)^3 | x = y \text{ or } y = z \text{ or } x = z \} \)

*This is gl-notation.*
| positive root | $\alpha_1 = (1, -1, 0)$ | $\alpha_2 = (0, 1, -1)$ | $\alpha_1 + \alpha_2 = (1, 0, -1)$ |
| reflection action | $x_1 \leftrightarrow x_2$ | $x_2 \leftrightarrow x_3$ | $x_1 \leftrightarrow x_3$ |
| $\perp$-hyperplane | $\{(x, x, 0)\}$ | $\{(0, y, y)\}$ | $\{(z, 0, z)\}$ |

Hyperplane equations: $\{(x, y, z) \in (\mathbb{R}^2)^3 \mid x = y \text{ or } y = z \text{ or } x = z\}$

Observe that this matches the diagonal of the configuration space picture.
<table>
<thead>
<tr>
<th>positive root</th>
<th>$\alpha_1 = (1, -1, 0)$</th>
<th>$\alpha_2 = (0, 1, -1)$</th>
<th>$\alpha_1 + \alpha_2 = (1, 0, -1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>reflection action</td>
<td>$x_1 \leftrightarrow x_2$</td>
<td>$x_2 \leftrightarrow x_3$</td>
<td>$x_1 \leftrightarrow x_3$</td>
</tr>
<tr>
<td>$\perp$-hyperplane</td>
<td>${(x, x, 0)}$</td>
<td>${(0, y, y)}$</td>
<td>${(z, 0, z)}$</td>
</tr>
</tbody>
</table>

Hyperplane equations: $\{(x, y, z) \in (\mathbb{R}^2)^3 \mid x = y$ or $y = z$ or $x = z\}$

<table>
<thead>
<tr>
<th>positive root</th>
<th>$\alpha'_1 = (1, 1, 0)$</th>
<th>$\alpha_1 = (1, -1, 0)$</th>
<th>more “type A-like”</th>
</tr>
</thead>
<tbody>
<tr>
<td>reflection action</td>
<td>$x'_1, x_1 \leftrightarrow -x'_1, -x_1$</td>
<td>$x_1 \leftrightarrow x_2$</td>
<td>more “type A-like”</td>
</tr>
<tr>
<td>$\perp$-hyperplane</td>
<td>${(x, -x, 0, 0)}$</td>
<td>${(x, x, 0, 0)}$</td>
<td>more “type A-like”</td>
</tr>
</tbody>
</table>

Hyperplane equations: $\{(x, y, z, w) \in \mathbb{C}^4 \mid x = \pm y$ etc.$\}$
| Positive root | $\alpha_1 = (1, -1, 0)$ | $\alpha_2 = (0, 1, -1)$ | $\alpha_1 + \alpha_2 = (1, 0, -1)$ |
| Reflection action | $x_1 \leftrightarrow x_2$ | $x_2 \leftrightarrow x_3$ | $x_1 \leftrightarrow x_3$ |
| $\perp$-Hyperplane | $\{(x, x, 0)\}$ | $\{(0, y, y)\}$ | $\{(z, 0, z)\}$ |

Hyperplane equations: $\{(x, y, z) \in (\mathbb{R}^2)^3 \mid x = y \text{ or } y = z \text{ or } x = z\}$

Observe that this matches the diagonal of the configuration space picture up to a 2-fold covering $(x, y, z, w) \mapsto (x^2, y^2, z^2, w^2)$.

| Positive root | $\alpha'_1 = (1, 1, 0)$ | $\alpha_1 = (1, -1, 0)$ | more “type A-like” |
| Reflection action | $x'_1, x_1 \leftrightarrow -x'_1, -x_1$ | $x_1 \leftrightarrow x_2$ | more “type A-like” |
| $\perp$-Hyperplane | $\{(x, -x, 0, 0)\}$ | $\{(x, x, 0, 0)\}$ | more “type A-like” |

Hyperplane equations: $\{(x, y, z, w) \in \mathbb{C}^4 \mid x = \pm y \text{ etc.}\}$
<table>
<thead>
<tr>
<th>(\alpha_1)</th>
<th>(\alpha_2)</th>
<th>(\alpha_1 + \alpha_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, -1, 0))</td>
<td>((0, 1, -1))</td>
<td>((1, 0, -1))</td>
</tr>
</tbody>
</table>

**Reflection Action**

- \(x_1 \leftrightarrow x_2\)
- \(x_2 \leftrightarrow x_3\)
- \(x_1 \leftrightarrow x_3\)

**\(\perp\)-hyperplane**

- \(\{(x, x, 0)\}\)
- \(\{(0, y, y)\}\)
- \(\{(z, 0, z)\}\)

Hyperplane equations: \(\{(x, y, z) \in (\mathbb{R}^2)^3 \mid x = y \text{ or } y = z \text{ or } x = z\}\)

---

Similarly in (affine) types ABCD.

