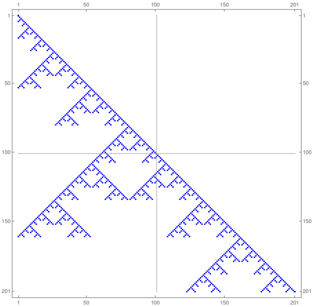
Why (modular) representation theory?

Or: Fractals and SL_2

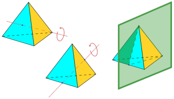
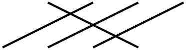
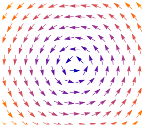
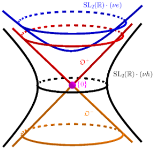
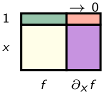
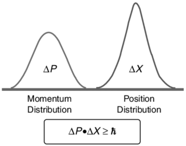
Daniel Tubbenhauer



Based on joint with Lousie Sutton, Paul Wedrich, Jieru Zhu

February 2021

Abstract vs. real life

	Abstract	Incarnation
Numbers	3	 or  or...
Finite groups	$\mathrm{S}_4 = \langle s, t, u \mid \text{some relations} \rangle$	 or  or...
Lie groups	$\mathrm{SL}_2 = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid ad - bc = 1 \}$	 or  or...
More (Lie algebras, algebras,) categories...)	$\mathrm{W} = \langle X, Y \mid XY = YX + 1 \rangle$	 or  or...

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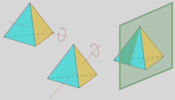
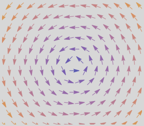
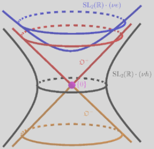
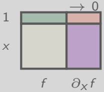
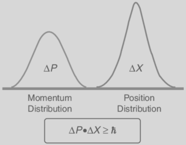
Finite groups	$\mathrm{S}_4 = \langle s, t, u \mid \text{some relations} \rangle$	 or  or...
Lie groups	$\mathrm{SL}_2 = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid ad - bc = 1 \}$	 or  or...
More (Lie algebras, algebras,) categories...)	$\mathrm{W} = \langle X, Y \mid XY = YX + 1 \rangle$	 or  or...

People and objects are eventually known by their actions.

Representation theory studies the right-hand side
using the power of linear algebra.

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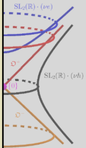
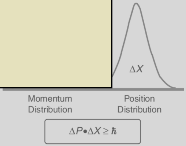
Lie groups		 or...
More (Lie algebras, algebras,) categories...)		f $\partial_X f$  or...

People and objects are eventually known by their actions.

Representation theory studies the right-hand side using the power of linear algebra.

The representation theory approach.

Reduce a non-linear problem to questions in linear algebra.

Problem involving an action $G \circlearrowright X$ $\xrightarrow{\text{"linearize"}}$ Problem involving a linear action $\mathbb{K}[G] \circlearrowright \mathbb{K}X$

$\longrightarrow$ Decomposition of the problem into simple/elements $\longrightarrow$ new insights? $\longrightarrow$ $G \circlearrowright X$

What are modules?

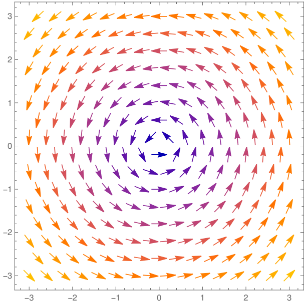
Frobenius ∼1895++, Burnside ∼1900++. ▸ Representation theory is the ▸ useful? study of linear actions of G (a finite group, a reductive group, an algebra...)

$$\mathcal{M}\colon \mathrm{G} \longrightarrow \mathcal{E}\mathrm{nd}(\mathtt{V}),$$

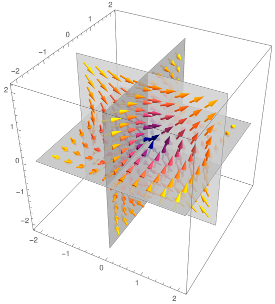
with $\mathtt{V}$ being some vector space. (Called modules or representations.)

Examples.

