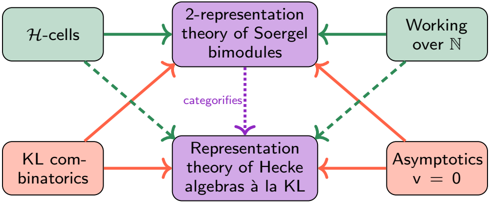
2-representations of Soergel bimodules I

Or: $\mathcal{H}$-cells and asymptotes

Daniel Tubbenhauer (Part II: Vanessa Miemietz)



$\mathcal{H}$-cells

2-representation theory of Soergel bimodules

Working over $\mathbb{N}$

categorifies

KL combinatorics

Representation theory of Hecke algebras à la KL

Asymptotics v = 0

Joint with Marco Mackaay, Volodymyr Mazorchuk and Xiaoting Zhang

December 2019

Clifford, Munn, Ponizovskiĭ, Green ∼1942++. Finite semigroups or monoids.

Example. $\mathbb{N}$, $\mathrm{Aut}(\{1,\dots,n\}) = S_n \subset T_n = \mathrm{End}(\{1,\dots,n\})$, groups, groupoids, categories, any $\cdot$ closed subsets of matrices, "everything" ▸ click, *etc.*

The cell orders and equivalences:

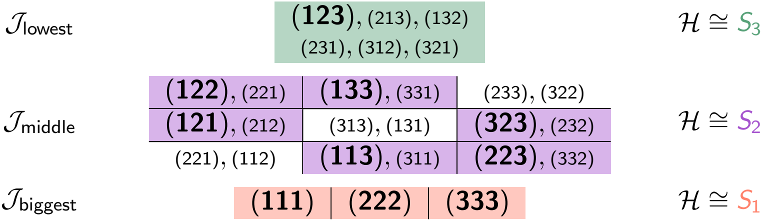
$$\begin{aligned} x \leq_L y \Leftrightarrow \exists z\colon y = zx, &\quad x \sim_L y \Leftrightarrow (x \leq_L y) \wedge (y \leq_L x), \\ x \leq_R y \Leftrightarrow \exists z'\colon y = xz', &\quad x \sim_R y \Leftrightarrow (x \leq_R y) \wedge (y \leq_R x), \\ x \leq_{LR} y \Leftrightarrow \exists z, z'\colon y = zxz', &\quad x \sim_{LR} y \Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x). \end{aligned}$$

Left, right and two-sided cells: Equivalence classes.

Example (group-like). The unit 1 is always in the lowest cell – *e.g.* $1 \leq_L y$ because we can take $z = y$. Invertible elements g are always in the lowest cell – *e.g.* $g \leq_L y$ because we can take $z = yg^{-1}$.

Clifford, Munn, Ponizovskiĭ, Green ∼1942++. Finite semigroups or monoids.

Example (the transformation monoid T_3). Cells – left $\mathcal{L}$ (columns), right $\mathcal{R}$ (rows), two-sided $\mathcal{J}$ (big rectangles), $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ (small rectangles).



$\mathcal{J}_{\text{lowest}}$		$(\mathbf{123}), (213), (132)$ $(231), (312), (321)$		$\mathcal{H} \cong S_3$
	$(\mathbf{122}), (221)$	$(\mathbf{133}), (331)$	$(233), (322)$	
$\mathcal{J}_{\text{middle}}$	$(\mathbf{121}), (212)$	$(313), (131)$	$(\mathbf{323}), (232)$	$\mathcal{H} \cong S_2$
	$(221), (112)$	$(\mathbf{113}), (311)$	$(\mathbf{223}), (332)$	
$\mathcal{J}_{\text{biggest}}$	$(\mathbf{111})$	$(\mathbf{222})$	$(\mathbf{333})$	$\mathcal{H} \cong S_1$

Cute facts.

- Each $\mathcal{H}$ contains precisely one idempotent e or no idempotent. Each e is contained in some $\mathcal{H}(e)$. (Idempotent separation.)
- Each $\mathcal{H}(e)$ is a maximal subgroup. (Group-like.)
- Each simple has a unique maximal $\mathcal{J}(e)$ whose $\mathcal{H}(e)$ does not kill it. (Apex.)

Clifford, M … monoids.

Example (… right $\mathcal{R}$ (rows), two …

Theorem. (Mind your groups!)—stated for monoids

There is a one-to-one correspondence

$$\left\{\begin{matrix}\text{simples with}\\ \text{apex } \mathcal{J}(e)\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{simples of (any)}\\ \mathcal{H}(e) \subset \mathcal{J}(e)\end{matrix}\right\}.$$

Thus, the maximal subgroups $\mathcal{H}(e)$ (semisimple over $\mathbb{C}$) control the whole representation theory (non-semisimple; even over $\mathbb{C}$).

$\mathcal{J}_{\text{lo}}$				$\cong S_3$
$\mathcal{J}_{\text{middle}}$	**(121)**, (212)	(313), (131)	**(323)**, (232)	$\cong S_2$
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Example. (T_3.)

$\mathcal{H}(e) = S_3, S_2, S_1$ gives $3 + 2 + 1 = 6$ associated simples.

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Cute facts.

This is a general philosophy in representation theory.

Buzz words. Idempotent truncations, Kazhdan–Lusztig cells, quasi-hereditary algebras, cellular algebras, *etc.*

- Each $\mathcal{H}$ Each e is contain
- Each $\mathcal{H}(e)$ is a maximal subgroup. (Group-like.)
- Each sim kill it. (Apex.)

Note. Whenever one has a (reasonable) antiinvolution *, the $\mathcal{H}$-cells to consider are the diagonals $\mathcal{H} = \mathcal{L} \cap \mathcal{L}^\star$.

Kazhdan–Lusztig (KL) and others $\sim$1979++. Green's theory in linear.

Choose a basis. For a finite-dimensional algebra S (over $\mathbb{Z}_{\mathsf{v}} = \mathbb{Z}[\mathsf{v}, \mathsf{v}^{-1}]$) fix a basis B_{S}. For $x, y, z \in B_{\mathrm{S}}$ write $y \Subset zx$ if y appears in zx with non-zero coefficient.

The cell orders and equivalences:

$$\begin{aligned} x \leq_L y &\Leftrightarrow \exists z\colon y \Subset zx, & x \sim_L y &\Leftrightarrow (x \leq_L y) \wedge (y \leq_L x),\\ x \leq_R y &\Leftrightarrow \exists z'\colon y \Subset xz', & x \sim_R y &\Leftrightarrow (x \leq_R y) \wedge (y \leq_R x),\\ x \leq_{LR} y &\Leftrightarrow \exists z, z'\colon y \Subset zxz', & x \sim_{LR} y &\Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x). \end{aligned}$$

Left, right and two-sided cells: Equivalence classes.

Example (group-like). For $\mathrm{S} = \mathbb{C}[G]$ and the choice of the group element basis $B_{\mathrm{S}} = G$, cell theory is boring.

Kazhdan–Lusztig (KL) and others ∼1979++. Green's theory in linear.

Example (▸ Coxeter group **of type B_2, B_{S}=KL basis).** Cells – left $\mathcal{L}$ (columns), right $\mathcal{R}$ (rows), two-sided $\mathcal{J}$ (big rectangles), $\mathcal{H} = \mathcal{L} \cap \mathcal{L}^{-1}$ (diagonal rectangles).

$\mathcal{J}_{\text{lowest}}$	$\mathbf{1}$	$\mathrm{S}_{\mathcal{H}} \cong \mathbb{Z}_{\mathsf{v}}$
$\mathcal{J}_{\text{middle}}$	$\begin{array}{c\|c} \mathbf{s}, sts & ts \\ \hline st & \mathbf{t}, tst \end{array}$	$\mathrm{S}_{\mathcal{H}}$‘ $\cong$ ’$\mathbb{Z}_{\mathsf{v}}[\mathbb{Z}/2\mathbb{Z}]$
$\mathcal{J}_{\text{biggest}}$	$\mathbf{w_0}$	$\mathrm{S}_{\mathcal{H}}$‘ $\cong$ ’$\mathbb{Z}_{\mathsf{v}}$

Everything crucially depends on the choice of B_{S}.

