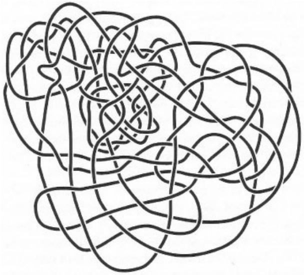
Why (categorical) representation theory?

Or: Representing symmetries

Daniel Tubbenhauer



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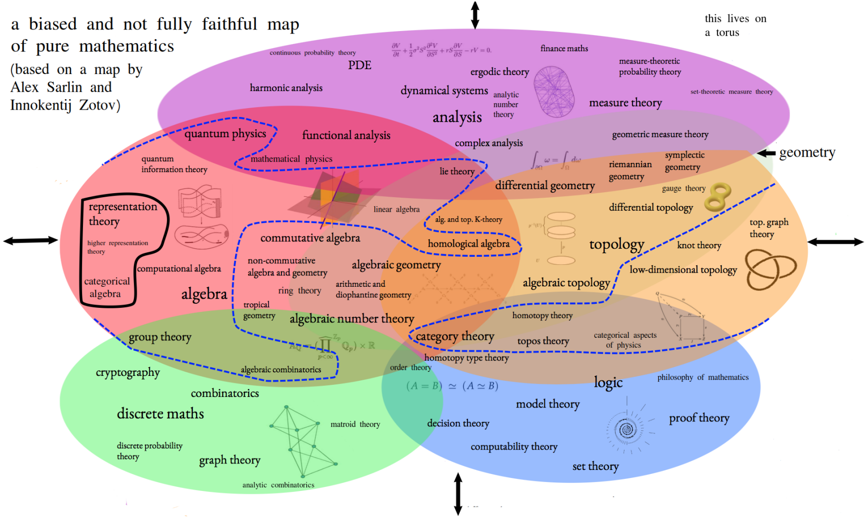
August 2021

Where are we?

a biased and not fully faithful map of pure mathematics

(based on a map by Alex Sarlin and Innokentij Zotov)

this lives on a torus



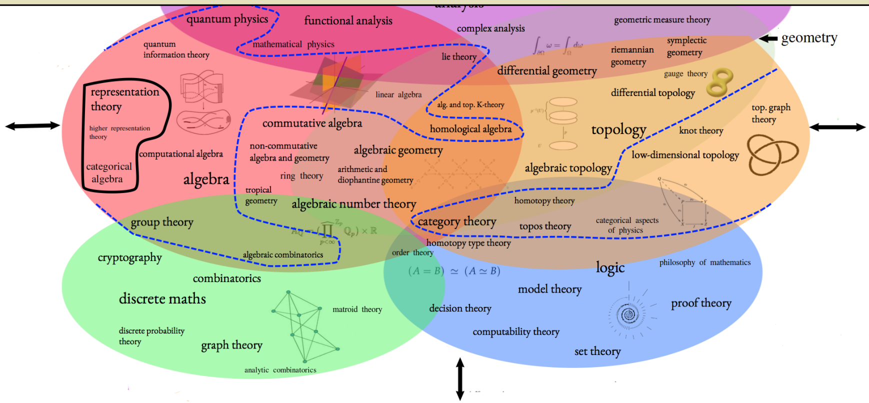
The six main fields of pure mathematics algebra, analysis, geometry, topology, logic, discrete mathematics

Today

Black box. Representation theory and its categorical analog (brief) My research area

Dashed box. Where I like to apply them My research outreach

Applications beyond my current research? The future (within OIST?)