$\mathrm{SL}_2(\mathbb{R}) \to \mathcal{E}\mathrm{nd}(\mathbb{R}^2)$, *e.g.* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



$\mathrm{SL}_2(\mathbb{R}) \to \mathcal{E}\mathrm{nd}(\mathbb{R}^3)$, *e.g.* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$



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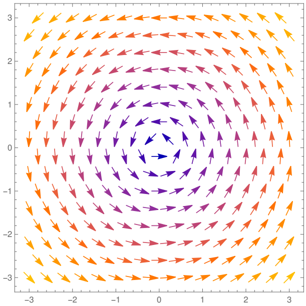

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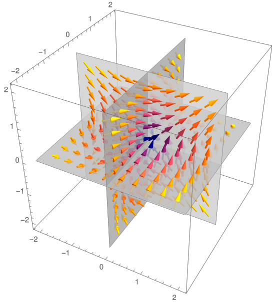
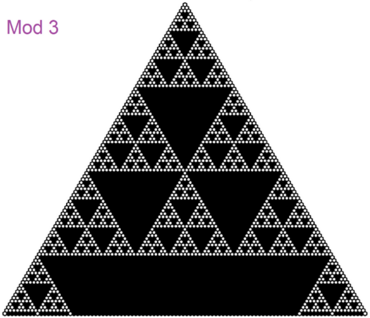
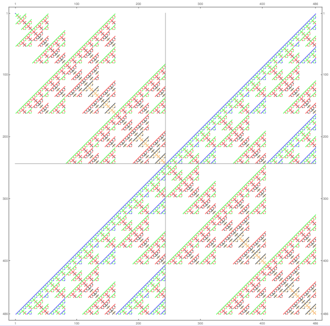


Question. What can we say about finite-dimensional modules of SL_2...

- ...in the context of the representation theory of classical groups? $\rightsquigarrow$ The modules and their structure.
- ...in the context of the representation theory of Hopf algebras? $\rightsquigarrow$ Fusion rules *i.e.* tensor products rules.
- ...in the context of categories? $\rightsquigarrow$ Morphisms of representations and their structure. (Not today – time, in general, flies!)

The most amazing things happen if the characteristic of the underlying field $\mathbb{K} = \overline{\mathbb{K}}$ of $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{K})$ is finite, and we will see (inverse) fractals, *e.g.*





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Spoiler: What will be the take away?

Well, in some sense modular (char $p < \infty$) representation theory so much harder than classical one (char ∞ a.k.a. char 0) because secretly we are doing fractal geometry.

In my toy example SL_2 we can do everything explicitly.

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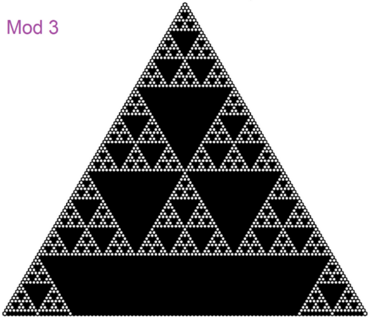

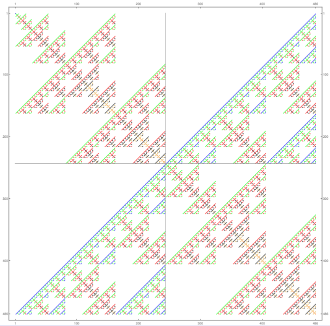

Weyl $\sim$1923. The SL_2 (dual) Weyl modules $\Delta(v-1)$.

$\Delta(1-1)$							X^0Y^0						
$\Delta(2-1)$						X^1Y^0		X^0Y^1					
$\Delta(3-1)$					X^2Y^0		X^1Y^1		X^0Y^2				
$\Delta(4-1)$				X^3Y^0		X^2Y^1		X^1Y^2		X^0Y^3			
$\Delta(5-1)$			X^4Y^0		X^3Y^1		X^2Y^2		X^1Y^3		X^0Y^4		
$\Delta(6-1)$		X^5Y^0		X^4Y^1		X^3Y^2		X^2Y^3		X^1Y^4		X^0Y^5	
$\Delta(7-1)$	X^6Y^0		X^5Y^1		X^4Y^2		X^3Y^3		X^2Y^4		X^1Y^5		X^0Y^6

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto$ matrix who's rows are expansions of $(aX+cY)^{v-i}(bX+dY)^{i-1}$.

The simples

Example $\Delta(7-1) = \mathbb{K}X^6Y^0 \oplus \cdots \oplus \mathbb{K}X^0Y^6$.