- $\mathrm{S}_{\mathcal{H}} = \mathbb{Z}_{\mathsf{v}}\{B_{\mathcal{H}}\}$ is an algebra modulo bigger cells, but the $\mathrm{S}_{\mathcal{H}}$ do not parametrize the simples of S. ▸ Example
- $\mathrm{S}_{\mathcal{H}}$ tends to have pseudo-idempotents $e^2 = \lambda \cdot e$ rather than idempotents. Even worse, $\mathrm{S}_{\mathcal{H}}$ could contain no (pseudo-)idempotent e at all.
- $\mathrm{S}_{\mathcal{H}}$ is not group-like in general.

Question.

What can one do to at least partially recover the $\mathcal{H}$-cell theorem?

Question.

Can we find good a basis for which $\mathrm{S}_{\mathcal{H}}$ is group-like?

Kazhdan–Lusz … n linear.

Example (▸ Coxe … (columns), right $\mathcal{R}$ (rows), two-s … ectangles).

$\mathcal{J}_{\mathrm{lowest}}$			$_{\mathcal{H}} \cong \mathbb{Z}_{\mathrm{v}}$
$\mathcal{J}_{\mathrm{middle}}$	st	**t**, tst	$\mathrm{S}_{\mathcal{H}}` \cong '\mathbb{Z}_{\mathrm{v}}[\mathbb{Z}/2\mathbb{Z}]$
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$\mathcal{J}_{\text{bi}}$

Spoiler.

On the categorified level the "basis problem" vanishes
– take the basis given by the equivalence classes of indecomposables –
and a version of the $\mathcal{H}$-cell theorem can be recovered.

However, $\mathrm{S}_{\mathcal{H}}$ still is not group-like.

Everyth

- $\mathrm{S}_{\mathcal{H}}$ parametrize the simples of S. Example
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However, $\mathrm{S}_{\mathcal{H}}$ still is not group-like.

In a few minutes (Vanessa’s talk).

The whole categorified story.

Now.

How to make $\mathrm{S}_{\mathcal{H}}$ group-like for the KL basis (a good basis).

Example (type B_2).

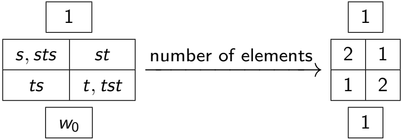
$W = \langle s, t \mid s^2 = t^2 = 1, tsts = stst \rangle$. Number of elements: 8. Number of cells: 3, named 0 (lowest) to 2 (biggest).

Cell order:

Size of the cells:

cell	0	1	0′
size	1	6	1

Cell structure:



1	

s, sts	st
ts	t, tst

w_0	

$\xrightarrow{\text{number of elements}}$

1	

2	1
1	2

1	

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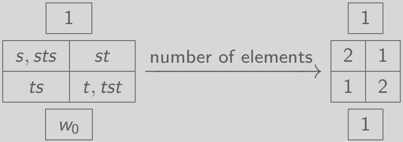

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Example (SAGEMath).

$$c_s \cdot c_s = (1{+}\text{bigger powers})c_s.$$
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$$c_{sts} \cdot c_{sts} = (1{+}\text{bigger powers})c_s{+}\text{higher cell elements}.$$
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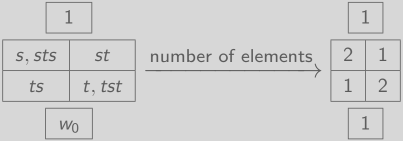

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1	

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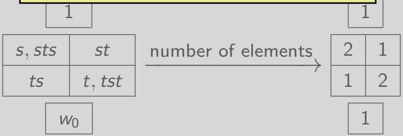

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Cell orde

Size of t

Fact (Lusztig ∼1984++, Soergel–Elias–Williamson ∼1990,2012).

For any(!) Coxeter group W
there is a well-defined function

$$\mathbf{a}\colon W \to \mathbb{N}$$

which is constant on two-sided cells such that for $v, w \in \mathcal{J}$

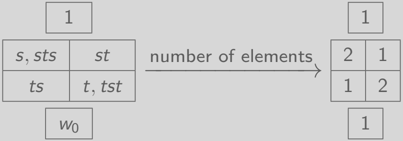
$$c_v \cdot c_w \in \mathbb{N}[v]\{c_x \mid x \in \mathcal{J}\} + \text{bigger friends.}$$

(Positively graded.)

▸ Big example

size	1	6	1

Cell structure:



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Idea (Lusztig ∼1984).

Ignore everything except the leading coefficient
of the classical KL basis shifted by $\mathbf{a}$(two-sided cell).
Those shifted versions are what I denote by c_w.

The asymptotic limit $\mathrm{A}^0(W)$ of $\mathrm{H}^{\mathsf{v}}(W)$ is defined as follows.

As a free $\mathbb{Z}$-module:

$$\mathrm{A}^0(W) = \bigoplus_{\mathcal{J}} \mathbb{Z}\{a_w \mid w \in \mathcal{J}\} \quad \text{vs.} \quad \mathrm{H}^{\mathsf{v}}(W) = \mathbb{Z}_{\mathsf{v}}\{c_w \mid w \in W\}.$$

Multiplication.

$$a_x a_y = \sum_{z \in \mathcal{J}} \gamma_{x,y}^{z} a_z \quad \text{vs.} \quad c_x c_y = \sum_{z \in \mathcal{J}} \mathsf{v}^{\mathbf{a}(z)} h_{x,y}^{z} c_z + \text{bigger friends.}$$

where

$$\gamma_{x,y}^{z} = (\mathsf{v}^{\mathbf{a}(z)} h_{x,y}^{z})(0) \in \mathbb{N}.$$

Think: “A crystal limit for the Hecke algebra” .

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As

Mu

whe

Example (type B_2).

The multiplication tables (empty entries are 0 and $[2] = 1 + \mathrm{v}^2$) in 1:

	a_s	a_{sts}	a_{st}	a_t	a_{tst}	a_{ts}
a_s	a_s	a_{sts}	a_{st}			
a_{sts}	a_{sts}	a_s	a_{st}			
a_{ts}	a_{ts}	a_{ts}	$a_t + a_{tst}$			
a_t				a_t	a_{tst}	a_{ts}
a_{tst}				a_{tst}	a_t	a_{ts}
a_{st}				a_{st}	a_{st}	$a_s + a_{sts}$

	c_s	c_{sts}	c_{st}	c_t	c_{tst}	c_{ts}
c_s	$[2]c_s$	$[2]c_{sts}$	$[2]c_{st}$	c_{st}	$c_{st}+c_{w_0}$	$c_s + c_{sts}$
c_{sts}	$[2]c_{sts}$	$[2]c_s+[2]^2c_{w_0}$	$[2]c_{st}+[2]c_{w_0}$	$c_s + c_{sts}$	$c_s+[2]^2c_{w_0}$	$c_s + c_{sts}+[2]c_{w_0}$
c_{ts}	$[2]c_{ts}$	$[2]c_{ts}+[2]c_{w_0}$	$[2]c_t + [2]c_{tst}$	$c_t + c_{tst}$	$c_t + c_{tst}+[2]c_{w_0}$	$2c_{ts}+c_{w_0}$
c_t	c_{ts}	$c_{ts}+c_{w_0}$	$c_t + c_{tst}$	$[2]c_t$	$[2]c_{tst}$	$[2]c_{ts}$
c_{tst}	$c_t + c_{tst}$	$c_t+[2]^2c_{w_0}$	$c_t + c_{tst}+[2]c_{w_0}$	$[2]c_{tst}$	$[2]c_t+[2]^2c_{w_0}$	$[2]c_{ts}+[2]c_{w_0}$
c_{st}	$c_s + c_{sts}$	$c_s + c_{sts} + [2]c_{w_0}$	$2c_{st}+c_{w_0}$	$[2]c_{st}$	$[2]c_{st}+[2]c_{w_0}$	$[2]c_s + [2]c_{sts}$

The asymptotic algebra is much simpler!

