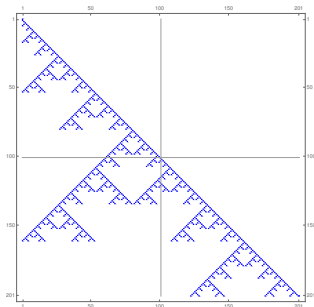


Why (modular) representation theory?

Or: Fractals and SL_2



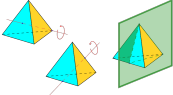

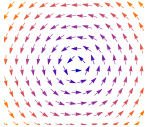
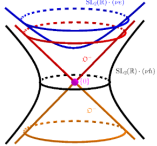
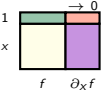
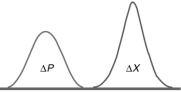
Daniel Tubbenhauer



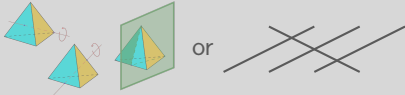
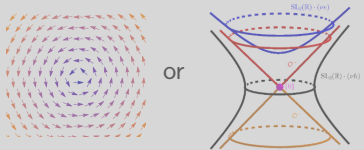
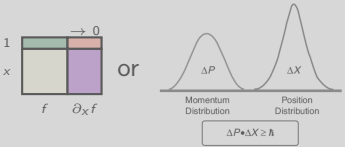
Based on joint work with Lousie Sutton, Paul Wedrich, Jieru Zhu

February 2021

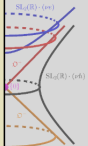

Abstract vs. real life

	Abstract	Incarnation
Numbers	3	 or  or...
Finite groups	$S_4 = \langle s, t, u \mid \text{some relations} \rangle$	 or  or...
Lie groups	$SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}$	 or  or...
More (Lie algebras, algebras, categories...)	$W = \langle X, Y \mid XY = YX + 1 \rangle$	 OR  OR... <div style="border: 1px solid black; padding: 2px; width: fit-content; margin: 10px auto;"> $\Delta P \cdot \Delta X \geq \hbar$ </div>

Abstract vs. real life

	Abstract	Incarnation
Numbers	<div style="border: 1px solid black; background-color: #ffffcc; padding: 10px; text-align: center;"> <p>People and objects are eventually known by their actions.</p> <p>Representation theory studies the right-hand side using the power of linear algebra.</p> </div>	
Finite groups	$S_4 = \langle s, t, u \mid \text{some relations} \rangle$	
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More (Lie algebras, algebras, categories...)	$W = \langle X, Y \mid XY = YX + 1 \rangle$	

Abstract vs. real life

	Abstract	Incarnation
Numbers	<p>People and objects are eventually known by their actions.</p> <p>Representation theory studies the right-hand side using the power of linear algebra.</p>	or...
Finite groups	<p>The representation theory approach.</p> <p>Reduce a non-linear problem to questions in linear algebra.</p>	or...
Lie groups	<p>Problem involving an action</p> $G \curvearrowright X$ <p>new insights?</p>	<p>Problem involving a linear action</p> $\mathbb{K}[G] \curvearrowright \mathbb{K}X$ <p>or...</p> 
More (Lie algebras, algebras, categories...)	<p>Decomposition of the problem into simple/elements</p>	<p>or...</p> 

What are modules?

Frobenius $\sim 1895++$, **Burnside** $\sim 1900++$. ▶ Representation theory is the ▶ useful? study of linear actions of G (a finite group, a reductive group, an algebra...)

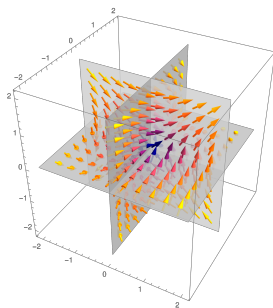
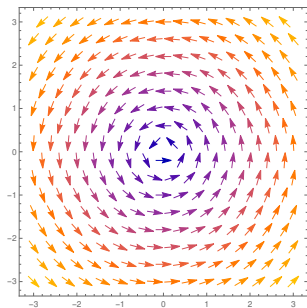
$$\mathcal{M}: G \longrightarrow \mathcal{E}\text{nd}(V),$$

with V being some vector space. (Called modules or representations.)

Examples.

$$\text{SL}_2(\mathbb{R}) \rightarrow \mathcal{E}\text{nd}(\mathbb{R}^2), \text{ e.g. } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\text{SL}_2(\mathbb{R}) \rightarrow \mathcal{E}\text{nd}(\mathbb{R}^3), \text{ e.g. } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



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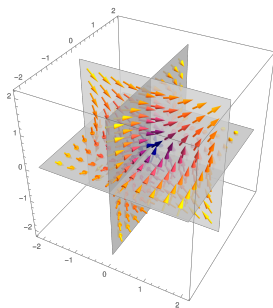
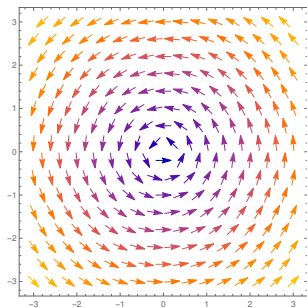
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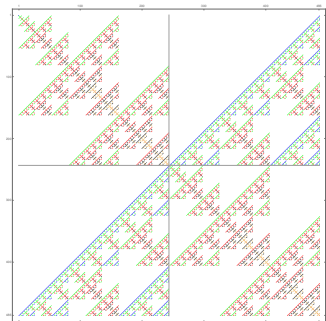
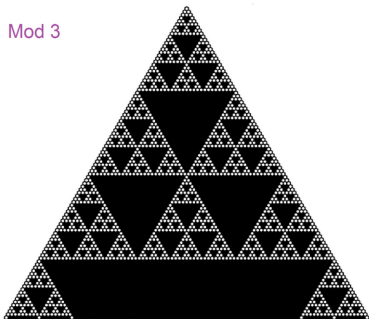