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts as

$$\begin{pmatrix}
a^6 & 6a^5c & 15a^4c^2 & 20a^3c^3 & 15a^2c^4 & 6ac^5 & c^6 \\
a^5b & 5a^4bc+a^5d & 10a^3bc^2+5a^4cd & 10a^2bc^3+10a^3c^2d & 5abc^4+10a^2c^3d & bc^5+5ac^4d & c^5d \\
a^4b^2 & 4a^3b^2c+2a^4bd & 6a^2b^2c^2+8a^3bcd+a^4d^2 & 4ab^2c^3+12a^2bc^2d+4a^3cd^2 & b^2c^4+8abc^3d+6a^2c^2d^2 & 2bc^4d+4ac^3d^2 & c^4d^2 \\
a^3b^3 & 3a^2b^3c+3a^3b^2d & 3ab^3c^2+9a^2b^2cd+3a^3bd^2 & b^3c^3+9ab^2c^2d+9a^2bcd^2+a^3d^3 & 3b^2c^3d+9abc^2d^2+3a^2cd^3 & 3bc^3d^2+3ac^2d^3 & c^3d^3 \\
a^2b^4 & 2ab^4c+4a^2b^3d & b^4c^2+8ab^3cd+6a^2b^2d^2 & 4b^3c^2d+12ab^2cd^2+4a^2bd^3 & 6b^2c^2d^2+8abcd^3+a^2d^4 & 4bc^2d^3+2acd^4 & c^2d^4 \\
ab^5 & b^5c+5ab^4d & 5b^4cd+10ab^3d^2 & 10b^3cd^2+10ab^2d^3 & 10b^2cd^3+5abd^4 & 5bcd^4+ad^5 & cd^5 \\
b^6 & 6b^5d & 15b^4d^2 & 20b^3d^3 & 15b^2d^4 & 6bd^5 & d^6
\end{pmatrix}$$

The rows are expansions of $(aX+cY)^{7-i}(bX+dY)^{i-1}$. Binomials!

$\Delta(3-1)$	X^2Y^0	X^1Y^1	X^0Y^2				
$\Delta(4-1)$	X^3Y^0	X^2Y^1	X^1Y^2	X^0Y^3			
$\Delta(5-1)$	X^4Y^0	X^3Y^1	X^2Y^2	X^1Y^3	X^0Y^4		
$\Delta(6-1)$	X^5Y^0	X^4Y^1	X^3Y^2	X^2Y^3	X^1Y^4	X^0Y^5	
$\Delta(7-1)$	X^6Y^0	X^5Y^1	X^4Y^2	X^3Y^3	X^2Y^4	X^1Y^5	X^0Y^6

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto$ matrix who's rows are expansions of $(aX+cY)^{v-i}(bX+dY)^{i-1}$.

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ acts as } \begin{pmatrix}
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\end{pmatrix}$$

The rows are expansions of $(aX + cY)^{7-i}(bX + dY)^{i-1}$. Binomials!

$\Delta(3{-}1)$ X^2Y^0 X^1Y^1 X^0Y^2

Example $\Delta(7{-}1)$, characteristic 0.

No common eigensystem $\Rightarrow$ $\Delta(7{-}1)$ simple.

Example $\Delta(7{-}1)$, characteristic 2.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ acts as } \begin{pmatrix}
a^6 & 0 & a^4c^2 & 0 & a^2c^4 & 0 & c^6 \\
a^5b & a^4bc+a^5d & a^4cd & 0 & abc^4 & bc^5+ac^4d & c^5d \\
a^4b^2 & 0 & a^4d^2 & 0 & b^2c^4 & 0 & c^4d^2 \\
a^3b^3 & a^2b^3c+a^3b^2d & ab^3c^2+a^2b^2cd+a^3bd^2 & b^3c^3+ab^2c^2d+a^2bcd^2+a^3d^3 & b^2c^3d+abc^2d^2+a^2cd^3 & bc^3d^2+ac^2d^3 & c^3d^3 \\
a^2b^4 & 0 & b^4c^2 & 0 & a^2d^4 & 0 & c^2d^4 \\
ab^5 & b^5c+ab^4d & b^4cd & 0 & abd^4 & bcd^4+ad^5 & cd^5 \\
b^6 & 0 & b^4d^2 & 0 & b^2d^4 & 0 & d^6
\end{pmatrix}$$

$(0, 0, 0, 1, 0, 0, 0)$ is a common eigenvector, so we found a submodule.

▸ The simples

Weyl $\sim$1923. The SL_2 (du).

When is $\Delta(v-1)$ simple?