The asym

As a free

Fact (Lusztig ∼1984++).

$\mathrm{A}^0(W) = \bigoplus_{\mathcal{J}} \mathrm{A}^0_{\mathcal{J}}(W)$ with the a_w basis
and all its summands $\mathrm{A}^0_{\mathcal{J}}(W) = \mathbb{Z}\{a_w \mid w \in \mathcal{J}\}$
are multifusion algebras. (Group-like.)

Multifusion algebras = decategorifications of multifusion categories.

Multiplication.

$$a_x a_y = \sum_{z\in\mathcal{J}} \gamma^z_{x,y} a_z \text{ vs. } c_x c_y = \sum_{z\in\mathcal{J}} \mathrm{v}^{\mathbf{a}(z)} h^z_{x,y} c_z + \text{bigger friends.}$$

where

$$\gamma^z_{x,y} = (\mathrm{v}^{\mathbf{a}(z)} h^z_{x,y})(0) \in \mathbb{N}.$$

Think: “A crystal limit for the Hecke algebra”.

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Surprising fact (Lusztig ∼1984++).
It seems one throws almost everything away, but:

There is an explicit embedding

$$\mathrm{H}^{\mathsf{v}}(W)\hookrightarrow\mathrm{A}^0(W)\otimes_{\mathbb{Z}}\mathbb{Z}_{\mathsf{v}}$$

which is an isomorphism after scalar extension to $\mathbb{C}(\mathsf{v})$.

Multiplication.

$a_x a_y =$

friends.

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Multiplication.

$a_x a_y =$

friends.

where

Surprising consequence (Lusztig ∼1984++).

There is a(n explicit) one-to-one correspondence

$$\{\text{simples of } \mathrm{H}^{\mathsf{v}}(W) \text{ with apex } \mathcal{J}\} \xleftrightarrow{\text{one-to-one}} \{\text{simples of } \mathrm{A}^0_{\mathcal{J}}(W)\}.$$

Thus, simples of W are ordered into cells (“families”).

The asymptotic limi

Calculation (Lusztig ∼1984++).

For almost all $\mathcal{H} \subset \mathcal{J}$ in finite Coxeter type

$$\mathrm{A}^0_{\mathcal{H}}(W) \cong \mathbb{Z}\big[(\mathbb{Z}/2\mathbb{Z})^{k=k(\mathcal{J})}\big].$$

As a free $\mathbb{Z}$-module:

$$\mathrm{A}^0(W) = \bigoplus_{\mathcal{J}} \mathbb{Z}\{a_w \mid w \in \mathcal{J}\} \text{ vs. } \mathrm{H}^{\mathrm{v}}(W) = \mathbb{Z}_{\mathrm{v}}\{c_w \mid w \in W\}.$$

Multiplication.

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where

$$\gamma^z_{x,y} = (\mathrm{v}^{\mathbf{a}(z)} h^z_{x,y})(0) \in \mathbb{N}.$$

Think: “A crystal limit for the Hecke algebra”.

The asymptotic limi

Calculation (Lusztig ∼1984++).

For almost all $\mathcal{H}\subset\mathcal{J}$ in finite Coxeter type

$$\mathrm{A}^0_{\mathcal{H}}(W)\cong\mathbb{Z}\big[(\mathbb{Z}/2\mathbb{Z})^{k=k(\mathcal{J})}\big].$$

As a free $\mathbb{Z}$-module:

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$$2^k\leq\#\{\text{simples with apex }\mathcal{J}\}\leq 2^{2k}.$$

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Example.

In type A one always has $k(\mathcal{J})=0$, so the $\mathcal{H}$-cell theorem holds.

In other types one only gets lower and upper bounds. ▸ Big example
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Not too bad: Idempotents in all $\mathcal{J}$, group-like $\mathrm{A}^0_{\mathcal{H}}(W)$ and "almost $\mathcal{H}$-cell theorem".

Spoiler. $\mathcal{H}$-cells and asymptotes are much nicer on the categorified level.

Categorified picture – Part 1.

Theorem (Soergel–Elias–Williamson $\sim$1990,2012).
There exists a graded, monoidal category $\mathscr{S}^{\mathsf{v}} = \mathscr{S}^{\mathsf{v}}(W)$ such that:

(1) For every $w \in W$, there exists an indecomposable object $\mathtt{C}_w$.

(2) The $\mathtt{C}_w$, for $w \in W$, form a complete set of pairwise non-isomorphic indecomposable objects up to shifts.

(3) The identity object is $\mathtt{C}_1$, where 1 is the unit in W.

(4) $\mathscr{S}^{\mathsf{v}}$ categorifies H^{v} with $[\mathtt{C}_w] = c_w$.

(5) $\mathrm{grdim}\big(\mathrm{hom}_{\mathscr{S}^{\mathsf{v}}}(\mathtt{C}_v, \mathsf{v}^k \mathtt{C}_w)\big) = \delta_{v,w}\delta_{0,k}$. (Soergel's hom formula *a.k.a.* positively graded.)

Let R- or R_W be the polynomial or the coinvariant algebra attached to the geometric representation of W. Soergel bimodules for me are defined as the additive Karoubi closure of the full subcategory of R- or R^W-bimodules generated by the Bott–Samelson bimodules, *e.g.* $\mathtt{B}_s = \mathrm{R} \otimes_{\mathrm{R}^s} \mathrm{R}$, and their shifts.

Categorified picture – Part 1.

Examples in type A_1; polynomial ring.

Let $\mathrm{R} = \mathbb{C}[x]$ with $\deg(x) = 2$ and $W = S_2$ action given by $s.x = -x$; $\mathrm{R}^s = \mathbb{C}[x^2]$.

The indecomposable Soergel bimodules over R are
$\mathtt{C}_1 = \mathbb{C}[x]$ and $\mathtt{C}_s = \mathbb{C}[x] \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x]$.

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The coinvariant algebra is $\mathrm{R}_W = \mathbb{C}[x]/x^2$.

The indecomposable Soergel bimodules over R_W are
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Examples in type A_1; coinvariant algebra.

$$\mathtt{C}_s \otimes_{\mathrm{R}_W} \mathtt{C}_s = \big(\mathbb{C}[x]/x^2 \otimes \mathbb{C}[x]/x^2\big) \otimes_{\mathbb{C}[x]/x^2} \big(\mathbb{C}[x]/x^2 \otimes \mathbb{C}[x]/x^2\big).$$

Which gives $\mathtt{C}_s\mathtt{C}_s \cong \mathtt{C}_s \oplus \mathtt{C}_s\langle 2\rangle = (1+\mathrm{v}^2)\mathtt{C}_s$.

Categorified picture – Part 2.

Theorem (Lusztig, Elias–Williamson $\sim$2012).
There exists a ▸ multifusion bicategory $\mathscr{A}^0 = \mathscr{A}^0(W)$ such that:

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2. The $\mathtt{A}_w$, for $w \in W$, form a complete set of pairwise non-isomorphic simple objects.
3. The 'identity objects' are $\mathtt{A}_d$, where d are Duflo involutions.
4. $\mathscr{A}^0$ categorifies A^0 with $[\mathtt{A}_w] = a_w$.
5. $\mathscr{A}^0$ is the degree zero part of $\mathscr{S}^{\mathsf{v}}$.

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Construction of $\mathscr{A}^0_{\mathcal{H}}$.

$\mathscr{A}^0_{\mathcal{H}} = \mathrm{add}\big(\{\mathrm{v}^k\mathtt{C}_w \mid w \in \mathcal{H}, k \geq 0\}\big)/\mathrm{add}\big(\{\mathrm{v}^k\mathtt{C}_w \mid w \in \mathcal{H}, k > 0\}\big)$ (Degree zero part.)

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Theorem (Bezrukavnikov–Finkelberg–Ostrik ∼2006).

For almost all $\mathcal{H} \subset \mathcal{J}$ in finite Coxeter type

$$\mathscr{A}^0_{\mathcal{H}}(W) \cong \mathscr{V}\mathrm{ect}\big((\mathbb{Z}/2\mathbb{Z})^{k=k(\mathcal{J})}\big).$$

Categorified picture – Part 2.