Question. What can we say about finite-dimensional modules of $SL_2...$

- ...in the context of the representation theory of classical groups? \rightsquigarrow The modules and their structure.
- ...in the context of the representation theory of Hopf algebras? \rightsquigarrow Fusion rules *i.e.* tensor products rules.
- ...in the context of categories? \rightsquigarrow Morphisms of representations and their structure. (Not today – time, in general, flies!)

The most amazing things happen if the characteristic of the underlying field $\mathbb{K} = \overline{\mathbb{K}}$ of $SL_2 = SL_2(\mathbb{K})$ is finite, and we will see (inverse) fractals, e.g.

Mod 3



Question. What can we say about finite-dimensional modules of SL_2 ...

- ...in the context of the representation theory of classical groups? \rightsquigarrow The modular representation theory of SL_2 is much harder than classical one (char ∞ a.k.a. char 0) because secretly we are doing fractal geometry.

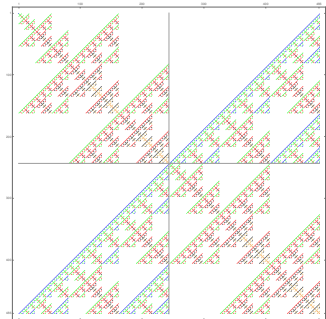
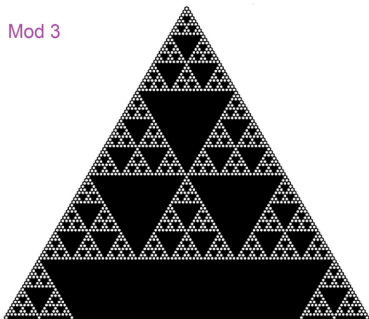
Spoiler: What will be the take away?

Well, in some sense modular (char $p < \infty$) representation theory is so much harder than classical one (char ∞ a.k.a. char 0) because secretly we are doing fractal geometry.

In my toy example SL_2 we can do everything explicitly.

The most amazing things happen if the characteristic of the underlying field $\mathbb{K} = \overline{\mathbb{K}}$ of $SL_2 = SL_2(\mathbb{K})$ is finite, and we will see (inverse) fractals, e.g.

Mod 3



Weyl ~ 1923 . The SL_2 (dual) Weyl modules $\Delta(v-1)$.

$\Delta(1-1)$

$x^0 y^0$

$\Delta(2-1)$

$x^1 y^0 \quad x^0 y^1$

$\Delta(3-1)$

$x^2 y^0 \quad x^1 y^1 \quad x^0 y^2$

$\Delta(4-1)$

$x^3 y^0 \quad x^2 y^1 \quad x^1 y^2 \quad x^0 y^3$

$\Delta(5-1)$

$x^4 y^0 \quad x^3 y^1 \quad x^2 y^2 \quad x^1 y^3 \quad x^0 y^4$

$\Delta(6-1)$

$x^5 y^0 \quad x^4 y^1 \quad x^3 y^2 \quad x^2 y^3 \quad x^1 y^4 \quad x^0 y^5$

$\Delta(7-1)$

$x^6 y^0 \quad x^5 y^1 \quad x^4 y^2 \quad x^3 y^3 \quad x^2 y^4 \quad x^1 y^5 \quad x^0 y^6$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$ matrix whose rows are expansions of $(aX + cY)^{v-i} (bX + dY)^{i-1}$.

► The simples

Example $\Delta(7-1) = \mathbb{K}X^6Y^0 \oplus \dots \oplus \mathbb{K}X^0Y^6$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts as

a^6	$6a^5c$	$15a^4c^2$	$20a^3c^3$	$15a^2c^4$	$6ac^5$	c^6
a^5b	$5a^4bc + a^5d$	$10a^3b^2c^2 + 5a^4cd$	$10a^2bc^3 + 10a^3c^2d$	$5abc^4 + 10a^2c^3d$	$bc^5 + 5ac^4d$	c^5d
a^4b^2	$4a^3b^2c + 2a^4bd$	$6a^2b^2c^2 + 8a^3bcd + a^4d^2$	$4ab^2c^3 + 12a^2b^2cd + 4a^3c^2d^2$	$b^2c^4 + 8abc^3d + 6a^2c^2d^2$	$2bc^4d + 4ac^3d^2$	c^4d^2
a^3b^3	$3a^2b^3c + 3a^3b^2d$	$3ab^3c^2 + 9a^2b^2cd + 3a^3b^2d^2$	$b^3c^3 + 9ab^2c^2d + 9a^2bcd^2 + a^3d^3$	$3b^2c^3d + 9abc^2d^2 + 3a^2cd^3$	$3bc^3d^2 + 3ac^2d^3$	c^3d^3
a^2b^4	$2ab^4c + 4a^2b^3d$	$b^4c^2 + 8ab^3cd + 6a^2b^2d^2$	$4b^3c^2d + 12ab^2c^2d + 4a^2bd^3$	$6b^2c^2d^2 + 8abc^2d^3 + a^2d^4$	$4bc^2d^3 + 2acd^4$	c^2d^4
ab^5	$b^5c + 5ab^4d$	$5b^4cd + 10ab^3d^2$	$10b^3cd^2 + 10ab^2d^3$	$10b^2cd^3 + 5abd^4$	$5bc^4d + ad^5$	cd^5
b^6	$6b^5d$	$15b^4d^2$	$20b^3d^3$	$15b^2d^4$	$6bd^5$	d^6