$\Delta(v-1)$ is simple

$\Leftrightarrow$

$\binom{v-1}{w-1} \neq 0$ for all $w \leq v$

$\Leftrightarrow$ (Lucas's theorem)

$v = [a_r, 0, \ldots, 0]_p.$

$\Delta(1-1)$

$\Delta(2-1)$

$\Delta(3-1)$ Y^2

$\Delta(4-1)$ X^0Y^3

$\Delta(5-1)$ X^0Y^4

$\Delta(6-1)$ X^0Y^5

$\Delta(7-1)$ Y^5 X^0Y^6

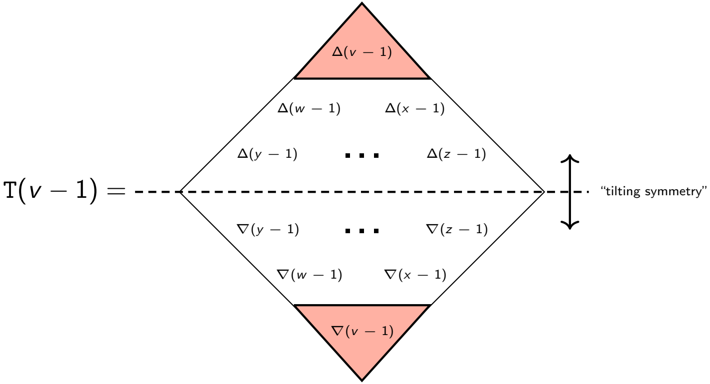
Lucas $\sim$1878.
"Binomials mod p are the product of binomials of the p-adic digits":
$\binom{a}{b} = \prod_{i=0}^{r} \binom{a_i}{b_i} \bmod p,$
where $a = [a_r, \ldots, a_0]_p = \sum_{i=0}^{r} a_i p^i$ *etc.*

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$ matrix who's rows are expansions of $(aX + cY)^{v-i}(bX + dY)^{i-1}$.

The simples

Ringel, Donkin ∼1991. There is a class of modules $\mathtt{T}(v-1)$ indexed by $\mathbb{N}$. They are a bit tricky to define, but:

- They have Δ- and ∇ filtrations, which look the same if you tilt your head:



- Play the role of projective modules.
- $\mathtt{T}(v-1) \cong \mathtt{L}(v-1) \cong \Delta(v-1) \cong \nabla(v-1)$ over $\mathbb{C}$.
- They are much more well-behaved than simples. ▸ Analogy

Ringel, Do[...] by $\mathbb{N}$. They are a bit tric[...]

- They ha[...] your head:

How many Weyl factors does $\mathtt{T}(v{-}1)$ have?

$\#$ Weyl factors of $\mathtt{T}(v{-}1)$ is 2^k where

$k = \max\{\nu_p(\binom{v-1}{w-1}), w \leq v\}$. (Order of vanishing of $\binom{v-1}{w-1}$.)

determined by (Lucas's theorem)

non-zero non-leading digits of $v = [a_r, a_{r-1}, ..., a_0]_p$.

$\mathtt{T}(v$ [...] ilting symmetry”

Example $\mathtt{T}(220540{-}1)$ for $p = 11$?

$v = 220540 = [1, 4, 0, 7, 7, 1]_{11}$;

Maximal vanishing for $w = 75594 = [0, 5, 1, 8, 8, 2]_{11}$;

$\binom{v-1}{w-1} = (HUGE) = [..., \neq 0, 0, 0, 0, 0]_{11}$.

$\Rightarrow \mathtt{T}(220540{-}1)$ has 2^4 Weyl factors.

- Play the ro[...]
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Ringel, Donkin ∼1991. There is a class of modules $\mathtt{T}(v{-}1)$ indexed by $\mathbb{N}$. They are a bit tricky to define, but:

- They have Δ- and ∇ filtrations, which look the same if you tilt your head:

Which Weyl factors does $\mathtt{T}(v{-}1)$ have a.k.a. the negative digits game?

Weyl factors of $\mathtt{T}(v{-}1)$ are

$\Delta([a_r, \pm a_{r-1}, \dots, \pm a_0]_p{-}1)$ where $v = [a_r, \dots, a_0]_p$ (appearing exactly once).

$\Delta(y-1)$ ... $\Delta(z-1)$

$\mathtt{T}(v$ ...

"tilting symmetry"

Example $\mathtt{T}(220540{-}1)$ for $p = 11$?

$v = 220540 = [1, 4, 0, 7, 7, 1]_{11}$;

has Weyl factors $[1, \pm 4, 0, \pm 7, \pm 7, \pm 1]_{11}$;

e.g. $\Delta(218690 = [1, 4, 0, -7, -7, -1]_{11}{-}1)$ appears.