Up next in Vanessa's talk. The categorification of Lusztig's "crystal approach" to the representation theory of H^{v} for W of finite type (proved in most cases):

A conjectural relationship between 2-representations of $\mathscr{A}^0$ and $\mathscr{S}^{\mathrm{v}}$ using $\mathscr{A}^0_{\mathcal{H}}$.

Here we use R^W to have finite-dimensional hom spaces.

Why is this awesome? Because, if true, then the conjectural relationship...

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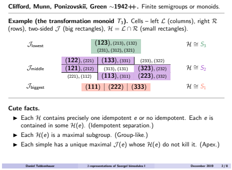

Clifford, Munn, Ponizovskiĭ, Green ∼1942++. Finite semigroups or monoids.

Example (the transformation monoid T_3). Cells – left $\mathcal{L}$ (columns), right $\mathcal{R}$ (rows), two-sided $\mathcal{J}$ (big rectangles), $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ (small rectangles).

Cute facts.

- Each $\mathcal{H}$ contains precisely one idempotent e or no idempotent. Each e is contained in some $\mathcal{H}(e)$. (Idempotent separation.)
- Each $\mathcal{H}(e)$ is a maximal subgroup. (Group-like.)
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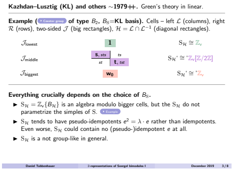

Kazhdan–Lusztig (KL) and others ∼1979++. Green's theory in linear.

Example (Coxeter group of type B_2, B_S=KL basis). Cells – left $\mathcal{L}$ (columns), right $\mathcal{R}$ (rows), two-sided $\mathcal{J}$ (big rectangles), $\mathcal{H} = \mathcal{L} \cap \mathcal{L}^{-1}$ (diagonal rectangles).

Everything crucially depends on the choice of B_S.

- $S_{\mathcal{H}} = \mathbb{Z}_v\{B_{\mathcal{H}}\}$ is an algebra modulo bigger cells, but the $S_{\mathcal{H}}$ do not parametrize the simples of S. Example
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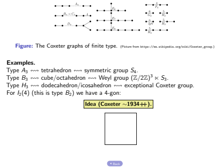

Figure: The Coxeter graphs of finite type.

Examples.
Type A_3 ⇝ tetrahedron ⇝ symmetric group S_4.
Type B_3 ⇝ cube/octahedron ⇝ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \times S_3$.
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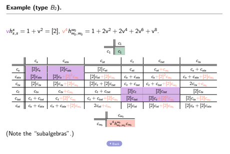
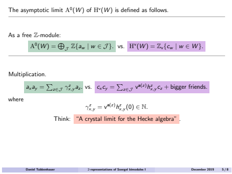
Idea (Coxeter ∼1934++).



Example (type B_2).

(Note the "subalgebras".)



The asymptotic limit $A^0(W)$ of $H^v(W)$ is defined as follows.

As a free $\mathbb{Z}$-module: $A^0(W) = \bigoplus_{\mathcal{J}} \mathbb{Z}\{a_w \mid w \in \mathcal{J}\}$. vs. $H^v(W) = \mathbb{Z}_v\{c_w \mid w \in W\}$.

Multiplication. $a_x a_y = \sum_{z \in \mathcal{J}} \gamma_{x,y}^z a_z$. vs. $c_x c_y = \sum_{z \in \mathcal{J}} v^{a(z)} h_{x,y}^z c_z$ + bigger friends.

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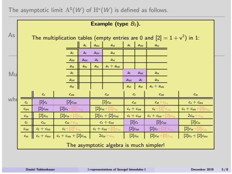
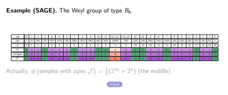


Example (type B_2).

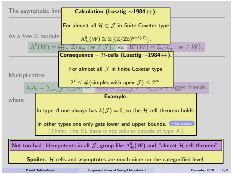
The multiplication tables (empty entries are 0 and $[2] = 1 + v^2$) in 1:

The asymptotic algebra is much simpler!



Example (SAGE). The Weyl group of type B_6.

Actually, $\#\{\text{simples with apex } \mathcal{J}\} = \frac{1}{2}(2^{2k} + 2^k)$ (the middle).



Calculation (Lusztig ∼1984++). For almost all $\mathcal{H} \subset \mathcal{J}$ in finite Coxeter type $A_{\mathcal{H}}^0(W) \cong \mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})^{k=k(\mathcal{J})}]$.

Consequence – $\mathcal{H}$-cells (Lusztig ∼1984++). For almost all $\mathcal{J}$ in finite Coxeter type $2^k \leq \#\{\text{simples with apex } \mathcal{J}\} \leq 2^{2k}$.

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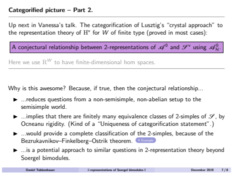

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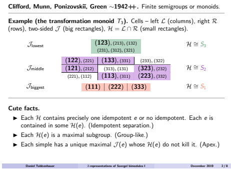
There is still much to do...



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$\mathcal{J}_{lowest}$ (123), (213), (132), (231), (312), (321) $\mathcal{H}\cong S_3$

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$\mathcal{J}_{biggest}$ (111) | (222) | (333) $\mathcal{H}\cong S_1$

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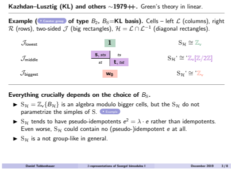

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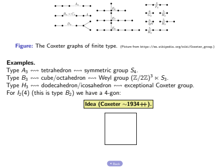

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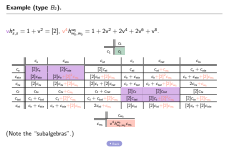
Back

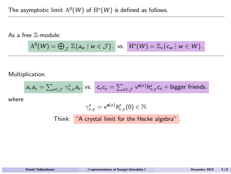


Example (type B_2).

$vh^s_{s,s}=1+v^2=[2]$, $v^4h^{w_0}_{w_0,w_0}=1+2v^2+2v^4+2v^6+v^8$.

(Note the "subalgebras".)

Back



The asymptotic limit $A^0(W)$ of $H^v(W)$ is defined as follows.

As a free $\mathbb{Z}$-module:

$A^0(W)=\bigoplus_{\mathcal{J}}\mathbb{Z}\{a_w\mid w\in\mathcal{J}\}$. vs. $H^v(W)=\mathbb{Z}_v\{c_w\mid w\in W\}$.

Multiplication.

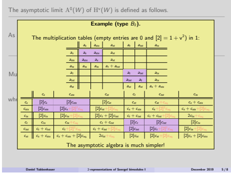
$a_xa_y=\sum_{z\in\mathcal{J}}\gamma^z_{x,y}a_z$. vs. $c_xc_y=\sum_{z\in\mathcal{J}}v^{a(z)}h^z_{x,y}c_z+$ bigger friends.

where

$\gamma^z_{x,y}=v^{a(z)}h^z_{x,y}(0)\in\mathbb{N}$.

Think: "A crystal limit for the Hecke algebra".

Daniel Tubbenhauer 2-representations of Soergel bimodules I December 2019 5 / 8



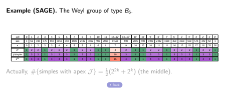
The asymptotic limit $A^0(W)$ of $H^v(W)$ is defined as follows.

Example (type B_2).

The multiplication tables (empty entries are 0 and $[2]=1+v^2$) in 1:

The asymptotic algebra is much simpler!

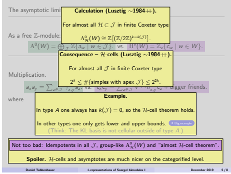
Daniel Tubbenhauer 2-representations of Soergel bimodules I December 2019 5 / 8



Example (SAGE). The Weyl group of type B_6.

Actually, $\#\{\text{simples with apex }\mathcal{J}\}=\frac{1}{2}(2^{2k}+2^k)$ (the middle).

Back



Calculation (Lusztig $\sim$1984++).