The rows are expansions of $(aX + cY)^{7-i}(bX + dY)^{i-1}$. Binomials!

$$\Delta(3-1) \quad X^2Y^0 \quad X^1Y^1 \quad X^0Y^2$$

$$\Delta(4-1) \quad X^3Y^0 \quad X^2Y^1 \quad X^1Y^2 \quad X^0Y^3$$

$$\Delta(5-1) \quad X^4Y^0 \quad X^3Y^1 \quad X^2Y^2 \quad X^1Y^3 \quad X^0Y^4$$

$$\Delta(6-1) \quad X^5Y^0 \quad X^4Y^1 \quad X^3Y^2 \quad X^2Y^3 \quad X^1Y^4 \quad X^0Y^5$$

$$\Delta(7-1) \quad X^6Y^0 \quad X^5Y^1 \quad X^4Y^2 \quad X^3Y^3 \quad X^2Y^4 \quad X^1Y^5 \quad X^0Y^6$$

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a^4b^2	$4a^3b^2c + 2a^4bd$	$6a^2b^2c^2 + 8a^3bcd + a^4d^2$	$4ab^2c^3 + 12a^2b^2cd + 4a^3c^2d^2$	$b^2c^4 + 8abc^3d + 6a^2c^2d^2$	$2bc^4d + 4ac^3d^2$	c^4d^2
a^3b^3	$3a^2b^3c + 3a^3b^2d$	$3ab^3c^2 + 9a^2b^2cd + 3a^3b^2d^2$	$b^3c^3 + 9ab^2c^2d + 9a^2bc^2d^2 + a^3d^3$	$3b^2c^3d + 9abc^2d^2 + 3a^2cd^3$	$3bc^3d^2 + 3ac^2d^3$	c^3d^3
a^2b^4	$2ab^4c + 4a^2b^3d$	$b^4c^2 + 8ab^3cd + 6a^2b^2d^2$	$4b^3c^2d + 12ab^2c^2d^2 + 4a^2bd^3$	$6b^2c^2d^2 + 8abc^2d^3 + a^2d^4$	$4bc^2d^3 + 2acd^4$	c^2d^4
ab^5	$b^5c + 5ab^4d$	$5b^4cd + 10ab^3d^2$	$10b^3cd^2 + 10ab^2d^3$	$10b^2cd^3 + 5abd^4$	$5bc^4d^4 + ad^5$	cd^5
b^6	$6b^5d$	$15b^4d^2$	$20b^3d^3$	$15b^2d^4$	$6bd^5$	d^6

The rows are expansions of $(aX + cY)^{7-i}(bX + dY)^{i-1}$. Binomials!

$\Delta(3-1)$

X^2Y^0

X^1Y^1

X^0Y^2

Example $\Delta(7-1)$, characteristic 0.

No common eigensystem $\Rightarrow \Delta(7-1)$ simple.

Example $\Delta(7-1)$, characteristic 2.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts as

a^6	0	a^4c^2	0	a^2c^4	0	c^6
a^5b	$a^4bc + a^5d$	a^4cd	0	abc^4	$b^5c^5 + ac^4d$	c^5d
a^4b^2	0	a^4d^2	0	b^2c^4	0	c^4d^2
a^3b^3	$a^2b^3c + a^3b^2d$	$ab^3c^2 + a^2b^2cd + a^3bd^2$	$b^3c^3 + ab^2c^2d + a^2bc^2d^2 + a^3d^3$	$b^2c^3d + abc^2d^2 + a^2cd^3$	$bc^3d^2 + ac^2d^3$	c^3d^3
a^2b^4	0	b^4c^2	0	a^2d^4	0	c^2d^4
ab^5	$b^5c + ab^4d$	b^4cd	0	abd^4	$bc^4d^4 + ad^5$	cd^5
b^6	0	b^4d^2	0	b^2d^4	0	d^6

$(0, 0, 0, 1, 0, 0, 0)$ is a common eigenvector, so we found a submodule.

► The simples

Weyl ~ 1923 . The SL_2 (du **When is $\Delta(v-1)$ simple?**).

$\Delta(1-1)$

$\Delta(v-1)$ is simple

$\Delta(2-1)$

\Leftrightarrow

$\Delta(3-1)$

$\binom{v-1}{w-1} \neq 0$ for all $w \leq v$

$\Delta(4-1)$

\Leftrightarrow (Lucas's theorem)

$v = [a_r, 0, \dots, 0]_p$.

$\Delta(5-1)$

Lucas ~ 1878 .

