2-rep^n theory of Soergel bimodules

Thanks — jt work w/ Mackaay, Mazorchuk, Tubbenhauer, Zhang

Goal: Classify "simple" 2-rep^{n}s for Soergel bimodules.

① Background $\Bbbk = \bar{\Bbbk}$ field

Let $\mathfrak{A}^f_{\Bbbk}$ be the 2-cat w/ obj: small cats $\simeq$ A-proj (or A-inj) A f.d. $\Bbbk$-alg, 1-mor: add^{ve} $\Bbbk$-linear functors, 2-mor: nat^l transf

A 2-category $\mathcal{C}$ with objects $i, j, \ldots$, 1-mor $F, G, \mathbb{1}_i$, 2-mor $\alpha, \beta, \mathrm{id}_F$ is finitary if

- $\forall i, j \in \mathcal{C}$: $\mathcal{C}(i,j) \in \mathfrak{A}^f_{\Bbbk}$
- horizontal comp is biadd^{ve} & $\Bbbk$-linear
- $\forall i \in \mathcal{C}$, $\mathbb{1}_i$ is indec.

It is fiat if each 1-mor $F \in \mathcal{C}(i,j)$ has a biadjoint $F^* \in \mathcal{C}(j,i)$.

Example
- finitary quotients of Kac-Moody 2-cats
- Soergel bimodules $\mathcal{S}_W$ over the coinvariant algebra, W fin^{te} Coxeter gp

(W,S,V) Cox sys, $C = \dfrac{\mathbb{R}[V]}{(\mathbb{R}[V]^W_+)}$, $C_s := C \otimes_{C^s} C$, $\mathcal{S}$ gen^d monoidally by C_s, $s \in S$

- fusion categories (s.s. rig^d ⊗-cat with indec $\mathbb{1}$)

From now on $\mathcal{C}$ finitary 2-cat.

A finitary 2-rep^n of $\mathcal{C}$ is a (strict) 2-functor $\mathcal{C} \to \mathfrak{A}^f_{\Bbbk}$, i.e.

$\forall i \in \mathcal{C}$ $\mathbf{M}(i) \simeq B_i$-proj, $\forall F \in \mathcal{C}(i,j)$ $\mathbf{M}(F): \mathbf{M}(i) \to \mathbf{M}(j)$ add^{ve} $\Bbbk$-lin functor

$\forall \alpha: F \to G$ $\mathbf{M}(\alpha): \mathbf{M}(F) \to \mathbf{M}(G)$ nat^l transf

Example • $\mathbf{P}_i = \mathcal{C}(i,-)$

A finitary 2-rep^n $\mathbf{M}$ of $\mathcal{C}$ is simple transitive if $\coprod_{i \in \mathcal{C}} \mathbf{M}(i)$ has no proper $\mathcal{C}$-stable ideals

⇝ analogue of Jordan-Hölder theorem (Mazorchuk-M)

Thm (MMMT) $\mathbf{M}$ simple trans 2-rep^n of $\mathcal{C}$ $\Rightarrow$ $\exists$ cosimple coalg 1-mor C in $\underline{\mathcal{C}}$ (injective abelianisation)

s.t. $\mathbf{M} \simeq \mathrm{inj}_{\underline{\mathcal{C}}}(C)$ (injective right C-comodules)

$\exists$ bij $\left\{ \begin{matrix} \text{simple trans} \\ \text{2-rep}^n\text{s of } \mathcal{C} \end{matrix} \right\} / \text{equiv} \simeq \left\{ \begin{matrix} \text{cosimple coalg} \\ \text{1-mor in } \underline{\mathcal{C}} \end{matrix} \right\} / \text{Morita-Takeuchi equiv}

② Cells - recap

On isoclasses of 1-mor define $F \leq_{\mathcal{L}} G$ if $\exists H$ s.t. $G \mid HF$ (direct summand of)

$\leq_{\mathcal{R}}$ … FH

$\leq_{\mathcal{J}}$ … $\exists H_1, H_2$ … $G \mid H_1 F H_2$

Equiv classes are left/right/two-sided ($\mathcal{J}$) - cells. $\mathcal{H}$-cells: intersection of left/right cells

Examples: see Daniel's talk!

Fact $\mathbf{M}$ simple trans 2-repⁿ $\Rightarrow \exists! \mathcal{J}$ two-sided cell s.t. $\mathbf{M}(\mathcal{J}') = 0 \ \forall \mathcal{J}' \not\leq \mathcal{J}$ (apex)

For a left cell $\mathcal{L}$, define cell 2-repⁿ $\mathbf{C}_{\mathcal{L}}$ as quotient of $\mathrm{add}\{FX \mid F \in \mathcal{C}, X \in \mathcal{L}\}$ (add^e closure) by its unique (Prop) max^l $\mathcal{C}$-stable ideal

Note: simple transitive by definition!

Interesting question: Under what conditions are all s.t. 2-repⁿs (equiv to) cell 2-repⁿs?

True for finite (→ Mazorchuk-M) and affine (→ MacPherson) Kac-Moody, but not Soergel bimodules!

Why?