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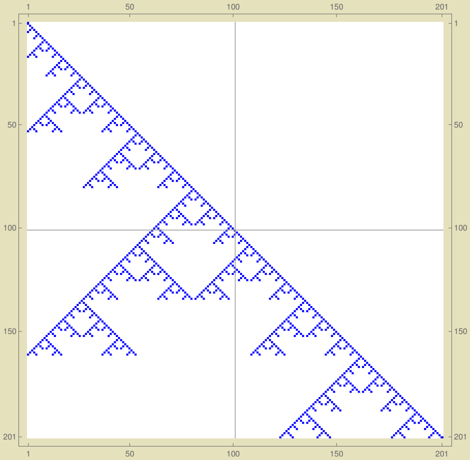
- They hav ... t your head:

$\mathtt{T}($ ... ng symmetry”

The tilting-Cartan matrix a.k.a. $\big(\mathtt{T}(v{-}1) : \Delta(w{-}1)\big)$?



This is characteristic 3.

- Play the ...
- $\mathtt{T}(v{-}1) \cong \mathtt{L}(v{-}1) \cong \Delta(v{-}1) \cong \nabla(v{-}1)$ over $\mathbb{C}$.
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General.
These facts hold in general, and
tilting modules form the "nicest possible" monoidal subcategory.

Tilting modules form a braided monoidal category $\mathcal{T}\mathrm{ilt}$.
Simple$\otimes$simple$\neq$simple, Weyl$\otimes$Weyl$\neq$Weyl, but tilting$\otimes$tilting$=$tilting.

The Grothendieck algebra $[\mathcal{T}\mathrm{ilt}]$ of $\mathcal{T}\mathrm{ilt}$ is a commutative algebra with basis $[\mathtt{T}(v-1)]$. So what I would like to answer on the object level, *i.e.* for $[\mathcal{T}\mathrm{ilt}]$:

- What are the fusion rules? Answer
- Find the $N_{v,w}^{x}\in\mathbb{N}[0]$ in $\mathtt{T}(v-1)\otimes\mathtt{T}(w-1)\cong\bigoplus_{x}N_{v,w}^{x}\mathtt{T}(x-1)$.
  - ▷ For $[\mathcal{T}\mathrm{ilt}]$ this means finding the structure constants.
- What are the thick $\otimes$-ideals? Answer
  - ▷ For $[\mathcal{T}\mathrm{ilt}]$ this means finding the ideals.

Tilting modules form a braided monoidal category $\mathcal{T}\mathrm{ilt}$.
Simple$\otimes$simple$\neq$simple, Weyl$\otimes$Weyl$\neq$Weyl, but tilting$\otimes$tilting$=$tilting.

The Grothendieck algebra $[\mathcal{T}\mathrm{ilt}]$ of $\mathcal{T}\mathrm{ilt}$ is a commutative algebra with basis $[\mathtt{T}(v-1)]$. So what I would like to answer on the object level, *i.e.* for $[\mathcal{T}\mathrm{ilt}]$:

- What are the fusion rules? Answer
- Find the $N_{v,w}^{x} \in \mathbb{N}[0]$ in $\mathtt{T}(v-1) \otimes \mathtt{T}(w-1) \cong \bigoplus_{x} N_{v,w}^{x}\mathtt{T}(x-1)$.
  - ▷ For $[\mathcal{T}\mathrm{ilt}]$ this means finding the structure constants.
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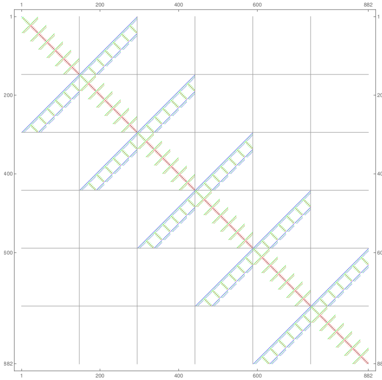
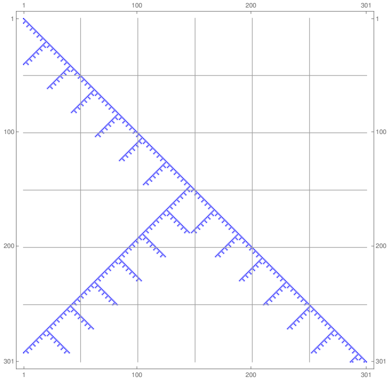
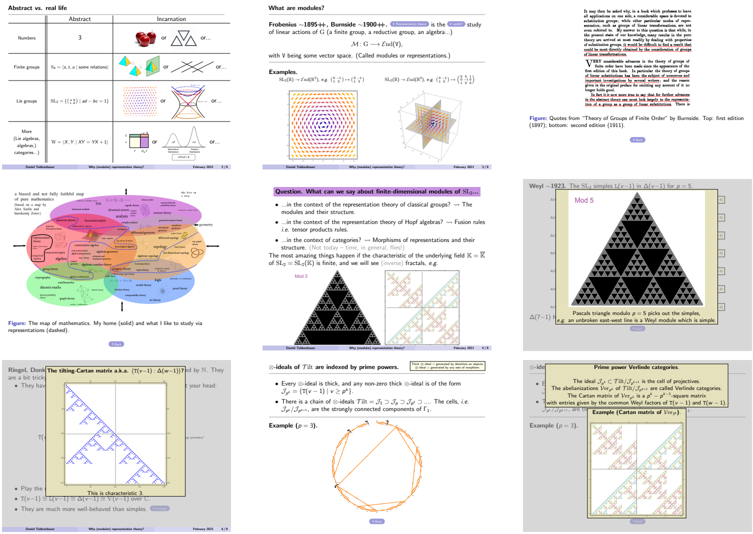
All of this generalizes to...