$\Delta(6-1)$

"Binomials mod p are the product of binomials of the p -adic digits":

$$\binom{a}{b} = \prod_{i=0}^r \binom{a_i}{b_i} \pmod{p},$$

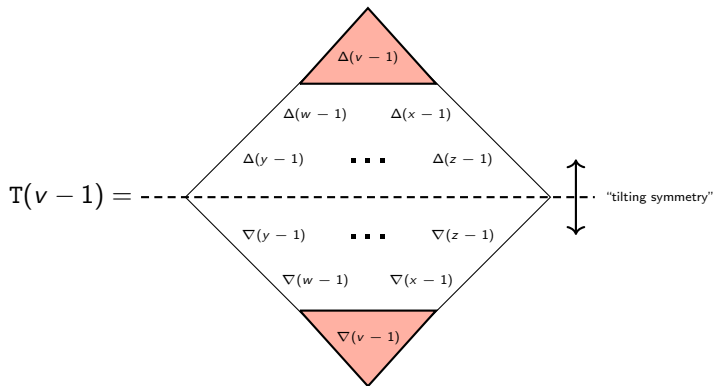
$\Delta(7-1)$

where $a = [a_r, \dots, a_0]_p = \sum_{i=0}^r a_i p^i$ etc.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$ matrix whose rows are expansions of $(aX + cY)^{v-i} (bX + dY)^{i-1}$.

Ringel, Donkin ~1991. There is a class of modules $T(v-1)$ indexed by \mathbb{N} . They are a bit tricky to define, but:

- They have Δ - and ∇ filtrations, which look the same if you tilt your head:



- Play the role of projective modules.
- $T(v-1) \cong L(v-1) \cong \Delta(v-1) \cong \nabla(v-1)$ over \mathbb{C} .
- They are much more well-behaved than simples. [▶ Analogy](#)

Ringel, Don: 1991. They are a bit tricky by \mathbb{N} . They

- They have

How many Weyl factors does $T(v-1)$ have?

Weyl factors of $T(v-1)$ is 2^k where

$$k = \max\{\nu_p\left(\binom{v-1}{w-1}\right), w \leq v\}. \text{ (Order of vanishing of } \binom{v-1}{w-1}\text{.)}$$

determined by (Lucas's theorem)

non-zero non-leading digits of $v = [a_r, a_{r-1}, \dots, a_0]_p$.

your head:

$T(v$

Example $T(220540-1)$ for $p = 11$?

$$v = 220540 = [1, 4, 0, 7, 7, 1]_{11};$$

Maximal vanishing for $w = 75594 = [0, 5, 1, 8, 8, 2]_{11};$

$$\binom{v-1}{w-1} = (\text{HUGE}) = [\dots, \neq 0, 0, 0, 0, 0]_{11}.$$

$\Rightarrow T(220540-1)$ has 2^4 Weyl factors.

filtering symmetry"

- Play the role
- $T(v-1) \cong L(v-1) = \Delta(v-1) = v(v-1)$ over \mathbb{C} .
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▶ Analogy

Ringel, Donkin ~1991. There is a class of modules $T(v-1)$ indexed by \mathbb{N} . They are a bit tricky to define, but:

- They have Δ - and ∇ filtrations, which look the same if you tilt your head:

Which Weyl factors does $T(v-1)$ have a.k.a. the negative digits game?

Weyl factors of $T(v-1)$ are

$\Delta([a_r, \pm a_{r-1}, \dots, \pm a_0]_p - 1)$ where $v = [a_r, \dots, a_0]_p$ (appearing exactly once).

$\Delta(y-1) \quad \dots \quad \Delta(z-1)$

$T(v)$ ↑ tilting symmetry"

Example $T(220540-1)$ for $p = 11$?

$v = 220540 = [1, 4, 0, 7, 7, 1]_{11};$

has Weyl factors $[1, \pm 4, 0, \pm 7, \pm 7, \pm 1]_{11};$

e.g. $\Delta(218690 = [1, 4, 0, -7, -7, -1]_{11} - 1)$ appears.

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▶ Analogy

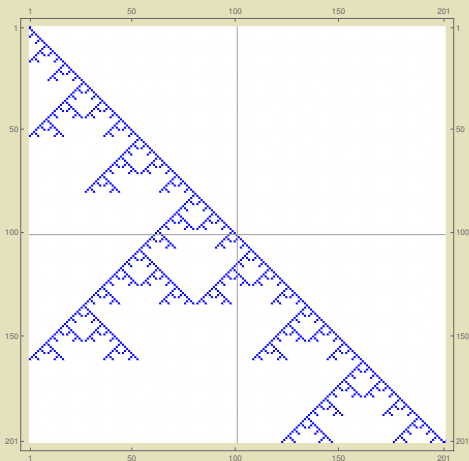
Ringel, Donkin The tilting-Cartan matrix a.k.a. $(T(v-1) : \Delta(w-1))?$ defined by \mathbb{N} . They are a bit tricky

- They have

at your head:

$T(v$

ing symmetry"



This is characteristic 3.

- Play the
- $T(v-1) \cong L(v-1) \cong \Delta(v-1) \cong V(v-1)$ over \mathbb{C} .
- They are much more well-behaved than simples.

▶ Analogy

General.
These facts hold in general, and
tilting modules form the “nicest possible” monoidal subcategory.

Tilting modules form a braided monoidal category \mathcal{Tilt} .

Simple \otimes simple \neq simple, Weyl \otimes Weyl \neq Weyl, but tilting \otimes tilting = tilting.