For almost all $\mathcal{H}\subset\mathcal{J}$ in finite Coxeter type

$A^0_{\mathcal{H}}(W)\cong\mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})^{k=k(\mathcal{J})}]$.

Consequence – $\mathcal{H}$-cells (Lusztig $\sim$1984++).

For almost all $\mathcal{J}$ in finite Coxeter type

$2^k\leq\#\{\text{simples with apex }\mathcal{J}\}\leq 2^{2k}$.

Example.

In type A one always has $k(\mathcal{J})=0$, so the $\mathcal{H}$-cell theorem holds.

In other types one only gets lower and upper bounds. Big example

(Think: The KL basis is not cellular outside of type A.)

Not too bad: Idempotents in all $\mathcal{J}$, group-like $A^0_{\mathcal{H}}(W)$ and "almost $\mathcal{H}$-cell theorem".

Spoiler. $\mathcal{H}$-cells and asymptotes are much nicer on the categorified level.

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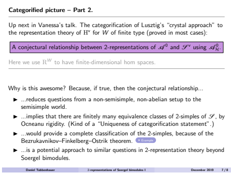

Categorified picture – Part 2.

Up next in Vanessa's talk. The categorification of Lusztig's "crystal approach" to the representation theory of H^v for W of finite type (proved in most cases):

A conjectural relationship between 2-representations of $\mathscr{A}^0$ and $\mathscr{S}^v$ using $\mathscr{A}^0_{\mathcal{H}}$.

Here we use R^W to have finite-dimensional hom spaces.

Why is this awesome? Because, if true, then the conjectural relationship...

- ...reduces questions from a non-semisimple, non-abelian setup to the semisimple world.
- ...implies that there are finitely many equivalence classes of 2-simples of $\mathscr{S}$, by Ocneanu rigidity. (Kind of a "Uniqueness of categorification statement".)
- ...would provide a complete classification of the 2-simples, because of the Bezrukavnikov–Finkelberg–Ostrik theorem. Example
- ...is a potential approach to similar questions in 2-representation theory beyond Soergel bimodules.

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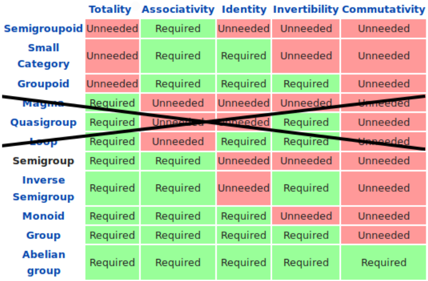
Thanks for your attention!



	Totality	Associativity	Identity	Invertibility	Commutativity
Semigroupoid	Unneeded	Required	Unneeded	Unneeded	Unneeded
Small Category	Unneeded	Required	Required	Unneeded	Unneeded
Groupoid	Unneeded	Required	Required	Required	Unneeded
~~Magma~~	~~Required~~	~~Unneeded~~	~~Unneeded~~	~~Unneeded~~	~~Unneeded~~
~~Quasigroup~~	~~Required~~	~~Unneeded~~	~~Unneeded~~	~~Required~~	~~Unneeded~~
~~Loop~~	~~Required~~	~~Unneeded~~	~~Required~~	~~Required~~	~~Unneeded~~
Semigroup	Required	Required	Unneeded	Unneeded	Unneeded
Inverse Semigroup	Required	Required	Unneeded	Required	Unneeded
Monoid	Required	Required	Required	Unneeded	Unneeded
Group	Required	Required	Required	Required	Unneeded
Abelian group	Required	Required	Required	Required	Required

Picture from `https://en.wikipedia.org/wiki/Semigroup`.

- There are zillions of semigroups, *e.g.* 1843120128 of order 8. (Compare: There are 5 groups of order 8.)
- Already the easiest of these are not semisimple – not even over $\mathbb{C}$.
- Almost all of them are of wild representation type.

Is the study of semigroups hopeless?

Green & co: No!

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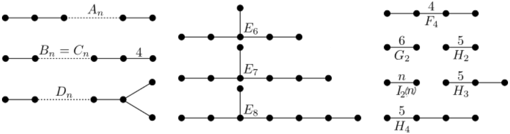
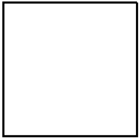


Figure: The Coxeter graphs of finite type. (Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Examples.

Type A_3 $\leftrightsquigarrow$ tetrahedron $\leftrightsquigarrow$ symmetric group S_4.
Type B_3 $\leftrightsquigarrow$ cube/octahedron $\leftrightsquigarrow$ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$.
Type H_3 $\leftrightsquigarrow$ dodecahedron/icosahedron $\leftrightsquigarrow$ exceptional Coxeter group.
For $I_2(4)$ (this is type B_2) we have a 4-gon:

Idea (Coxeter $\sim$1934++).



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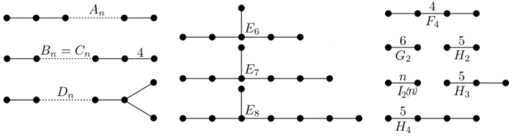

Figure: The Coxeter graphs of finite type. (Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Examples.

Fact. The symmetries are given by exchanging flags.

Type A_3 $\leftrightsquigarrow$ tetrahedron $\leftrightsquigarrow$ symmetric group S_4.

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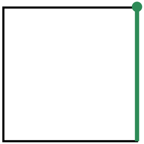
Type H_3 $\leftrightsquigarrow$ dodecahedron/icosahedron $\leftrightsquigarrow$ exceptional Coxeter group.

For $I_2(4$ e $B_2)$ we have a 4-gon:

Fix a flag F.

Idea (Coxeter ∼1934++).



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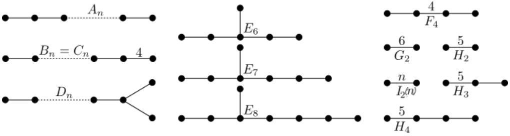

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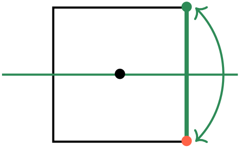
For I_2(4 e B_2) we have a 4-gon:

Fix a flag F.

Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Idea (Coxeter ∼1934++).



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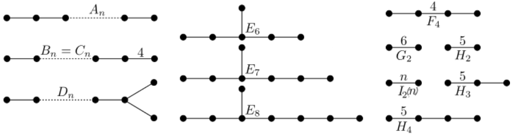

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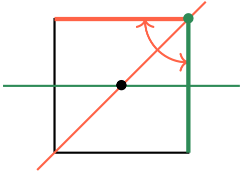
For $I_2(4$ pe $B_2)$ we have a 4-gon:

Fix a flag F.

Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Idea (Coxeter ∼1934++).



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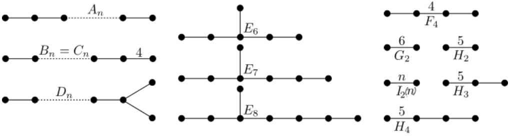

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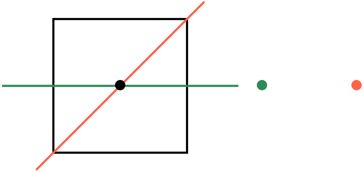
Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Write a vertex i for each H_i.

Idea (Coxeter ∼1934++).



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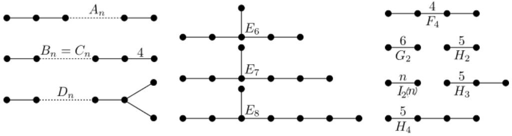

Figure: The Coxeter graphs of finite type. (Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

This gives a generator-relation presentation.

Examples.

And the braid relation measures the angle between hyperplanes.

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Type B_3 ⇝ cube/octahedron ⇝ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$.

Type H_3 ⇝ dodecahedron/icosahedron ⇝ exceptional Coxeter group.

For $I_2(4$ pe B_2) we have a 4-gon:

Fix a flag F.

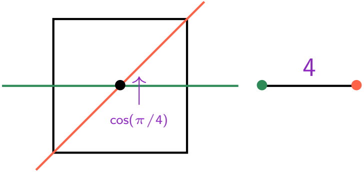
Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Write a vertex i for each H_i.