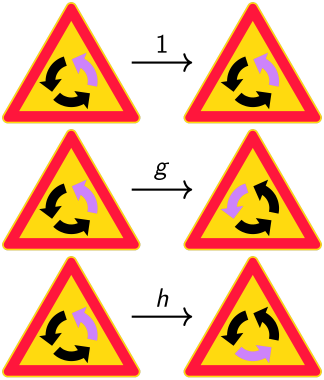


The six main fields of pure mathematics algebra, analysis, geometry, topology, logic, discrete mathematics

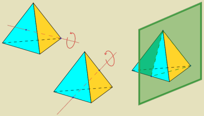
Representation theory – symmetries in vector spaces

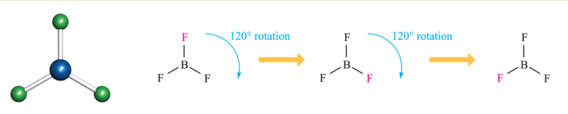
$\cdot$	1	g	h
1	1	g	h
g	g	h	1
h	h	1	g

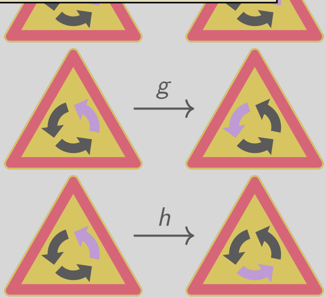
e.g. $gh = 1$



Symmetry is everywhere







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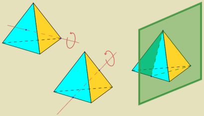
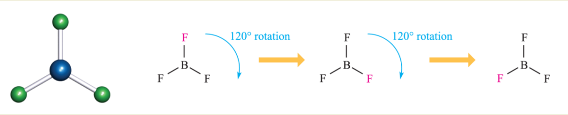




Symmetry in mathematics

Discrete symmetries (finite groups)

Smooth symmetries (Lie groups and Lie algebras) I like these

Algebras I like these

More...

Representation theory – symmetries in vector spaces

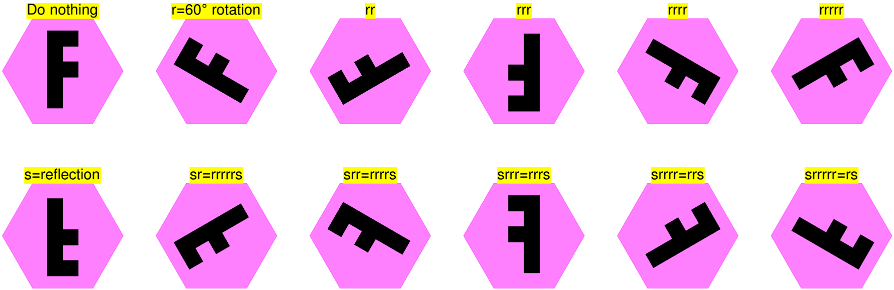
What symmetries satisfy

(a) We have a composition rule $\circ(g,h) = gh$ Multiplication

(b) We have $g(hf) = (gh)f$ Associativity

(c) There is a do nothing operation $1g = g = g1$ Unit

(d) There is an undo operation $gg^{-1} = 1 = g^{-1}g$ Inverse



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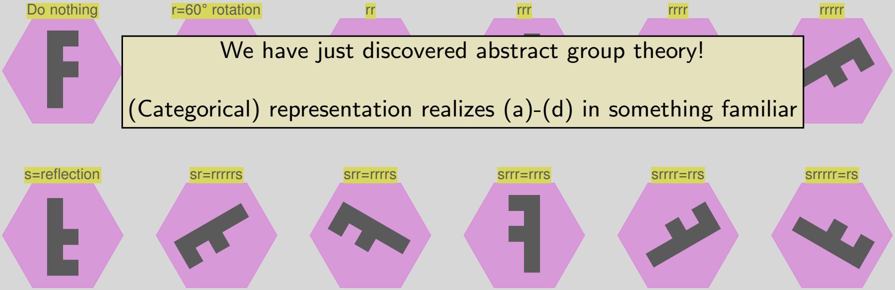
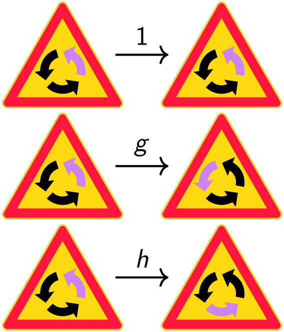


We have just discovered abstract group theory!