Fix $\mathcal{L} \subset \mathcal{J}$ (left cell of two-sided cell), set $\mathcal{H} = \mathcal{L} \cap \mathcal{L}^*$ e.g. $\mathcal{J}_{B_2}$ $\mathcal{L}$: 1; s, sts, st; ts, tst; $stst$

Define $\mathcal{C}_{\mathcal{H}}$ as follows: • take quotient by all $\mathcal{J}' \not\leq \mathcal{J}$;

• inside quotient take additive closure of $\mathbb{1}$, and 1-mor in $\mathcal{H}$.

(quotient out by max^l 2-ideal not containing id_F for $F \in \mathcal{H}$.)

Thm (Mackaay-Mazorchuk-M-Zhang)

$\exists$ bijection

$$\left\{ \begin{matrix} \text{s.t. 2-rep}^{\text{n}}\text{s of } \mathcal{C} \\ \text{w apex } \mathcal{J} \end{matrix} \right\} \Big/ \text{equiv} \quad \xleftrightarrow{1-1} \quad \left\{ \begin{matrix} \text{s.t. 2-rep}^{\text{n}}\text{s of } \mathcal{C}_{\mathcal{H}} \\ \text{w/ apex } \mathcal{H} \end{matrix} \right\}$$

(analogy semigroups Daniel's talk)

→ concentrate on $\mathcal{C}_{\mathcal{H}}$: smaller!!

(Remark: reason why Kac-Moody ~ only cell 2-repⁿ: all $\mathcal{H}$-cells are trivial)

Soergel bimodules: suffices to classify 2-rep^{n}s for $\mathcal{S}_{\mathcal{H}}$.

Recall (Daniel): To $\mathcal{H}$, associate bicategory $\mathcal{A}_{\mathcal{H}}$ (fusion (s.s.) simple transitive 2-rep's classified in most cases all Weyl types)

C_w = indec Soergel bimodule $\sim w \in W$
with top in deg zero (ie $C_w \mid C_1^{s_{i_1}} \otimes C_1^{s_{i_2}} \otimes \cdots \otimes C_1^{s_{i_r}}$, C_1 coinv. alg, $s_{i_1} \cdots s_{i_r} = w$ red exp)

$\mathcal{A}_{\mathcal{H}} = \mathrm{add}\{C_w\langle t\rangle \mid t \leq 0\} \,/\, \mathrm{add}\{C_w\langle t\rangle \mid t < 0\}$

C_w — deg 0, — deg 1, $C_w\langle -1\rangle$

$\mathcal{A}_{\mathcal{H}} \ni [C_w] =: A_w$

Fact
$\exists$ oplax bifunctor $\textcircled{H}\colon \mathcal{A}_{\mathcal{H}} \longrightarrow \mathcal{S}_{\mathcal{H}}$

(by general nonsense $\rightsquigarrow$ $A \in \mathcal{A}_{\mathcal{H}}$ coalg 1-mor $\Rightarrow$ $\textcircled{H}(A) \in \mathcal{S}_{\mathcal{H}}$ coalg 1-mor)

<u>Prop</u> $\textcircled{H}$ preserves cosimplicity & MT-equiv & MT-inequiv

$\therefore$ Have injection $\hat{\textcircled{H}}\colon \left\{\begin{matrix}\text{simple trans}\\ \text{2-rep's of } \mathcal{A}_{\mathcal{H}}\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{s.t. 2-rep}^n\text{s}\\ \text{of } \mathcal{S}_{\mathcal{H}} \text{ w/ apex } \mathcal{H}\end{matrix}\right\} \left(\overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\text{s.t. 2-rep}^n\text{s}\\ \text{of } \mathcal{S} \text{ w/ apex}\end{matrix}\right\}\right)$

<u>Conj</u> $\hat{\textcircled{H}}$ is bijective.

Proved:
- type A — ancient MM
- type B/D — $(\mathbb{Z}/2\mathbb{Z})^k$ (new unpublished) via End Joey-Walter
- type F_4, E_6, E_7 — $(\mathbb{Z}/2\mathbb{Z})^k$ ($\exists \mathrm{Vec}_{\mathbb{Z}/2\mathbb{Z}}$ but contains w_0^I) or $w_0^I \in \mathcal{J}$
- type E_8 — one open case $\mathrm{Vec}^{\xi}\,\mathbb{Z}/2\mathbb{Z}$ no w_0^I
- H_3 — all but one $\mathcal{J}$-cell (no group-like $\mathcal{H}$-cell)
- H_4 — all but three $\mathcal{J}$-cells " (no idea what $\mathcal{S}_{\mathcal{H}}$ is)
- $I_2(n)$ ✓ $w_0^I \in \mathcal{J}$

ingredients: C_d Duflo inv is a Frobenius alg 1-mor in $\mathcal{S}_{\mathcal{H}}$

$\mathrm{End}_{\mathcal{S}_{\mathcal{H}}}(\mathbb{C}_{\mathcal{H}}) \approx \mathcal{A}_{\mathcal{H}}$ $\qquad$ $\mathrm{End}^{p,r}_{\mathcal{A}_{\mathcal{H}}}(\mathbb{C}_{\mathcal{H}}) \approx \mathcal{S}_{\mathcal{H}}$