- ...higher ranks, *e.g.* SL_3, where higher dimensional fractals show up. (We are very far away from understanding this!)
- ...quantum groups, *e.g.* quantum SL_2, where "distorted" fractals show up. (We do understanding this!)

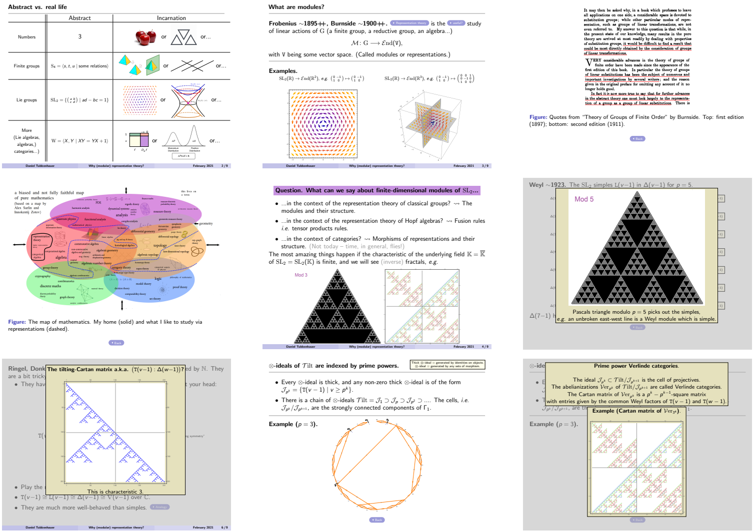
Two distorted fractals:







There is still much to do...



Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

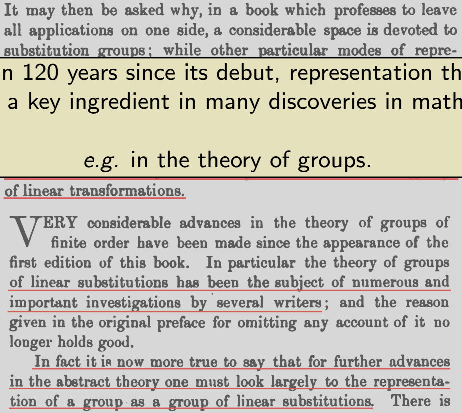
In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

Back

More than 120 years since its debut, representation theory has served as a key ingredient in many discoveries in mathematics,

e.g. in the theory of groups.



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I will however take a different stance:

Representations are sometimes more interesting than groups.

Today. SL_2 (easy) *vs.* its representations (fun).

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Back

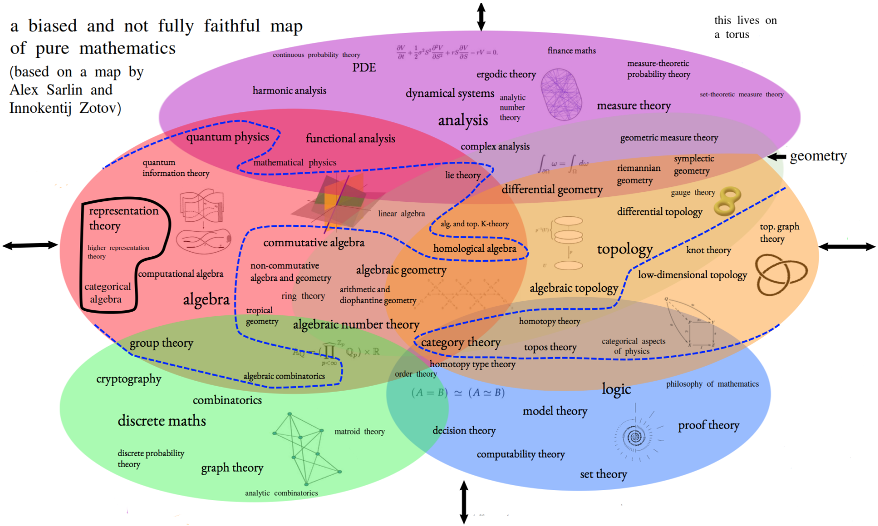


Figure: The map of mathematics. My home (solid) and what I like to study via representations (dashed).