(Categorical) representation realizes (a)-(d) in something familiar

Representation theory – symmetries in vector spaces

Representation theory associates linear objects to symmetries, *e.g.*



$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g \mapsto \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

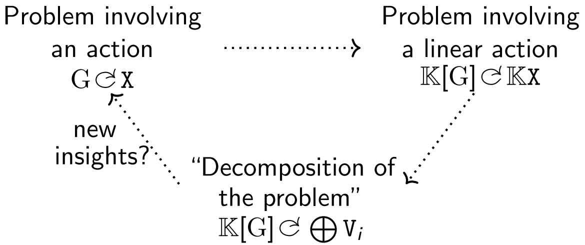
$$h \mapsto \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

- Representations are in the realm of linear algebra (matrices, vector spaces, *etc.*)
- Upshots. We can now talk about simple representations (the elements of the theory), we can vary the underlying scalars, and play other games

Representation theory – symmetries in vector spaces

The study of symmetries/actions is of fundamental importance in mathematics and related field, but it is also very hard

Representation theory approach. The analog linear problems, *e.g.* classifying simple representations, have often satisfying answers

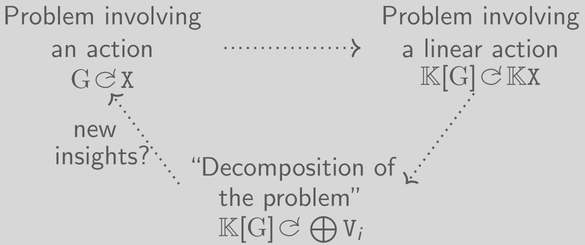


Representation theory has been unreasonable successful in mathematics

Wikipedia quote *"Representation theory is pervasive across fields of mathematics"*
In all the six main fields!

I will zoom into one aspect that I like and have worked on momentarily

simple representations, have often satisfying answers



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Representation theory has been unreasonable successful in mathematics

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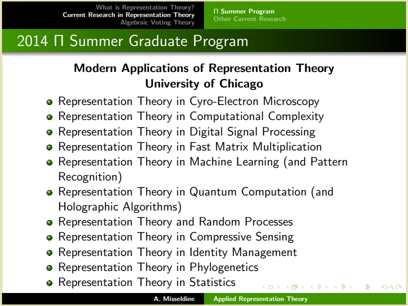
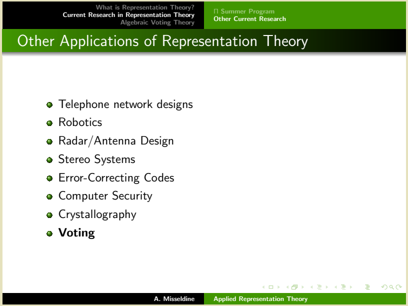
Representation theory has been unreasonable successful in the sciences



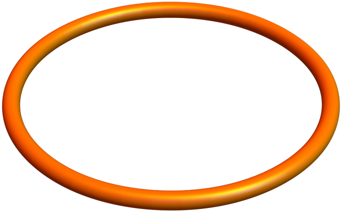


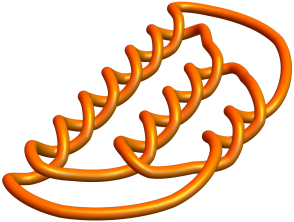
Also nice: `https://yetanothermathblog.com/2016/08/06/real-world-applications-of-representation-theory/`

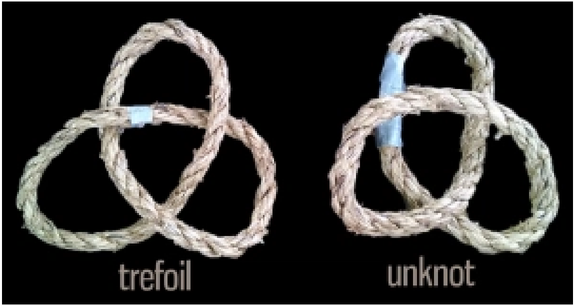
I would like to learn more about these!