Connect i, j by an n-edge for H_i, H_j having angle $\cos(\pi/n)$.

Idea (Coxeter ∼1934++).



◂ Back

Example (type B_2).

$$W \begin{cases} = \langle s, t \mid s^2 = t^2 = 1, tsts = stst \rangle \\ = \{1, s, t, st, ts, sts, tst, w_0\}. \end{cases}$$

$$\mathrm{H}^{\mathrm{v}}(W) \begin{cases} = \langle h_s, h_t \mid h_s^2 = (\mathrm{v}^{-1}-\mathrm{v})h_s + 1, h_t^2 = (\mathrm{v}^{-1}-\mathrm{v})h_t + 1, h_t h_s h_t h_s = h_s h_t h_s h_t \rangle \\ = \mathbb{Z}_{\mathrm{v}}\{h_1, h_s, h_t, h_{st}, h_{ts}, h_{sts}, h_{tst}, h_{w_0}\}. \end{cases}$$

In general, $\mathrm{H}^{\mathrm{v}}(W = (W|S))$ is generated by h_s for $s \in S$, which satisfy the quadratic relations and the braid relations.

KL basis:

$$\mathrm{H}^{\mathrm{v}}(W) = \mathbb{Z}_{\mathrm{v}}\{c_1 = 1, c_s = \mathrm{v}(h_s + \mathrm{v}), c_t = \mathrm{v}(h_t + \mathrm{v}), c_{st}, c_{ts}, c_{sts}, c_{tst}, c_{w_0}\}.$$

$c_s^2 = (1 + \mathrm{v}^2)c_s = [2]c_s$. (Quasi-idempotent, but "positively graded".)

◂ Back

Example (type B_2).

$$\mathrm{v}h^{s}_{s,s} = 1 + \mathrm{v}^2 = [2],\ \mathrm{v}^4 h^{w_0}_{w_0,w_0} = 1 + 2\mathrm{v}^2 + 2\mathrm{v}^4 + 2\mathrm{v}^6 + \mathrm{v}^8.$$

	c_1
c_1	c_1

	c_s	c_{sts}	c_{st}	c_t	c_{tst}	c_{ts}
c_s	$[2]c_s$	$[2]c_{sts}$	$[2]c_{st}$	c_{st}	$c_{st}+c_{w_0}$	$c_s + c_{sts}$
c_{sts}	$[2]c_{sts}$	$[2]c_s+[2]^2c_{w_0}$	$[2]c_{st}+[2]c_{w_0}$	$c_s + c_{sts}$	$c_s+[2]^2c_{w_0}$	$c_s + c_{sts}+[2]c_{w_0}$
c_{ts}	$[2]c_{ts}$	$[2]c_{ts}+[2]c_{w_0}$	$[2]c_t + [2]c_{tst}$	$c_t + c_{tst}$	$c_t + c_{tst}+[2]c_{w_0}$	$2c_{ts}+c_{w_0}$
c_t	c_{ts}	$c_{ts}+c_{w_0}$	$c_t + c_{tst}$	$[2]c_t$	$[2]c_{tst}$	$[2]c_{ts}$
c_{tst}	$c_t + c_{tst}$	$c_t+[2]^2c_{w_0}$	$c_t + c_{tst}+[2]c_{w_0}$	$[2]c_{tst}$	$[2]c_t+[2]^2c_{w_0}$	$[2]c_{ts}+[2]c_{w_0}$
c_{st}	$c_s + c_{sts}$	$c_s + c_{sts}+[2]c_{w_0}$	$2c_{st}+c_{w_0}$	$[2]c_{st}$	$[2]c_{st}+[2]c_{w_0}$	$[2]c_s + [2]c_{sts}$

	c_{w_0}
c_{w_0}	$\mathrm{v}^4 h^{w_0}_{w_0,w_0} c_{w_0}$

(Note the "subalgebras".)

◂ Back

Example (type B_2).

$\mathrm{v}h^{s}_{s,s}$

Thus, up to scaling(!), the $\mathrm{S}_{\mathcal{H}}$ are $\mathbb{C}(\mathrm{v})$, $\mathbb{C}(\mathrm{v})[\mathbb{Z}/2\mathbb{Z}]$ and $\mathbb{C}(\mathrm{v})$.
So $1+2+1$ simples, ordered by apex.
However, the Weyl group of type B_2 has $1+3+1$ simples, ordered by apex.

c_1	c_1

	c_s	c_{sts}	c_{st}	c_t	c_{tst}	c_{ts}
c_s	$[2]c_s$	$[2]c_{sts}$	$[2]c_{st}$	c_{st}	$c_{st}+c_{w_0}$	c_s+c_{sts}
c_{sts}	$[2]c_{sts}$	$[2]c_s+[2]^2c_{w_0}$	$[2]c_{st}+[2]c_{w_0}$	c_s+c_{sts}	$c_s+[2]^2c_{w_0}$	$c_s+c_{sts}+[2]c_{w_0}$
c_{ts}	$[2]c_{ts}$	$[2]c_{ts}+[2]c_{w_0}$	$[2]c_t+[2]c_{tst}$	c_t+c_{tst}	$c_t+c_{tst}+[2]c_{w_0}$	$2c_{ts}+c_{w_0}$
c_t	c_{ts}	$c_{ts}+c_{w_0}$	c_t+c_{tst}	$[2]c_t$	$[2]c_{tst}$	$[2]c_{ts}$
c_{tst}	c_t+c_{tst}	$c_t+[2]^2c_{w_0}$	$c_t+c_{tst}+[2]c_{w_0}$	$[2]c_{tst}$	$[2]c_t+[2]^2c_{w_0}$	$[2]c_{ts}+[2]c_{w_0}$
c_{st}	c_s+c_{sts}	$c_s+c_{sts}+[2]c_{w_0}$	$2c_{st}+c_{w_0}$	$[2]c_{st}$	$[2]c_{st}+[2]c_{w_0}$	$[2]c_s+[2]c_{sts}$

	c_{w_0}
c_{w_0}	$\mathrm{v}^4h^{w_0}_{w_0,w_0}c_{w_0}$

(Note the "subalgebras".)

◂ Back

Example (type B_2).

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Thus, up to scaling(!), the $\mathrm{S}_{\mathcal{H}}$ are $\mathbb{C}(\mathrm{v})$, $\mathbb{C}(\mathrm{v})[\mathbb{Z}/2\mathbb{Z}]$ and $\mathbb{C}(\mathrm{v})$.
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	c_1
c_1	c_1

	c_s	c_{sts}	c_{st}	c_t	c_{tst}	c_{ts}
c_s	$[2]c_s$	$[2]c_{sts}$	$[2]c_{st}$	c_{st}	$c_{st}+c_{w_0}$	c_s+c_{sts}
c_{sts}	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$+[2]c_{w_0}$
c_{ts}	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$+c_{w_0}$
c_t	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$]c_{ts}$
c_{tst}	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$+[2]c_{w_0}$
c_{st}	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$+[2]c_{sts}$

Crucial: "Up to scaling" is not a good notion for
the categorified world as we should work over $\mathbb{N}_{\mathrm{v}}=\mathbb{N}[\mathrm{v},\mathrm{v}^{-1}]$ or $\mathbb{Z}_{\mathrm{v}}$.

Using appropriate versions of simple $\mathbb{N}_{\mathrm{v}}$-representations,
one almost recovers the $\mathcal{H}$-cell theorem.

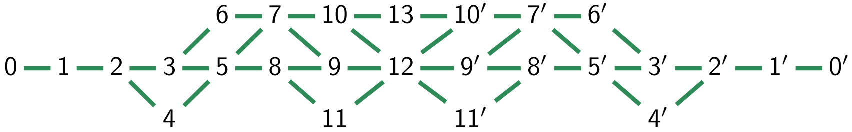
However, $\mathrm{S}_{\mathcal{H}}$ are still not nice over $\mathbb{N}_{\mathrm{v}}$ or $\mathbb{Z}_{\mathrm{v}}$.

(Note the "subalgebras".)