The Grothendieck algebra $[\mathcal{Tilt}]$ of \mathcal{Tilt} is a commutative algebra with basis $[T(v-1)]$. So what I would like to answer on the object level, *i.e.* for $[\mathcal{Tilt}]$:

- What are the fusion rules? [▶ Answer](#)
- Find the $N_{v,w}^x \in \mathbb{N}[0]$ in $T(v-1) \otimes T(w-1) \cong \bigoplus_x N_{v,w}^x T(x-1)$.
 - ▷ For $[\mathcal{Tilt}]$ this means finding the structure constants.
- What are the thick \otimes -ideals? [▶ Answer](#)
 - ▷ For $[\mathcal{Tilt}]$ this means finding the ideals.

Tilting modules form a braided monoidal category \mathcal{Tilt} .

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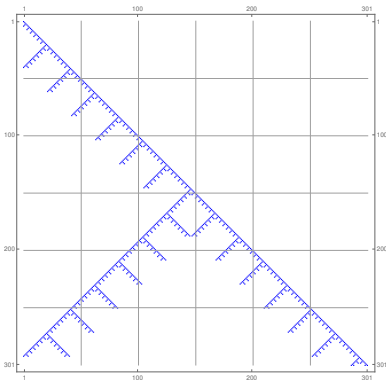
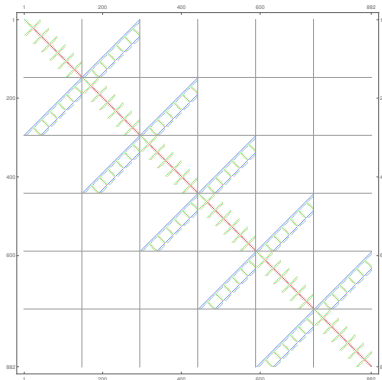
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




All of this generalizes to...

- ...higher ranks, e.g. SL_3 , where higher dimensional fractals show up. (We are very far away from understanding this!)
- ...quantum groups, e.g. quantum SL_2 , where “distorted” fractals show up. (We do understand this!)

Two distorted fractals:



Abstract vs. real life

	Abstract	Incarnation
Numbers	3	 OR  OR ...
Finite groups	$S_3 = \langle s, t \mid s^2 = t^2 = 1, st = ts \rangle$	 OR  OR ...
Lin groups	$SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}$	 OR  OR ...
More (See algebraic geometry.)	$W = \langle X, Y \mid XY = YX \rangle$	 OR  OR ...

David Tannenbaum Why (modular) representation theory? February 2016 9/19



Figure: The map of mathematics. My home (solid) and what I like to study via representations (dashed).

David Tannenbaum Why (modular) representation theory? February 2016 9/19

Rings, Domains, The **sliding Cartan matrix** a.k.a. $(\mathbb{T}(n-1) \Delta(n-1))$ by N. They are a bit tricky

- They live in your head.

- Play this
- $\mathbb{T}(n-1) \Delta(n-1) \cong \Delta(n, n-1) \cong \mathbb{T}(n-1) \Delta(n-1)$ This is characteristic 3.
- They are much more well-behaved than simples.

David Tannenbaum Why (modular) representation theory? February 2016 9/19

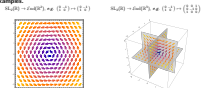
What are modules?

Frobenius – 1895+, Burnside – 1900+. **Representation theory** is the study of linear actions of G (a finite group, a reductive group, an algebra...)

$$M: G \rightarrow GL(V),$$

with V being some vector space. (Called **modules** or **representations**.)

Examples.

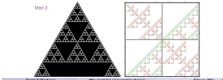


David Tannenbaum Why (modular) representation theory? February 2016 9/19

Question. What can we say about finite-dimensional modules of SL_2 ?

- ...in the context of the representation theory of classical groups? → The modules and their structure.
- ...in the context of the representation theory of Hopf algebras? → Fusion rules i.e. tensor products rules.
- ...in the context of categories? → Morphisms of representations and their structure. (Not today – time, in general, fine)

The most amazing things happen if the characteristic of the underlying field $K = \mathbb{F}$ of $SL_2 = SL_2(K)$ is finite, and we will see (inverse) fractals, e.g.



David Tannenbaum Why (modular) representation theory? February 2016 9/19

\mathfrak{sl}_2 -ideals of $\mathbb{F}[t]$ are indexed by prime powers.

- Every \mathfrak{sl}_2 -ideal is thick, and any non-zero thick \mathfrak{sl}_2 -ideal is of the form $\mathcal{J}_p = \langle t^p - 1 \rangle$ with $p \geq 1$.
- There is a chain of \mathfrak{sl}_2 -ideals $\mathbb{F}[t] \supset \mathcal{J}_2 \supset \mathcal{J}_3 \supset \mathcal{J}_4 \supset \dots$. The cells, i.e. $\mathcal{J}_p / \mathcal{J}_{p+1}$, are the strongly connected components of $\mathbb{F}[t]$.

Example ($p = 3$).



David Tannenbaum Why (modular) representation theory? February 2016 9/19

It may then be asked why is a book which pretends to have an application to one field a worthwhile topic to discuss to mathematicians from other particular fields of representation theory? In fact, the answer is that this is the power view of our knowledge, they realize in the process they are not so much ready to study with precision of abstract groups, it is not difficult to find a book that will be more deeply oriented by the combination of groups of finite order.