Representation theory and knots – this is green and red

A knot/link is a string in three-space

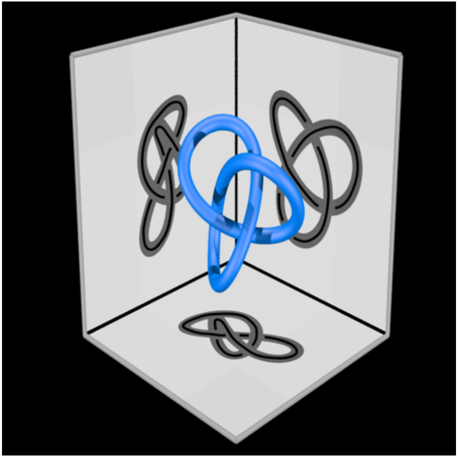


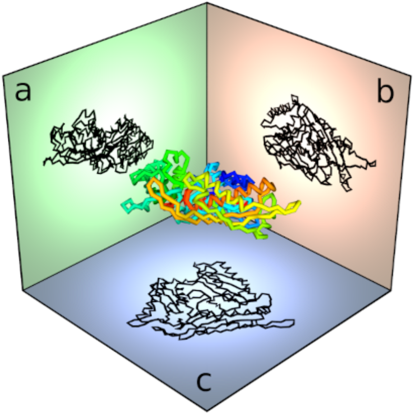




Representation theory and knots – this is green and red

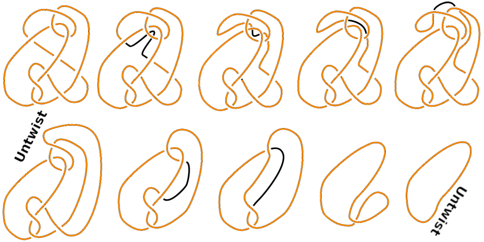
A projection (called a knot/link diagram) is a 2d shadow





Left Trefoil Right Knotted protein

Projections might vary drastically

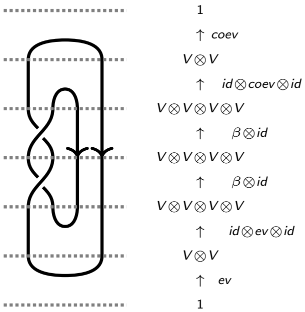


Knot theory is mostly the search for knot invariants – numerical data computed from a projection that depends only on the knot:

$$\text{invariants different} \;\Rightarrow\; \text{knots different}$$

Representation theory and knots – this is green and red

Witten–Reshetikhin–Turaev (WRT, $\sim$1990): use representation theory!



(a) Put the projection in a Morse position

(b) To each generic horizontal slice associate a representation of a quantum group "A non-commutative symmetry"

(c) To each basic piece associate a linear map

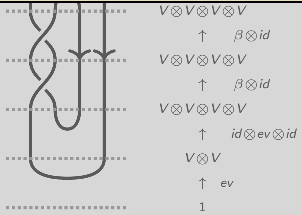
(d) The whole construction gives a family of invariants

Representation theory and knots – this is green and red

Witten–Resh … y!

WRT opened a new field of mathematics – quantum algebra

If you ask me at 2am in the morning
"What are you doing for a living?"
I might answer "Quantum algebra"



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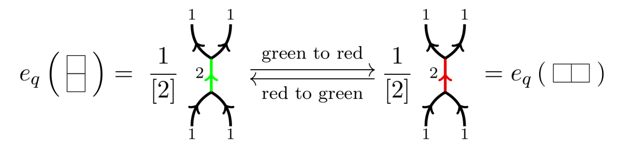
$V \otimes V \otimes V \otimes V$

Quantum algebra is strongly merged with diagrammatic representation theory

For example, there are colorful proofs of symmetries with knot invariants:



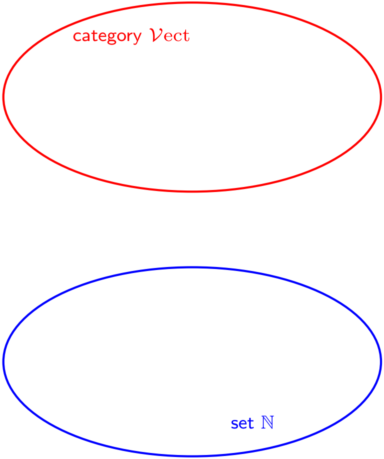
(Picture from one of my papers – using representations of Hecke and super Lie algebras)