<table>
<thead>
<tr>
<th>(\alpha_1')</th>
<th>(\alpha_1)</th>
<th>more “type A-like”</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 1, 0))</td>
<td>((1, -1, 0))</td>
<td>more “type A-like”</td>
</tr>
</tbody>
</table>

**Reflection Action**

- \(x_1', x_1 \leftrightarrow -x_1', -x_1\)
- \(x_1 \leftrightarrow x_2\)

**\(\perp\)-hyperplane**

- \(\{(x, -x, 0, 0)\}\)
- \(\{(x, x, 0, 0)\}\)

Hyperplane equations: \(\{(x, y, z, w) \in \mathbb{C}^4 \mid x = \pm y \text{ etc.}\}\)
Noumi–Sugitani \(\sim 1994\), Letzter \(\sim 1999\). Quantum groups have few Hopf subalgebras, but plenty of coideal subalgebras.

\(c\mathcal{U}_v\) is not a Hopf algebra, but rather a right coideal (subalgebra) of \(\mathcal{U}_v\):

\[
\Delta(B) = B \otimes K^{-1} + 1 \otimes B \in c\mathcal{U}_v \otimes \mathcal{U}_v,
\]

which gives \(\text{Rep}(c\mathcal{U}_v)\) the structure of a right \(\text{Rep}(\mathcal{U}_v)\)-category \(\Rightarrow\) right handedness of diagrams, e.g.:

\begin{align*}
\text{Ok from this picture} & \quad \text{Not ok from this picture}
\end{align*}
\textbf{Noumi–Sugitani} \sim 1994, \textbf{Letzter} \sim 1999. Quantum groups have few Hopf subalgebras, but plenty of coideal subalgebras.

\( \mathcal{U}_v \) is not a Hopf algebra, but rather a right coideal (subalgebra) of \( \mathcal{U}_v \):

\textbf{Example.} The vector representations of \( \mathfrak{gl}_n \), \( \mathfrak{so}_n \) and \( \mathfrak{sp}_n \) all agree, and indeed \( \mathfrak{so}_n \hookrightarrow \mathfrak{gl}_n \) and \( \mathfrak{sp}_n \hookrightarrow \mathfrak{gl}_n \).

But the quantum vector representations do not agree, i.e.
\[ \mathcal{U}_v(\mathfrak{so}_n) \not\hookrightarrow \mathcal{U}_v(\mathfrak{gl}_n) \text{ and } \mathcal{U}_v(\mathfrak{sp}_n) \not\hookrightarrow \mathcal{U}_v(\mathfrak{gl}_n). \]

This is bad. Idea: Invent new quantizations such that \( \mathcal{U}_v'(\mathfrak{so}_n) \hookrightarrow \mathcal{U}_v(\mathfrak{gl}_n) \) and \( \mathcal{U}_v'(\mathfrak{sp}_n) \hookrightarrow \mathcal{U}_v(\mathfrak{gl}_n) \).

\[ \xrightarrow{\text{Ok from this picture}} \quad \xrightarrow{\text{Not ok from this picture}} \]
Noumi–Sugitani \(\sim 1994\), Letzter \(\sim 1999\). Quantum groups have few Hopf subalgebras, but plenty of coideal subalgebras.

\(\mathfrak{c}U_v\) is not a Hopf algebra, but rather a right coideal (subalgebra) of \(\mathfrak{c}U_v\):

**Example.** The vector representations of \(\mathfrak{gl}_n\), \(\mathfrak{so}_n\) and \(\mathfrak{sp}_n\) all agree, and indeed \(\mathfrak{so}_n \hookrightarrow \mathfrak{gl}_n\) and \(\mathfrak{sp}_n \hookrightarrow \mathfrak{gl}_n\).