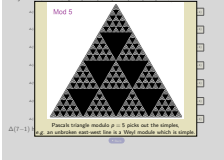
For the module theory in the theory of groups of finite order, one has to look into the representation theory of finite groups. It is possible that some of the groups in the original paper for writing my version of it is too large to read.

In fact it is not even true to say that for finite groups in the theory of finite order, one has to look into the representation theory of finite groups. It is possible that some of the groups in the original paper for writing my version of it is too large to read.

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

David Tannenbaum Why (modular) representation theory? February 2016 9/19

Weyl – 1923. The SL_2 simple $(n-1) \times (n-1)$ for $p = 5$.



David Tannenbaum Why (modular) representation theory? February 2016 9/19

Prime power Verma categories.

- The ideal $\mathcal{J}_p \subset \mathbb{F}[t]$ is the cell of projection.
- The subalgebras $U_{\mathcal{J}_p}$ of $U(\mathfrak{sl}_2)$ are called Verma categories.
- The Cartan matrix of $U_{\mathcal{J}_p}$ is a $p^2 \times p^2$ -square matrix.
- Both entries give by the common Weyl factors of $\mathbb{T}(n-1)$ and $\mathbb{T}(n-1)$.

Example ($p = 3$).



David Tannenbaum Why (modular) representation theory? February 2016 9/19

There is still much to do...

Abstract vs. real life

	Abstract	Incarnation
Numbers	3	
Finite groups	$S_3 = \{s, t, st\}$ (some relations)	
Lin groups	$SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}$	
More (See algebraic algebraic categories...)	$W = \langle X, Y \mid XY = YX \rangle$	



Figure: The map of mathematics. My home (red) and what I like to study via representations (dashed).

Ringsel, Dots The **tiling-Cartan matrix** a.k.a. $(\mathbb{T}(n-1), \Delta(n-1))$ by N. They are a bit tricky

- They live in your head.

- Play this
- $\mathbb{T}(n-1) = \mathbb{T}(n-1) - \Delta(n-1) \cong \mathbb{T}(n-1) \otimes \mathbb{T}(n-1)$ over \mathbb{Z} .
- They are much more well-behaved than simples.

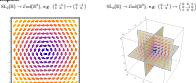
What are modules?

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$$M: G \rightarrow \text{Lin}(V),$$

with V being some vector space. (Called **modules** or **representations**.)

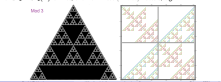
Examples.



Question. What can we say about finite-dimensional modules of SL_2 ?

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The most amazing things happen if the characteristic of the underlying field $K = \mathbb{C}$ of $SL_2 = SL_2(K)$ is finite, and we will see (inverse) fractals, e.g.



◻-ideals of \mathbb{T} : are indexed by prime powers.

- Every \mathfrak{p} -ideal is thick, and any non-zero thick \mathfrak{p} -ideal is of the form $\mathcal{J}_p = (\mathbb{T}(p-1) \mid v \geq p^k)$.
- There is a chain of \mathfrak{p} -ideals $\mathbb{T} \supset \mathcal{J}_p \supset \mathcal{J}_{p^2} \supset \dots$. The cells, i.e. $\mathcal{J}_{p^k} \setminus \mathcal{J}_{p^{k+1}}$, are the strongly connected components of \mathbb{T} .

Example ($p = 3$).

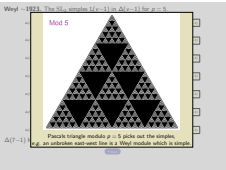


It may then be asked why is a book which pretends to have an application to one with a non-trivial open to research in mathematics groups which other mathematicians of representation theory are not interested in. My answer to this question is that this is the power of the knowledge they have in the area they are asked to work on. They are not really interested in the problem of representation theory, but they are interested in the problem of representation theory of finite groups. This is not a new discovery by the combination of groups of finite order.

Very readable reference in the theory of groups of finite order. It has been used as the reference of the author of this book. It contains a lot of information and is very readable. It is a good reference for the original papers for reading any version of it is very hard to find.

In fact it is not even true to say that the author is interested in the theory of finite groups. It is the representation theory of finite groups which is the subject of his research.

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).



Prime power VerKlede categories.

- The ideal $\mathcal{J}_p \subset \mathbb{T} \setminus \mathcal{J}_{p^2}$ is the cell of projection.
- The subalgebras \mathbb{T}_{p^k} of \mathbb{T} , \mathcal{J}_{p^k} are called VerKlede categories.
- The Cartan matrix of \mathbb{T}_{p^k} is a $p^k \times p^k$ -square matrix.
- Both entries give by the common Weyl factors of $\mathbb{T}(n-1)$ and $\mathbb{T}(n-1)$.

Example ($p = 3$).



Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

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VERY considerable advances in the theory of groups of

I will however take a different stance:

Representations are sometimes more interesting than groups.

Today. SL_2 (easy) vs. its representations (fun).