A non-commutative symmetry

(c) To each basic piece associate a linear map

(d) The whole construction gives a family of invariants

Categorification in a nutshell



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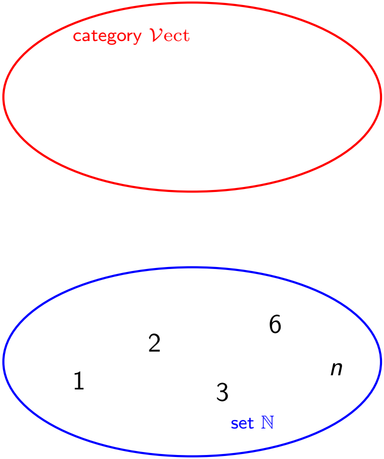

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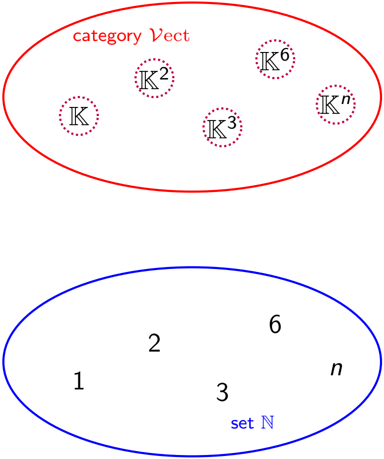

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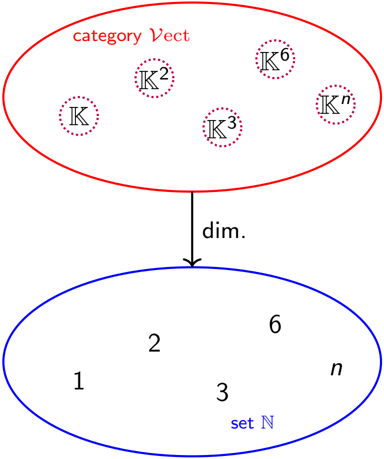

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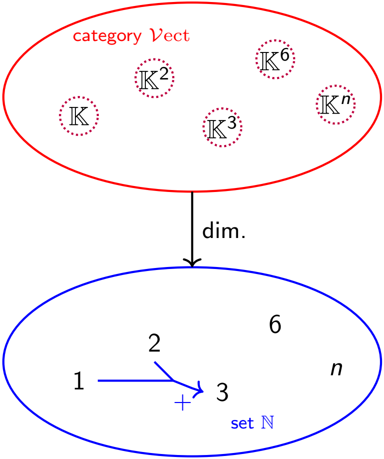

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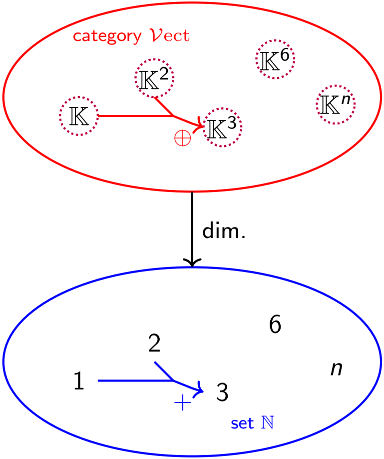

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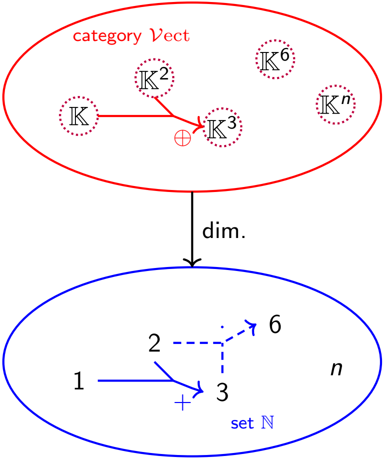

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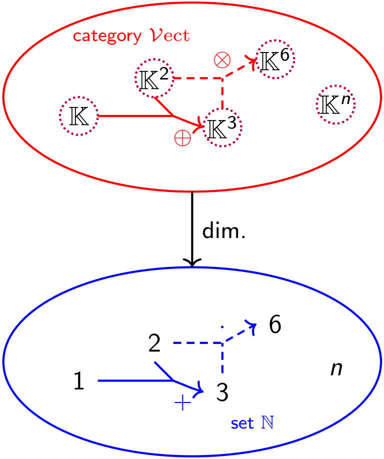