But the quantum vector representations do not agree, i.e.

\[\mathfrak{c}U_v(\mathfrak{so}_n) \nrightarrow \mathfrak{c}U_v(\mathfrak{gl}_n)\] and \[\mathfrak{c}U_v(\mathfrak{sp}_n) \nrightarrow \mathfrak{c}U_v(\mathfrak{gl}_n)\].

This is bad. Idea: Invent new quantizations such that

\[\mathfrak{c}U'_v(\mathfrak{so}_n) \hookrightarrow \mathfrak{c}U_v(\mathfrak{gl}_n)\] and \[\mathfrak{c}U'_v(\mathfrak{sp}_n) \hookrightarrow \mathfrak{c}U_v(\mathfrak{gl}_n)\].

**Observation.** This happens repeatedly.
Noumi–Sugitani \~1994, Letzter \~1999. Quantum groups have few Hopf subalgebras, but plenty of coideal subalgebras.

\( c \mathcal{U}_v \) is not a Hopf algebra, but rather a right coideal (subalgebra) of \( \mathcal{U}_v \):

\[
\Delta(B) = B \otimes K^{-1} + 1 \otimes B \in c \mathcal{U}_v \otimes \mathcal{U}_v,
\]

which gives

\[
\text{This happens really often. In our case we have basically right handedness on}
\]

\[
\mathfrak{gl}_1 \hookrightarrow \mathfrak{sl}_2, \quad (t) \mapsto \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}
\]

which does not quantize properly...

\textbf{Observation.} This happens repeatedly.

Ok from this picture

Not ok from this picture
A version of Schur’s remarkable duality.

Plain old $\mathfrak{sl}_2$: Acts by matrices.
The symmetric group: Acts by permutation.

$\mathcal{U}_1(\mathfrak{sl}_2) \circ \underbrace{V_1 \otimes \cdots \otimes V_1}_d \circ \mathcal{H}_1(A)$

**Schur $\sim 1901$.** The natural actions of $\mathcal{U}_1(\mathfrak{sl}_2)$ and $\mathcal{H}_1(A)$ on $V_1^\otimes d = (\mathbb{C}^2)^\otimes d$ commute and generate each other’s centralizer.
A version of Schur’s remarkable duality.

\[ \mathcal{U}_1(\mathfrak{sl}_2) \curvearrowright V_1 \otimes \cdots \otimes V_1 \curvearrowright \mathcal{H}_1(A) \]

\[ \parallel \]

\[ V_1 \otimes \cdots \otimes V_1 \]

\( d \) times
A version of Schur's remarkable duality.

\[ \mathcal{U}_1(\mathfrak{sl}_2) \circ \bigotimes_{d \text{ times}} V_1 \circ \mathcal{H}_1(A) \]

\[ \bigcap \bigotimes_{d \text{ times}} V_1 \quad \mathcal{H}_1(D) \times \mathbb{Z}/2\mathbb{Z} \]

Ignore the component group \( \mathbb{Z}/2\mathbb{Z} \).
A version of Schur’s remarkable duality.

\[ \mathcal{U}_1(\mathfrak{sl}_2) \cap V_1 \otimes \cdots \otimes V_1 \cap \mathcal{H}_1(A) \]

\[ \mathcal{H}_1(D) \times \mathbb{Z}/2\mathbb{Z} \]

\[ V_1 \otimes \cdots \otimes V_1 \cap \mathcal{H}_1(D) \times \mathbb{Z}/2\mathbb{Z} \]

\[ d \text{ times} \]

Acts by signed permutations.
A version of Schur's remarkable duality.

\[ \mathcal{U}_1(\mathfrak{sl}_2) \Join V_1 \otimes \cdots \otimes V_1 \Join \mathcal{H}_1(A) \]

\[ \mathcal{U}_1(\mathfrak{gl}_1) \Join V_1 \otimes \cdots \otimes V_1 \Join \mathcal{H}_1(D) \rtimes \mathbb{Z}/2\mathbb{Z} \]

\[ d \text{ times} \]

\[ \mathcal{U}_1(\mathfrak{gl}_1) \Join V_1 \otimes \cdots \otimes V_1 \Join \mathcal{H}_1(D) \rtimes \mathbb{Z}/2\mathbb{Z} \]
A version of Schur’s remarkable duality.

\[ \mathcal{U}_1(\mathfrak{sl}_2) \supset V_1 \otimes \cdots \otimes V_1 \supset \mathcal{H}_1(\mathfrak{A}) \]

\[ \mathcal{U}_1(\mathfrak{gl}_1) \supset V_1 \otimes \cdots \otimes V_1 \supset \mathcal{H}_1(\mathfrak{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \]

The antidiagonal embedding: \( \mathfrak{gl}_1 \hookrightarrow \mathfrak{sl}_2, \ (t) \mapsto \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} \)

Acts by restriction.