Back

Weyl ∼1923. The SL_2 simples $\mathtt{L}(v-1)$ in $\Delta(v-1)$ for $p = 5$.

$\Delta(1-1)$	X^0Y^0							$L(1-1)$
$\Delta(2-1)$	X^1Y^0	X^0Y^1						$L(2-1)$
$\Delta(3-1)$	X^2Y^0	X^1Y^1	X^0Y^2					$L(3-1)$
$\Delta(4-1)$	X^3Y^0	X^2Y^1	X^1Y^2	X^0Y^3				$L(4-1)$
$\Delta(5-1)$	X^4Y^0	X^3Y^1	X^2Y^2	X^1Y^3	X^0Y^4			$L(5-1)$
$\Delta(6-1)$	X^5Y^0	X^4Y^1	X^3Y^2	X^2Y^3	X^1Y^4	X^0Y^5		$L(6-1)$
$\Delta(7-1)$	X^6Y^0	X^5Y^1	X^4Y^2	X^3Y^3	X^2Y^4	X^1Y^5	X^0Y^6	$L(7-1)$

$\Delta(7-1)$ has (its head) $\mathtt{L}(7-1)$ and $\mathtt{L}(3-1)$ as factors.

Back

Weyl ∼1923. The SL_2 simples $\mathrm{L}(v-1)$ in $\Delta(v-1)$ for $p = 5$.

$\Delta(1$ … $-1)$

$\Delta(2$ … $-1)$

$\Delta(3$ … $-1)$

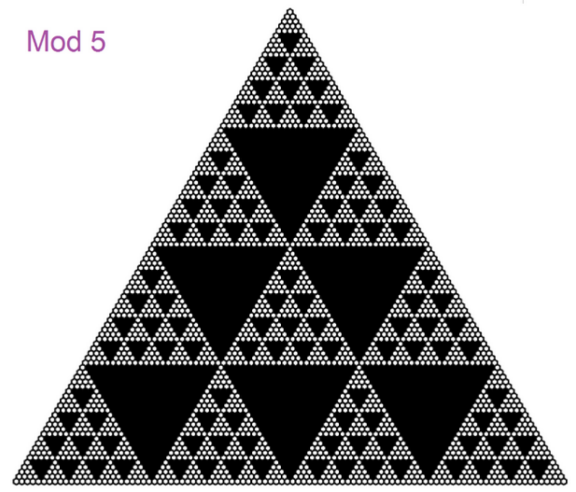
$\Delta(4$ … $-1)$

$\Delta(5$ … $-1)$

$\Delta(6$ … $-1)$

$\Delta(7$ … $-1)$

$\Delta(7-1)$ h



Pascals triangle modulo $p = 5$ picks out the simples,
e.g. an unbroken east-west line is a Weyl module which is simple.

Back

Two notions of “elements”

No substructure	Does not decompose
Simples	Indecomposables
$(*)\mathtt{V} \subset \mathtt{L} \Rightarrow \mathtt{V} \cong 0$ or $\mathtt{V} \cong \mathtt{L}$	$\mathtt{T} \cong \mathtt{V} \oplus \mathtt{W} \Rightarrow \mathtt{V} \cong 0$ or $\mathtt{V} \cong \mathtt{T}$

Both are legit elements of which one would like a periodic table.

G finite group, $\mathbb{K}[\mathrm{G}]$ the regular module (G acting on itself).

No substructure	Does not decompose
Simples	Projective indecomposables
$(*)$	$\oplus$-summands of $\mathbb{K}[\mathrm{G}]$

SL_2, $\Delta(1)$ the regular module (matrices acting by matrices).

No substructure	Does not decompose
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Back

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In good cases:
Simple=indecomposable
but not always.

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Back

Fusion graphs.

The fusion graph $\Gamma_v = \Gamma_{\mathtt{T}(v-1)}$ of $\mathtt{T}(v-1)$ is:

- Vertices of Γ_v are $w \in \mathbb{N}$, and identified with $\mathtt{T}(w-1)$.
- k edges $w \xrightarrow{k} x$ if $\mathtt{T}(x-1)$ appears k times in $\mathtt{T}(v-1) \otimes \mathtt{T}(w-1)$.
- $\mathtt{T}(v-1)$ is a $\otimes$-generator if Γ_v is strongly connected.
- This works for any reasonable monoidal category, with vertices being indecomposable objects and edges count multiplicities in $\otimes$-products.