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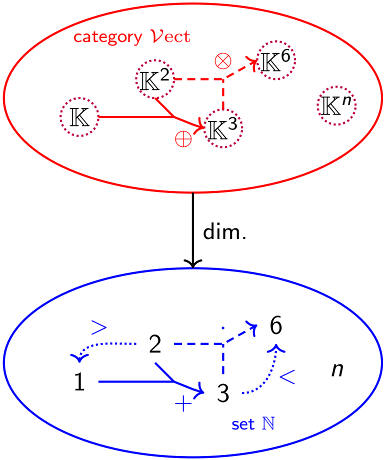

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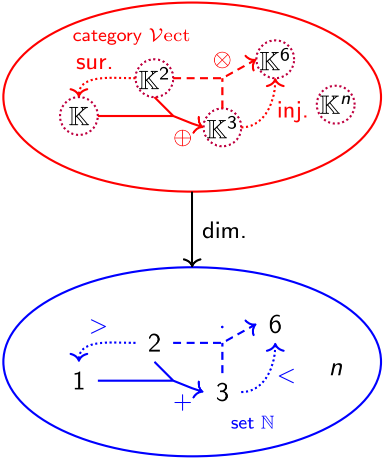

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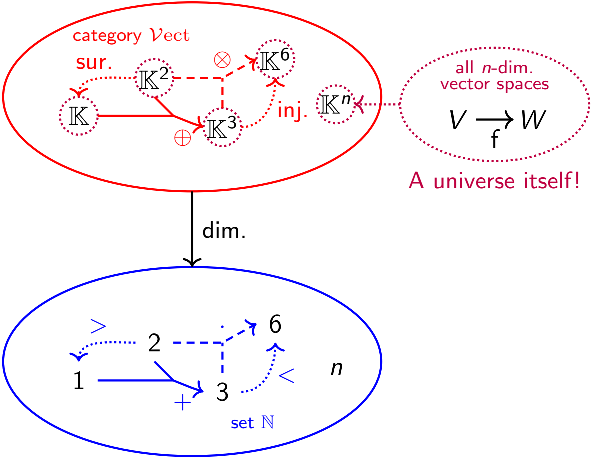

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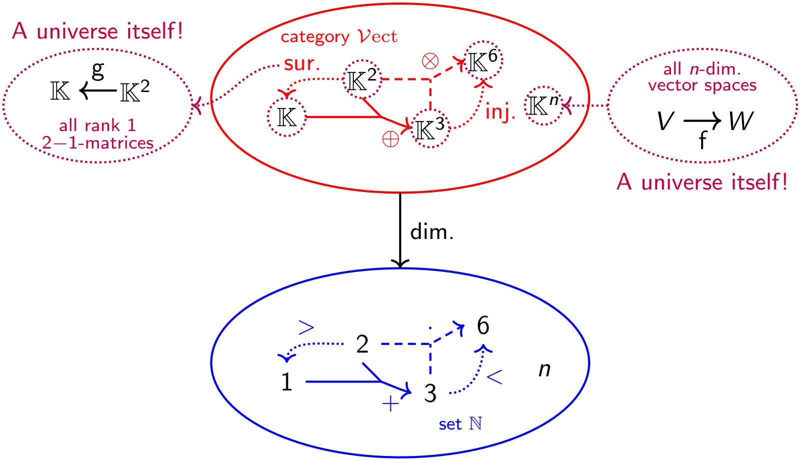

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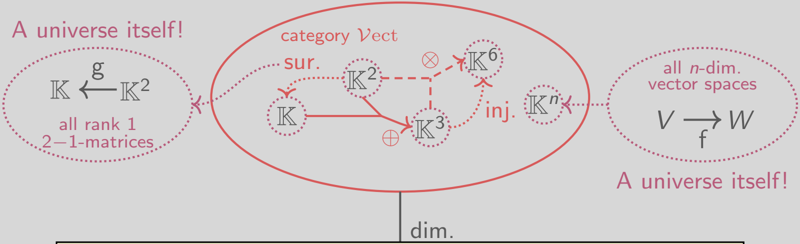