◂ Back

Example (SAGEMath). The Weyl group of type B_6. Number of elements: 46080. Number of cells: 26, named 0 (lowest) to 25 (biggest).

Cell order:



Size of the cells and **a**-value:

cell	0	1	2	3	4	5	6	7	8	9	10	11	12=12′	13=13′	11′	10′	9′	8′	7′	6′	5′	4′	3′	2′	1′	0′
size	1	62	342	576	650	3150	350	1600	2432	3402	900	2025	14500	600	2025	900	3402	2432	1600	350	3150	650	576	342	62	1
a	0	1	2	3	3	4	4	5	5	6	6	6	7	9	10	10	10	11	11	16	12	15	17	18	25	36

◂ Back

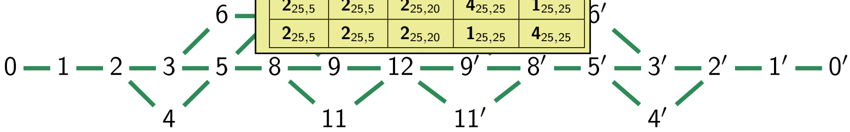
Example (cell 12).

Cell 12 is a bit scary:

$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{5,5}$	$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{20,5}$	$\mathbf{1}_{20,5}$	$\mathbf{4}_{20,20}$	$\mathbf{2}_{20,25}$	$\mathbf{2}_{20,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{4}_{25,25}$	$\mathbf{1}_{25,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{1}_{25,25}$	$\mathbf{4}_{25,25}$

Example (SAGEMath). umber of elements: 46080.
Number of cells: 26, nam

Cell order:



Size of the cells and **a**-value:

cell	0	1	2	3	4	5	6	7	8	9	10	11	12=12′	13=13′	11′	10′	9′	8′	7′	6′	5′	4′	3′	2′	1′	0′
size	1	62	342	576	650	3150	350	1600	2432	3402	900	2025	14500	600	2025	900	3402	2432	1600	350	3150	650	576	342	62	1
a	0	1	2	3	3	4	4	5	5	6	6	6	7	9	10	10	10	11	11	16	12	15	17	18	25	36

◂ Back

Example (SAGEMath). Here is a random calculation in the cell 12 for type B_6.

Graph:

Elements (shorthand $s_i = i$):

$$d = d^{-1} = 132123565.$$

◂ Back

Example (SAGEMath). Here is a random calculation in the cell 12 for type B_6.

$$\begin{gathered} c_d c_d = \\ (1 + 5v^2 + 12v^4 + 18v^6 + 18v^8 + 12v^{10} + 5v^{12} + v^{14}) c_d \\ +(v^2 + 4v^4 + 7v^6 + 7v^8 + 4v^{10} + v^{12}) c_{12132123565} \\ +(v^{-4} + 5v^{-2} + 11 + 14v^2 + 11v^4 + 5v^6 + v^8) c_{121232123565} \end{gathered}$$

Graph:

Elements (shorthand $s_i = i$):

$$d = d^{-1} = 132123565.$$

◂ Back

Example (SAGEMath). Here is a random calculation in the cell 12 for type B_6.

$$
\begin{gathered}
a_d a_d = \\
(1 + 5\mathrm{v}^2 + 12\mathrm{v}^4 + 18\mathrm{v}^6 + 18\mathrm{v}^8 + 12\mathrm{v}^{10} + 5\mathrm{v}^{12} + \mathrm{v}^{14})c_d \\
+(\mathrm{v}^2 + 4\mathrm{v}^4 + 7\mathrm{v}^6 + 7\mathrm{v}^8 + 4\mathrm{v}^{10} + \mathrm{v}^{12})c_{12132123565} \\
+(\mathrm{v}^{-4} + 5\mathrm{v}^{-2} + 11 + 14\mathrm{v}^2 + 11\mathrm{v}^4 + 5\mathrm{v}^6 + \mathrm{v}^8)c_{121232123565}
\end{gathered}
$$

Graph:

Elements (shorthand $s_i = i$):

$$d = d^{-1} = 132123565.$$

◂ Back

Example (SAGEMath). Here is a random calculation in the cell 12 for type B_6.

$$\begin{gathered} a_d a_d = \\ (1 + 5\mathrm{v}^2 + 12\mathrm{v}^4 + 18\mathrm{v}^6 + 18\mathrm{v}^8 + 12\mathrm{v}^{10} + 5\mathrm{v}^{12} + \mathrm{v}^{14})c_d \\ +(\mathrm{v}^2 + 4\mathrm{v}^4 + 7\mathrm{v}^6 + 7\mathrm{v}^8 + 4\mathrm{v}^{10} + \mathrm{v}^{12})c_{12132123565} \\ +(\mathrm{v}^{-4} + 5\mathrm{v}^{-2} + 11 + 14\mathrm{v}^2 + 11\mathrm{v}^4 + 5\mathrm{v}^6 + \mathrm{v}^8)c_{121232123565} \end{gathered}$$

Bigger friends.

Graph:

Elements (shorthand $s_i = i$):

$$d = d^{-1} = 132123565.$$

Back

Example (SAGEMath). Here is a random calculation in the cell 12 for type B_6.

$$
\begin{gathered}
a_d a_d = \\
(1 + 5\mathrm{v}^2 + 12\mathrm{v}^4 + 18\mathrm{v}^6 + 18\mathrm{v}^8 + 12\mathrm{v}^{10} + 5\mathrm{v}^{12} + \mathrm{v}^{14})c_d \\
+(\mathrm{v}^2 + 4\mathrm{v}^4 + 7\mathrm{v}^6 + 7\mathrm{v}^8 + 4\mathrm{v}^{10} + \mathrm{v}^{12})c_{12132123565}
\end{gathered}
$$

Graph:

$$1 \overset{4}{\textbf{—}} 2 \textbf{—} 3 \textbf{—} 4 \textbf{—} 5 \textbf{—} 6$$

Elements (shorthand $s_i = i$):

$$d = d^{-1} = 132123565.$$

◂ Back

Example (SAGEMath). Here is a random calculation in the cell 12 for type B_6.

$$a_d a_d =$$
$$(1+5v^2+12v^4+18v^6+18v^8+12v^{10}+5v^{12}+v^{14})c_d$$
$$+(v^2+4v^4+7v^6+7v^8+4v^{10}+v^{12})c_{12132123565}$$

Killed in the limit v → 0.

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = i$):

$$d = d^{-1} = 132123565.$$

◂ Back

Example (SAGEMath). Here is a random calculation in the cell 12 for type B_6.

$$a_d a_d =$$

$$a_d$$

Looks much simpler.

Graph:

$$1 \overset{4}{\text{—}} 2 \text{—} 3 \text{—} 4 \text{—} 5 \text{—} 6$$

Elements (shorthand $s_i = i$):

$$d = d^{-1} = 132123565.$$

◀ Back

Example (SAGEMath). The Weyl group of type B_6.

cell	0	1	2	3	4	5	6	7	8	9	10	11	12=12′	13=13′	11′	10′	9′	8′	7′	6′	5′	4′	3′	2′	1′	0′
size	1	62	342	576	650	3150	350	1600	2432	3402	900	2025	14500	600	2025	900	3402	2432	1600	350	3150	650	576	342	62	1
a	0	1	2	3	3	4	4	5	5	6	6	6	7	9	10	10	10	11	11	16	12	15	17	18	25	36
2^k	1	2	2	1	2	2	2	1	2	2	1	1	4	2	1	1	2	2	1	2	2	2	1	2	2	1
#simples	1	3	3	1	3	3	3	1	3	3	1	1	10	3	1	1	3	3	1	3	3	3	1	3	3	1
2^{2k}	1	4	4	1	4	4	4	1	4	4	1	1	16	4	1	1	4	4	1	4	4	4	1	4	4	1

Actually, $\#\{\text{simples with apex } \mathcal{J}\} = \frac{1}{2}(2^{2k} + 2^k)$ (the middle).