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a biased and not fully faithful map
of pure mathematics

(based on a map by
Alex Sarlin and
Innokentij Zotov)

this lives on
a torus

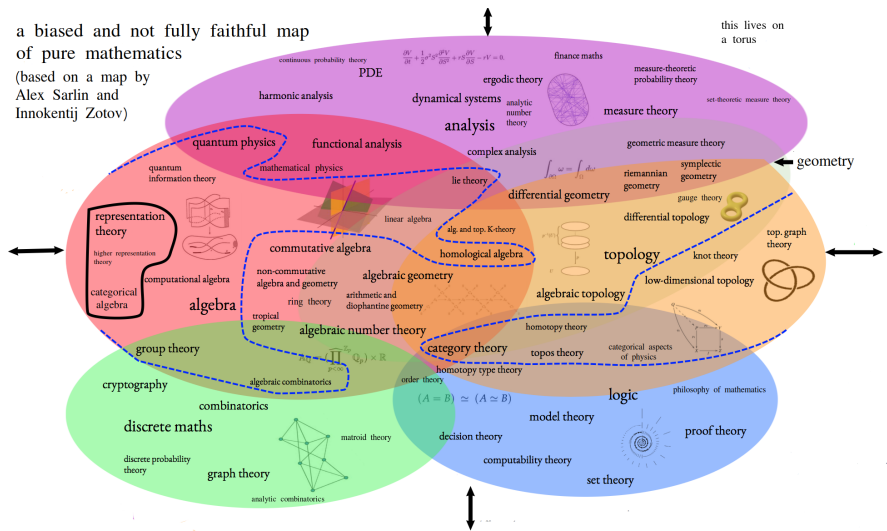


Figure: The map of mathematics. My home (solid) and what I like to study via representations (dashed).

Weyl ~ 1923 . The SL_2 simples $L(v-1)$ in $\Delta(v-1)$ for $p = 5$.

$$\Delta(1-1) \quad x^0 y^0 \quad L(1-1)$$

$$\Delta(2-1) \quad x^1 y^0 \quad x^0 y^1 \quad L(2-1)$$

$$\Delta(3-1) \quad x^2 y^0 \quad x^1 y^1 \quad x^0 y^2 \quad L(3-1)$$

$$\Delta(4-1) \quad x^3 y^0 \quad x^2 y^1 \quad x^1 y^2 \quad x^0 y^3 \quad L(4-1)$$

$$\Delta(5-1) \quad x^4 y^0 \quad x^3 y^1 \quad x^2 y^2 \quad x^1 y^3 \quad x^0 y^4 \quad L(5-1)$$

$$\Delta(6-1) \quad x^5 y^0 \quad x^4 y^1 \quad x^3 y^2 \quad x^2 y^3 \quad x^1 y^4 \quad x^0 y^5 \quad L(6-1)$$

$$\Delta(7-1) \quad x^6 y^0 \quad x^5 y^1 \quad x^4 y^2 \quad x^3 y^3 \quad x^2 y^4 \quad x^1 y^5 \quad x^0 y^6 \quad L(7-1)$$

$\Delta(7-1)$ has (its head) $L(7-1)$ and $L(3-1)$ as factors.

Weyl ~ 1923 . The SL_2 simples $L(\nu-1)$ in $\Delta(\nu-1)$ for $p = 5$.

$\Delta(1)$

Mod 5

$-1)$

$\Delta(2)$

$-1)$

$\Delta(3)$

$-1)$

$\Delta(4)$

$-1)$

$\Delta(5)$

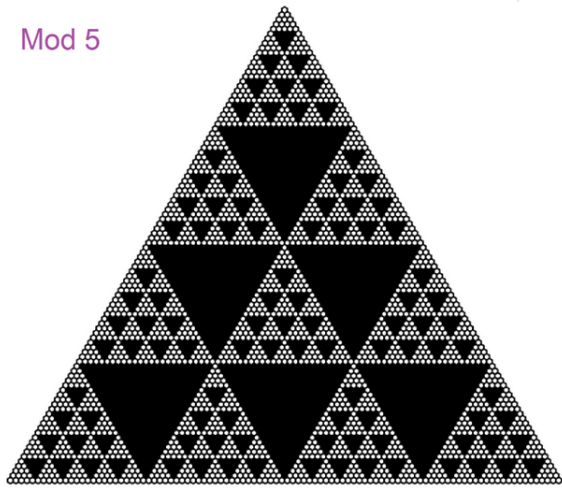
$-1)$

$\Delta(6)$

$-1)$

$\Delta(7)$

$-1)$



$\Delta(7-1)$ h

Pascal's triangle modulo $p = 5$ picks out the simples, e.g. an unbroken east-west line is a Weyl module which is simple.

← Back

Two notions of “elements”

No substructure	Does not decompose
Simplex	Indecomposables
$(*) V \subset L \Rightarrow V \cong 0 \text{ or } V \cong L$	$T \cong V \oplus W \Rightarrow V \cong 0 \text{ or } V \cong T$

Both are legit elements of which one would like a periodic table.

G finite group, $\mathbb{K}[G]$ the regular module (G acting on itself).

No substructure	Does not decompose
Simplex	Projective indecomposables
$(*)$	\oplus -summands of $\mathbb{K}[G]$

SL_2 , $\Delta(1)$ the regular module (matrices acting by matrices).

No substructure	Does not decompose
Simplex	Tilting modules
$(*)$	\oplus -summands of $\Delta(1)^{\otimes k}$

Two notions of “elements”

No substructure	Does not decompose
Simples	Indecomposables
(*) $V \subset L \Rightarrow V \cong 0$ or $V \cong L$	$T \cong V \oplus W \Rightarrow V \cong 0$ or $V \cong T$

Both are legit elements of which periodic table.

G finite group, $\mathbb{K}[G]$ the regular module (in itself).

In good cases:
Simple=indecomposable
but not always.