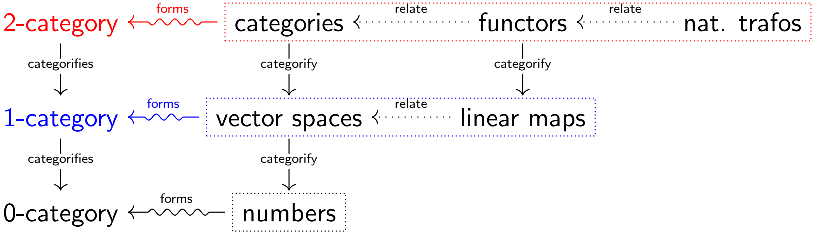
The idea underlying categorification

The category $\mathcal{V}$ect has the whole power of linear algebra at hand!

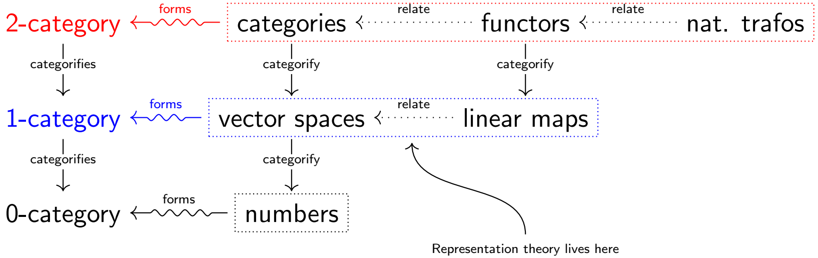
There is nothing comparable for $\mathbb{N}$:
$\mathbb{N}$ is just a shadow of $\mathcal{V}$ect

set $\mathbb{N}$

Categorical representation theory

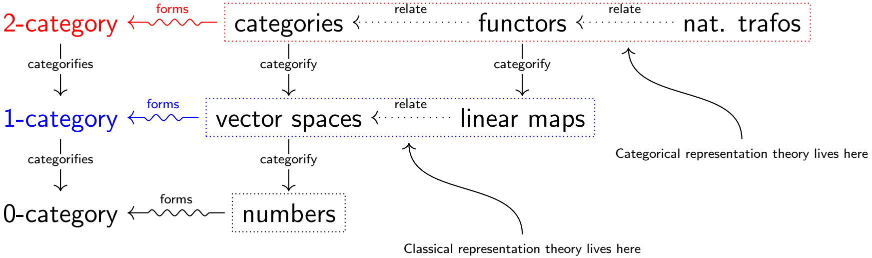


Categorical representation theory



Categorical representation theory – this is green and red

Categorical representation theory



Categorical representation theory – this is green and red

Categorical rep

2-category ← ... t. trafos

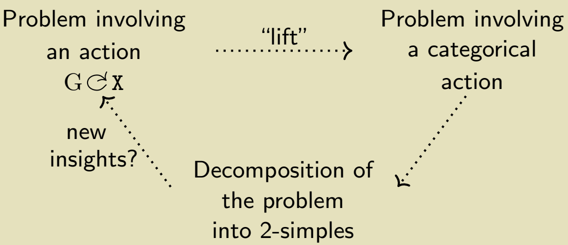
categorifies

1-category ←

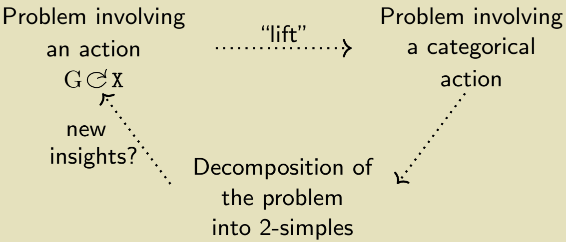
categorifies

0-category ←

What one can hope for.



What one can hope for.