**Regev \sim 1983.** The actions of \( \mathcal{U}_1(\mathfrak{gl}_1) \) and \( \mathcal{H}_1(\mathfrak{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \) on \( V_1 \otimes^d \) commute and generate each other’s centralizer.
A version of Schur’s remarkable duality.

\[ \mathcal{U}_v(\mathfrak{sl}_2) \circ V_v \otimes \cdots \otimes V_v \circ \mathcal{H}_v(A) \]

**Jimbo \sim 1985.** The natural actions of \( \mathcal{U}_v(\mathfrak{sl}_2) \) and \( \mathcal{H}_v(A) \) on \( V_v^\otimes d = (\mathbb{C}(v)^2)^\otimes d \) commute and generate each other’s centralizer.
A version of Schur’s remarkable duality.

\[ \mathcal{U}_v(\mathfrak{sl}_2) \circ V_v \otimes \cdots \otimes V_v \circ \mathcal{H}_v(A) \]

\[ \| \]

\[ V_v \otimes \cdots \otimes V_v \]

\[ d \text{ times} \]
A version of Schur’s remarkable duality.

\[ \mathcal{U}_v(\mathfrak{sl}_2) \otimes V_v \otimes \cdots \otimes V_v \subset \mathcal{H}_v(A) \]

\[ \mathcal{U}_v(\mathfrak{sl}_2) \otimes V_v \otimes \cdots \otimes V_v \subset \mathcal{H}_v(D) \rtimes \mathbb{Z}/2\mathbb{Z} \]

\[ d \text{ times} \]
A version of Schur’s remarkable duality.

\[ \mathcal{U}_v(\mathfrak{sl}_2) \supseteq V_v \otimes \cdots \otimes V_v \supseteq \mathcal{H}_v(A) \]

\[ \mathcal{V}_v \otimes \cdots \otimes \mathcal{V}_v \supseteq \mathcal{H}_v(D) \rtimes \mathbb{Z}/2\mathbb{Z} \]

Quantizes nicely.
A version of Schur's remarkable duality.

\[ \mathcal{U}_v(\mathfrak{sl}_2) \supseteq V_v \otimes \cdots \otimes V_v \supseteq \mathcal{H}_v(A) \]

\[ \bigcup \quad \| \quad \bigcap \]

\[ \overset{\text{??}}{\mathcal{U}_v(\mathfrak{sl}_2)} \supseteq V_v \otimes \cdots \otimes V_v \supseteq \mathcal{H}_v(D) \rtimes \mathbb{Z}/2\mathbb{Z} \]

\[ d \text{ times} \]
A version of Schur’s remarkable duality.

\[ \mathcal{U}_v(sl_2) \circ V_v \otimes \cdots \otimes V_v \circ \mathcal{H}_v(A) \]

\[ \mathcal{U}_v(gl_1) \circ V_v \otimes \cdots \otimes V_v \circ \mathcal{H}_v(D) \ltimes \mathbb{Z}/2\mathbb{Z} \]

d times
A version of Schur’s remarkable duality.

\[
\mathcal{U}_v(sl_2) \circ V_v \otimes \cdots \otimes V_v \circ \mathcal{H}_v(A)
\]

\[
\mathcal{U}_v(gl_1) \circ V_v \otimes \cdots \otimes V_v \circ \mathcal{H}_v(D) \rtimes \mathbb{Z}/2\mathbb{Z}
\]

Does not embed.

\[d \text{ times}\]
A version of Schur’s remarkable duality.

\[ \mathcal{U}_v(sl_2) \circ V_v \otimes \cdots \otimes V_v \circ \mathcal{H}_v(A) \]

\[ \mathcal{U}_v(gl_1) \circ V_v \otimes \cdots \otimes V_v \circ \mathcal{H}_v(D) \times \mathbb{Z}/2\mathbb{Z} \]

No commuting action.

\[ d \text{ times} \]

Back
A version of Schur’s remarkable duality.