Baby example. Assume that we have two indecomposable objects $\mathbb{1}$ and $\mathtt{X}$, with $\mathtt{X}^{\otimes 2} = \mathbb{1} \oplus \mathtt{X}$. Then:

$$\underset{\text{not a }\otimes\text{-generator}}{\Gamma_{\mathbb{1}} = \circlearrowright \mathbb{1} \qquad \mathtt{X} \circlearrowleft}\;,\qquad \underset{\text{a }\otimes\text{-generator}}{\Gamma_{\mathtt{X}} = \mathbb{1} \rightleftarrows \mathtt{X} \circlearrowleft}$$

Back

Fusion graphs.

The fusion graph Γ

- Vertices of Γ_v .
- k edges $w \xrightarrow{k}$ $) \otimes \mathtt{T}(w-1)$.
- $\mathtt{T}(v-1)$ is a $\otimes$
- This works for vertices being indecomposabl in $\otimes$-products.

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$\Gamma_{\mathbb{1}}$ $\mathtt{X} \circlearrowleft$

tor

The fusion graph of $\mathtt{T}(1) \cong \mathbb{K}^2$ for $p = \infty$:

The fusion graph of $\mathtt{T}(1) \cong \mathbb{K}^2$ for $p = 2$:

Fusion graphs.

The fusion graph Γ

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The fusion graph of $\mathtt{T}(1) \cong \mathbb{K}^2$ for $p = 2$:

In general, there is are cycles of length p with edges jumping $1 = p^0, p^1, p^2, \ldots$, units, reaping every $1 = p^0, p^1, p^2, \ldots$, steps.

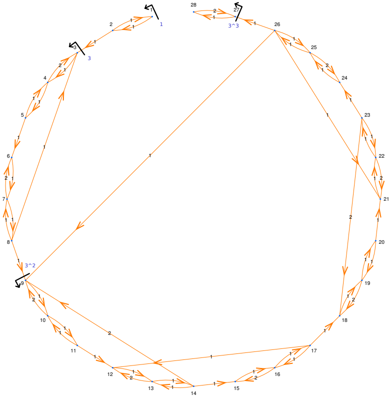
Back

$\otimes$-ideals of $\mathcal{T}\mathrm{ilt}$ are indexed by prime powers.

Thick $\otimes$-ideal = generated by identities on objects.
$\otimes$-ideal = generated by any sets of morphism.

- Every $\otimes$-ideal is thick, and any non-zero thick $\otimes$-ideal is of the form $\mathcal{J}_{p^k} = \{\mathtt{T}(v-1) \mid v \geq p^k\}$.
- There is a chain of $\otimes$-ideals $\mathcal{T}\mathrm{ilt} = \mathcal{J}_1 \supset \mathcal{J}_p \supset \mathcal{J}_{p^2} \supset \dots$. The cells, *i.e.* $\mathcal{J}_{p^k}/\mathcal{J}_{p^{k+1}}$, are the strongly connected components of Γ_1.

Example ($p = 3$).



Back

$\otimes$-ide

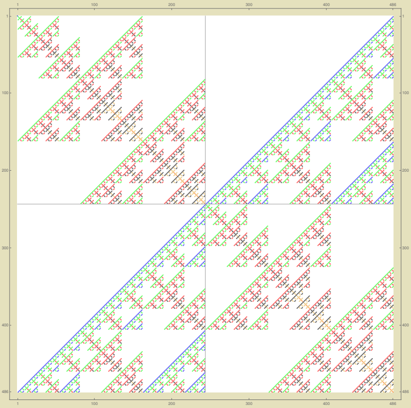
- E
  $\mathcal{J}$
- T
  $\mathcal{J}_{p^k}/\mathcal{J}_{p^{k+1}}$, are th 1.

Example ($p = 3$).

Prime power Verlinde categories.

The ideal $\mathcal{J}_{p^k} \subset \mathcal{T}\mathrm{ilt}/\mathcal{J}_{p^{k+1}}$ is the cell of projectives.
The abelianizations $\mathcal{V}\mathrm{er}_{p^k}$ of $\mathcal{T}\mathrm{ilt}/\mathcal{J}_{p^{k+1}}$ are called Verlinde categories.
The Cartan matrix of $\mathcal{V}\mathrm{er}_{p^k}$ is a $p^k - p^{k-1}$-square matrix
with entries given by the common Weyl factors of $\mathtt{T}(v-1)$ and $\mathtt{T}(w-1)$.

Example (Cartan matrix of $\mathcal{V}\mathrm{er}_{3^4}$).



Back