These ideas opened a new field of mathematics – categorification

If you ask me at 2am in the morning
"What are you doing for a living?"
I might answer "Categorification"

Categorical representation theory – this is green and red

Applications of categorical representation theory:

Khovanov & others $\sim$1999++. Knot homologies are instances of 2-representation theory. Low-dim. topology & Math. Physics

Khovanov–Seidel & others $\sim$2000++. Faithful 2-modules of braid groups. Low-dim. topology & Symplectic geometry

Chuang–Rouquier $\sim$2004. Proof of the Broué conjecture using 2-representation theory. p-RT of finite groups & Geometry & Combinatorics

Elias–Williamson $\sim$2012. Proof of the Kazhdan–Lusztig conjecture using ideas from 2-representation theory. Combinatorics & RT & Geometry

Riche–Williamson $\sim$2015. Tilting characters using 2-representation theory. p-RT of reductive groups & Geometry

Many more...

Categorical representation theory – this is green and red

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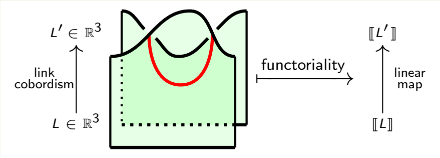
Functoriality of Khovanov–Rozansky's invariants ∼2017



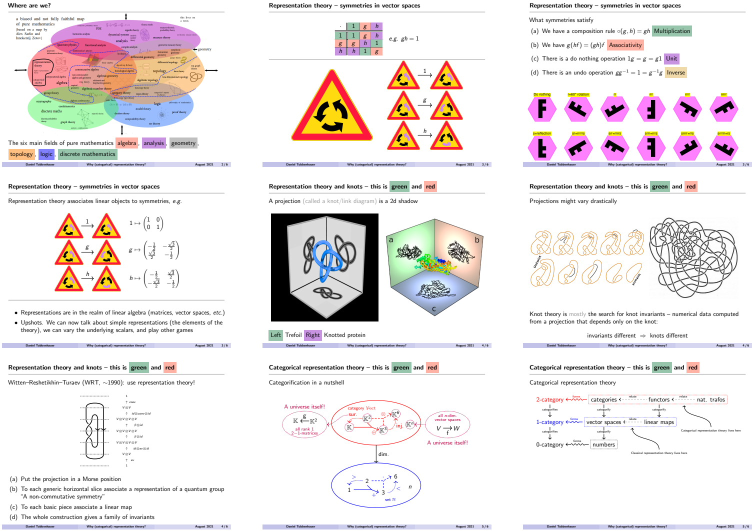
(Picture from one of my papers – using categorical representation theory)

This was conjectured from about 10 years,
but seemed infeasible to prove,
and has some impact on 4-dim. topology

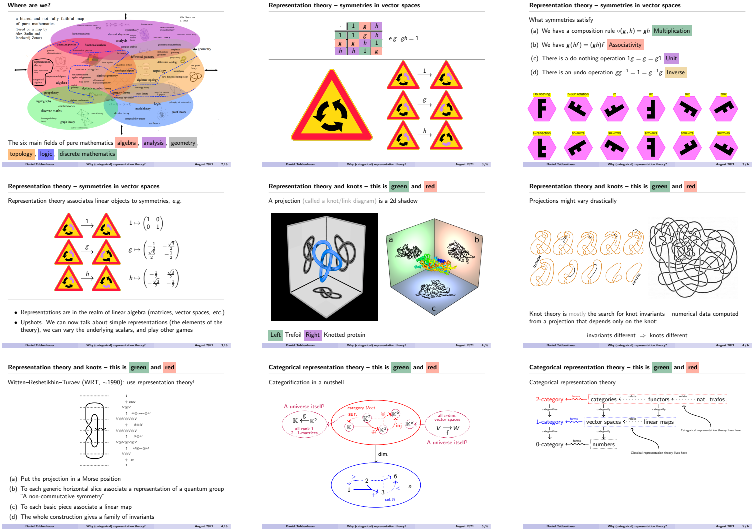
Riche–Williamson ∼2015. Tilting characters using 2-representation theory.

p-RT of reductive groups & Geometry

Many more...



There is still much to do...



Thanks for your attention!