\[ U_v(sl_2) \otimes V_v \otimes \cdots \otimes V_v \otimes \mathcal{H}_v(A) \]

\[ \mathcal{U}_v(sl_1) \otimes V_v \otimes \cdots \otimes V_v \otimes \mathcal{H}_v(D) \times \mathbb{Z}/2\mathbb{Z} \]

\[ d \text{ times} \]
A version of Schur’s remarkable duality.

\[ \mathcal{U}_v(\mathfrak{sl}_2) \circ V_v \otimes \cdots \otimes V_v \circ \mathcal{H}_v(A) \]

\[ \mathcal{U}_v(\mathfrak{gl}_1) \]

\[ \mathcal{C} \mathcal{U}_v(\mathfrak{gl}_1) \]

\[ \text{d times} \]

\[ \mathcal{H}_v(D) \times \mathbb{Z}/2\mathbb{Z} \]
A version of Schur's remarkable duality.

\[ \mathcal{U}_v(\mathfrak{sl}_2) \circlearrowleft V_v \otimes \cdots \otimes V_v \circlearrowright \mathcal{H}_v(A) \]

Is a subalgebra.

\[ c \mathcal{U}_v(\mathfrak{gl}_1) \]

\[ V_v \otimes \cdots \otimes V_v \circlearrowright \mathcal{H}_v(D) \rtimes \mathbb{Z}/2\mathbb{Z} \]

d times

\[ \Gamma = \Gamma((\tilde{B}_4, \tilde{D}_4, D_4, \tilde{C}_3, D_4)) \]

But, again, only in the special case of type ABCD this is known.

Message to take away. Coideal naturally appear in Schur–Weyl-like games. And these pull the strings from the background for tangle and link invariants.
A version of Schur’s remarkable duality.

\[ \mathcal{U}_v(\mathfrak{sl}_2) \circ V_v \otimes \cdots \otimes V_v \circ \mathcal{H}_v(A) \]

\[ \cup \quad \| \quad \cap \]

\[ c \mathcal{U}_v(\mathfrak{gl}_1) \circ V_v \otimes \cdots \otimes V_v \circ \mathcal{H}_v(D) \times \mathbb{Z}/2\mathbb{Z} \]

Act by restriction.

\[ \text{d times} \]

Message to take away. Coideal naturally appear in Schur–Weyl-like games. And these pull the strings from the background for tangle and link invariants.
A version of Schur’s remarkable duality.

\[ \mathcal{U}_v(\mathfrak{sl}_2) \circ V_v \otimes \cdots \otimes V_v \circ \mathcal{H}_v(A) \]

\[ \bigcup \big\| \bigcap \]

\[ \mathfrak{c} \mathcal{U}_v(\mathfrak{gl}_1) \circ V_v \otimes \cdots \otimes V_v \circ \mathcal{H}_v(D) \rtimes \mathbb{Z}/2\mathbb{Z} \]

\[ d \text{ times} \]

Ehrig–Stroppel, Bao–Wang \( \sim 2013 \). The actions of \( \mathfrak{c} \mathcal{U}_v(\mathfrak{gl}_1) \) and \( \mathcal{H}_v(D) \rtimes \mathbb{Z}/2\mathbb{Z} \) on \( V_v^\otimes d \) commute and generate each other’s centralizer.
A version of Schur’s remarkable duality.

\[ \mathcal{U}_\mathfrak{v}(\mathfrak{sl}_2) \subset V_\mathfrak{v} \otimes \cdots \otimes V_\mathfrak{v} \subset \mathcal{H}_\mathfrak{v}(A) \]

Hope.

The same works for the Coxeter diagrams

\[ \Gamma = \Gamma((\tilde{B}_4, \tilde{D}_4, D_4, \tilde{C}_3, D_4)) \]

\[ x(\Gamma) = \times + + c + c + + \times + + \times + + c + \]

But, again, only in the special case of type ABCD this is known.
A version of Schur’s remarkable duality.

\[
\mathcal{U}_v(\mathfrak{sl}_2) \ominus V_v \otimes \cdots \otimes V_v \ominus \mathcal{H}_v(A) \\
\bigcup \quad \quad \quad \quad \quad \bigcap \\
\mathcal{c} \mathcal{U}_v(\mathfrak{gl}_1) \ominus V_v \otimes \cdots \otimes V_v \ominus \mathcal{H}_v(D) \rtimes \mathbb{Z}/2\mathbb{Z}
\]

Message to take away. Coideal naturally appear in Schur–Weyl-like games. And these pull the strings from the background for tangle and link invariants.