◂ Back

Fusion categories. (Multi)fusion categories $\mathscr{C}$ over $\mathbb{C}$ are as easy as possible while being interesting:

- By definition, they are monoidal, rigid, semisimple, $\mathbb{C}$-linear categories with finitely many simple objects.
- They decategorify to (multi)fusion rings.
- **Ocneanu rigidity.** The number of multifusion categories (up to equivalence) with a given Grothendieck ring is finite.
- **Ocneanu rigidity.** The number of equivalence classes of simple transitive 2-representations over a given multifusion category is finite.
- **Crucial.** The latter two points are wrong if one drops the semisimplicity. (Counterexamples are known.)

Fusion categories—complete classification.

- Group-like. $\mathscr{C} \cong \mathscr{R}\mathrm{ep}(G)$ or twists; G finite group.
- Group-like. $\mathscr{C} \cong \mathscr{V}\mathrm{ect}(G)$ or twists; G finite group.
- Quantum groups. Semisimplifications of quantum group representations at roots of unity or twist of such.
- Exotic fusion categories. Coming *e.g.* from subfactors or Soergel bimodules.

Back

Folk theorem(?). The simple transitive 2-representations of $\mathscr{R}\mathrm{ep}(G)$ and $\mathscr{V}\mathrm{ect}(G)$ are classified by subgroups $H \subset G$ and $\phi \in H^2(H, \mathbb{C}^\times)$, up to conjugacy.

The classification is thus a numerical problem.

For example, for $\mathscr{R}\mathrm{ep}(S_5)$ (appears in type E_8) we have:

$\mathscr{R}\mathrm{ep}(S_5)$

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/5\mathbb{Z}$	S_3	$\mathbb{Z}/6\mathbb{Z}$	D_4	D_5	A_4	D_6	$GA(1,5)$	S_4	A_5	S_5
$\#$	1	2	1	1	2	1	2	1	1	1	1	1	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	$4, 1$	5	3	6	$5, 2$	$4, 2$	$4, 3$	$6, 3$	5	$5, 3$	$5, 4$	$7, 5$

This is completely different from their classical representation theory.

◀ Back

Example (type B_2).

$$W \begin{cases} = \langle s, t \mid s^2 = t^2 = 1, tsts = stst \rangle \\ = \{1, s, t, st, ts, sts, tst, w_0\}. \end{cases}$$

$$\mathrm{H}^{\mathrm{v}}(W) \begin{cases} = \langle h_s, h_t \mid h_s^2 = (\mathrm{v}^{-1} - \mathrm{v})h_s + 1, h_t^2 = (\mathrm{v}^{-1} - \mathrm{v})h_t + 1, h_t h_s h_t h_s = h_s h_t h_s h_t \rangle \\ = \mathbb{Z}_{\mathrm{v}}\{h_1, h_s, h_t, h_{st}, h_{ts}, h_{sts}, h_{tst}, h_{w_0}\}. \end{cases}$$

In general, $\mathrm{H}^{\mathrm{v}}(W = (W|S))$ is generated by h_s for $s \in S$, which satisfy the quadratic relations and the braid relations.

KL basis:

$$\mathrm{H}^{\mathrm{v}}(W) = \mathbb{Z}_{\mathrm{v}}\{c_1 = 1, c_s = \mathrm{v}(h_s + \mathrm{v}), c_t = \mathrm{v}(h_t + \mathrm{v}), c_{st}, c_{ts}, c_{sts}, c_{tst}, c_{w_0}\}.$$

$c_s^2 = (1 + \mathrm{v}^2)c_s = [2]c_s$. (Quasi-idempotent, but "positively graded".)

◂ Back

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$ (G a finite group).
- $\mathscr{C}$ is fusion (fiat and semisimple). For any $\mathtt{M}, \mathtt{N} \in \mathscr{C}$, we have $\mathtt{M} \otimes \mathtt{N} \in \mathscr{C}$:

  $$g(m \otimes n) = gm \otimes gn$$

  for all $g \in G, m \in \mathtt{M}, n \in \mathtt{N}$. There is a trivial representation 1.
- The regular 2-representation $\mathscr{M} \colon \mathscr{C} \to \mathscr{E}\mathrm{nd}(\mathscr{C})$:

$$\begin{tikzcd}
\mathtt{M} \arrow[r] \arrow[d, "f"'] & \mathtt{M} \otimes \_ \arrow[d, "f \otimes \_"] \\
\mathtt{N} \arrow[r] & \mathtt{N} \otimes \_
\end{tikzcd}.$$

- The decategorification is a $\mathbb{N}$-representation, the regular representation.
- The associated (co)algebra object is $\mathtt{A}_{\mathscr{M}} = 1 \in \mathscr{C}$.

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $K \subset G$ be a subgroup.
- $\mathcal{R}\mathrm{ep}(K)$ is a 2-representation of $\mathscr{R}\mathrm{ep}(G)$, with action

  $$\mathcal{R}es^G_K \otimes \_ \colon \mathscr{R}\mathrm{ep}(G) \to \mathscr{E}\mathrm{nd}(\mathcal{R}\mathrm{ep}(K))$$

  which is indeed a 2-action because $\mathcal{R}es^G_K$ is a $\otimes$-functor.
- The decategorifications are $\mathbb{N}$-representations.
- The associated (co)algebra object is $\mathtt{A}_{\mathscr{M}} = \mathcal{I}nd^G_K(1_K) \in \mathscr{C}$.

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $\psi \in H^2(K, \mathbb{C}^*)$. Let $\mathcal{V}(K,\psi)$ be the category of projective K-modules with Schur multiplier ψ, *i.e.*vector spaces $\mathtt{V}$ with $\rho\colon K \to \mathcal{E}\mathrm{nd}(\mathtt{V})$ such that

$$\rho(g)\rho(h) = \psi(g,h)\rho(gh), \text{ for all } g, h \in K.$$

- Note that $\mathcal{V}(K,1) = \mathcal{R}\mathrm{ep}(K)$ and

$$\otimes\colon \mathcal{V}(K,\phi) \boxtimes \mathcal{V}(K,\psi) \to \mathcal{V}(K,\phi\psi).$$

- $\mathcal{V}(K,\psi)$ is also a 2-representation of $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$:

$$\mathscr{R}\mathrm{ep}(G) \boxtimes \mathcal{V}(K,\psi) \xrightarrow{\mathcal{R}es^G_K \boxtimes \mathrm{Id}} \mathcal{R}\mathrm{ep}(K) \boxtimes \mathcal{V}(K,\psi) \xrightarrow{\otimes} \mathcal{V}(K,\psi).$$

- The decategorifications are $\mathbb{N}$-representations.
- The associated (co)algebra object is $\mathtt{A}_{\mathscr{M}} = \mathcal{I}nd^G_K(1_K) \in \mathscr{C}$, but with ψ-twisted multiplication.

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $\psi \in H^2(K, \mathbb{C}^*)$. Let $\mathcal{V}(K,\psi)$ be the category of projective K-modules with Schur multiplier ψ, *i.e.*vector spaces $\mathtt{V}$ with $\rho\colon K \to \mathcal{E}\mathrm{nd}(\mathtt{V})$ such that

Theorem (folklore?).

Completeness. All 2-simples of $\mathscr{R}\mathrm{ep}(G)$ are of the form $\mathcal{V}(K,\psi)$.

Non-redundancy. We have $\mathcal{V}(K,\psi) \cong \mathcal{V}(K',\psi')$

$\Leftrightarrow$

the subgroups are conjugate or $\psi' = \psi^g$, where $\psi^g(k,l) = \psi(gkg^{-1}, glg^{-1})$.

◂ Back

$$\mathscr{R}\mathrm{ep}(G) \boxtimes \mathcal{V}(K,\psi) \xrightarrow{\mathrm{Res}_K \boxtimes \mathrm{id}} \mathcal{R}\mathrm{ep}(K) \boxtimes \mathcal{V}(K,\psi) \xrightarrow{\otimes} \mathcal{V}(K,\psi).$$

- The decategorifications are $\mathbb{N}$-representations.
- The associated (co)algebra object is $\mathtt{A}_{\mathscr{M}} = \mathcal{I}nd_K^G(1_K) \in \mathscr{C}$, but with ψ-twisted multiplication.