No substructure	Does not decompose
Simples	Projective indecomposables
(*)	\oplus -summands of $\mathbb{K}[G]$

SL_2 , $\Delta(1)$ the regular module (matrices acting by matrices).

No substructure	Does not decompose
Simples	Tilting modules
(*)	\oplus -summands of $\Delta(1)^{\otimes k}$

Fusion graphs.

The fusion graph $\Gamma_v = \Gamma_{T(v-1)}$ of $T(v-1)$ is:

- Vertices of Γ_v are $w \in \mathbb{N}$, and identified with $T(w-1)$.
 - k edges $w \xrightarrow{k} x$ if $T(x-1)$ appears k times in $T(v-1) \otimes T(w-1)$.
 - $T(v-1)$ is a \otimes -generator if Γ_v is strongly connected.
 - This works for any reasonable monoidal category, with vertices being indecomposable objects and edges count multiplicities in \otimes -products.
-

Baby example. Assume that we have two indecomposable objects $\mathbb{1}$ and X , with $X^{\otimes 2} = \mathbb{1} \oplus X$. Then:

$$\Gamma_{\mathbb{1}} = \begin{array}{c} \curvearrowright \mathbb{1} \\ \text{not a } \otimes\text{-generator} \end{array}, \quad \Gamma_X = \begin{array}{c} X \curvearrowright \\ \mathbb{1} \rightleftarrows X \curvearrowright \\ \text{a } \otimes\text{-generator} \end{array}$$

Fusion graphs.

The fusion graph Γ

- Vertices of Γ_v
- k edges $w \xrightarrow{k}$
- $T(v-1)$ is a \otimes
- This works for indecomposable

Baby example. As $X^{\otimes 2} = \mathbb{1} \oplus X$. Then

$\Gamma_{\mathbb{1}}$

The fusion graph of $T(1) \cong \mathbb{K}^2$ for $p = \infty$:

The fusion graph of $T(1) \cong \mathbb{K}^2$ for $p = 2$:

$) \otimes T(w-1)$.

vertices being in \otimes -products.

objects $\mathbb{1}$ and X , with

$X \curvearrowright$

tor

Fusion graphs.

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Γ_1

The fusion graph of $T(1) \cong \mathbb{K}^2$ for $p = \infty$:

The fusion graph of $T(1) \cong \mathbb{K}^2$ for $p = 2$:

$\otimes T(w-1)$.

In general, there is are cycles of length p with edges jumping $1 = p^0, p^1, p^2, \dots$, units, reaping every $1 = p^0, p^1, p^2, \dots$, steps.

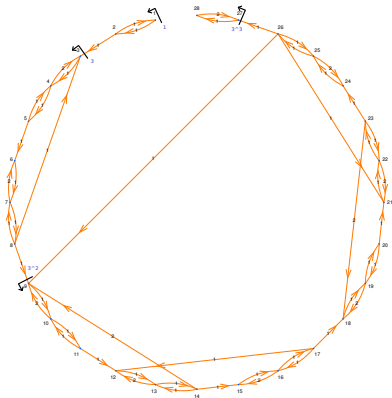
[◀ Back](#)

\otimes -ideals of \mathcal{T}_{ilt} are indexed by prime powers.

Thick \otimes -ideal = generated by identities on objects.
 \otimes -ideal = generated by any sets of morphism.

- Every \otimes -ideal is thick, and any non-zero thick \otimes -ideal is of the form $\mathcal{J}_{p^k} = \{\mathbb{T}(v-1) \mid v \geq p^k\}$.
- There is a chain of \otimes -ideals $\mathcal{T}_{\text{ilt}} = \mathcal{J}_1 \supset \mathcal{J}_p \supset \mathcal{J}_{p^2} \supset \dots$. The cells, i.e. $\mathcal{J}_{p^k} / \mathcal{J}_{p^{k+1}}$, are the strongly connected components of Γ_1 .

Example ($p = 3$).



⊗-ide

Prime power Verlinde categories.

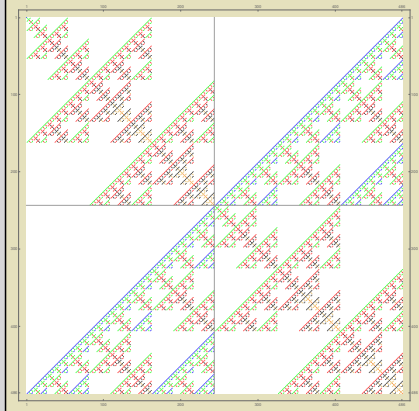
The ideal $\mathcal{J}_{p^k} \subset \mathcal{Tilt}/\mathcal{J}_{p^{k+1}}$ is the cell of projectives.

The abelianizations $\mathcal{V}_{\text{er}_{p^k}}$ of $\mathcal{Tilt}/\mathcal{J}_{p^{k+1}}$ are called Verlinde categories.

The Cartan matrix of $\mathcal{V}_{\text{er}_{p^k}}$ is a $p^k - p^{k-1}$ -square matrix with entries given by the common Weyl factors of $\mathbb{T}(v-1)$ and $\mathbb{T}(w-1)$.

$\mathcal{J}_{p^k}/\mathcal{J}_{p^{k+1}}$, are th

Example (Cartan matrix of $\mathcal{V}_{\text{er}_{3^4}}$).



Example ($p = 3$).