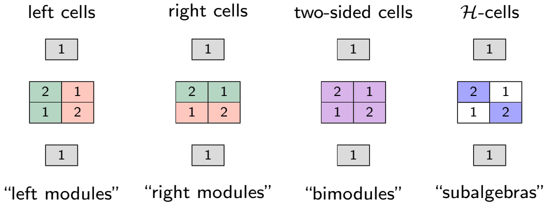
Dihedral groups, $\mathrm{SL}(2)_q$ and beyond

Or: Who colored my Dynkin diagrams?

Daniel Tubbenhauer



Joint with Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz and Xiaoting Zhang

July 2019

Let $A(\mathbf{\Gamma})$ be the adjacency matrix of a finite, connected, loopless graph $\mathbf{\Gamma}$. Let $\mathrm{U}_{e+1}(\mathtt{X})$ be the ▸ Chebyshev polynomial.

Classification problem (CP). Classify all $\mathbf{\Gamma}$ such that $\mathrm{U}_{e+1}(A(\mathbf{\Gamma}))=0$.

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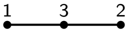
$$U_3(\mathtt{X})=(\mathtt{X}-2\cos(\tfrac{\pi}{4}))\mathtt{X}(\mathtt{X}-2\cos(\tfrac{3\pi}{4}))$$

$A_3 =$



$$\rightsquigarrow \quad A(A_3)=\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix} \quad \rightsquigarrow \quad S_{A_3}=\{2\cos(\tfrac{\pi}{4}),0,2\cos(\tfrac{3\pi}{4})\}$$

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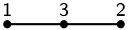

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$$\mathrm{U}_5(\mathtt{X}) = (\mathtt{X} - 2\cos(\tfrac{\pi}{6}))(\mathtt{X} - 2\cos(\tfrac{2\pi}{6}))\mathtt{X}(\mathtt{X} - 2\cos(\tfrac{4\pi}{6}))(\mathtt{X} - 2\cos(\tfrac{5\pi}{6}))$$

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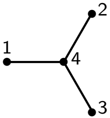

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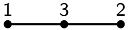

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✓ for $e = 2$

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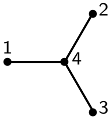

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✓ for $e = 4$

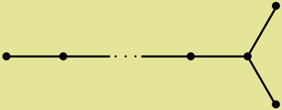
Let $A(\Gamma)$ be the adjacency matrix of a finite, connected, loopless graph Γ. Let $U_{e+1}(X)$ be the

Smith ∼1969. The graphs solutions to (CP) are precisely ADE graphs for $e+2$ being (at most) the Coxeter number.

Type A_m:

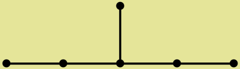
✓ for $e = m - 1$

Type D_m:



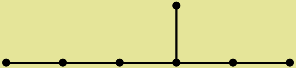
✓ for $e = 2m - 4$

Type E_6:



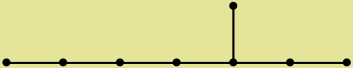
✓ for $e = 10$

Type E_7:



✓ for $e = 16$

Type E_8:



✓ for $e = 28$

Clas $= 0.$

$A_3 =$ 1 $\cos(\frac{3\pi}{4})\}$

$D_4 =$ 1 $\cos(\frac{5\pi}{6})\}$

1. **Dihedral representation theory**
    - Classical vs. $\mathbb{N}$-representation theory
    - Dihedral $\mathbb{N}$-representation theory

2. **Non-semisimple fusion rings**
    - The asymptotic limit
    - The limit $\mathrm{v} \to 0$ of the $\mathbb{N}$-representations

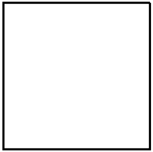
3. **Beyond**

The dihedral groups are of Coxeter type $\mathrm{I}_2(e+2)$:

$$W_{e+2} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \overline{\mathtt{s}}_{e+2} = \underbrace{\ldots \mathtt{sts}}_{e+2} = w_0 = \underbrace{\ldots \mathtt{tst}}_{e+2} = \overline{\mathtt{t}}_{e+2} \rangle,$$

$$e.g.\ :\ W_4 = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \mathtt{tsts} = w_0 = \mathtt{stst} \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, *e.g.* for $e=2$:



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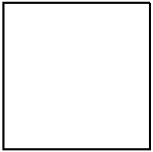

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$$W_{e+2} = \langle s \quad \ldots \quad = \overline{t}_{e+2}\rangle,$$

Fact. The symmetries are given by exchanging flags.

$$e.g. : \; W_4 = \langle s, t \mid s^2 = t^2 = 1, \; tsts = w_0 = stst\rangle$$

Fix a flag F.

Idea (Coxeter ∼1934++).

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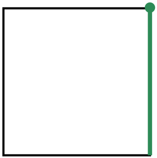

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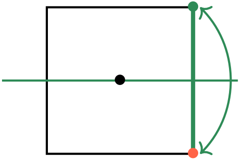
Fix a flag F.

Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Examp e the symmetry groups of regular $e+2$-gons, *e.g.* for $e=2$:



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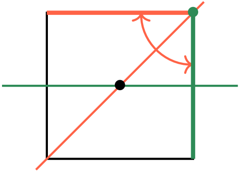
Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.



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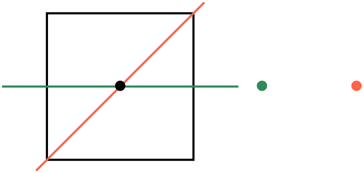

The dihedral groups are of Coxeter type $I_2(e+2)$:

This gives a generator-relation presentation.

$$W_{e+2} = \langle s, t \mid s^2 = t^2 = 1, \overline{s}_{e+2} = \ldots sts = w_0 = \ldots tst = \overline{t}_{e+2} \rangle,$$

And the braid relation measures the angle between hyperplanes.

$$e.g.: \; W_4 = \langle s, t \mid s^2 = t^2 = 1, \; tsts = w_0 = stst \rangle$$

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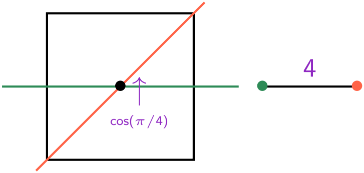
Connect i, j by an n-edge for H_i, H_j having angle $\cos(\pi/(e+2))$.



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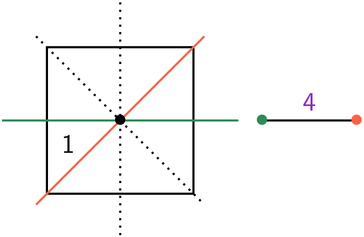

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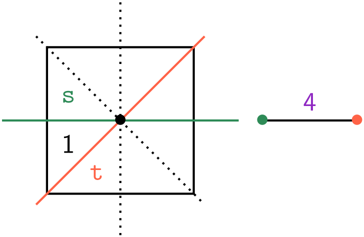

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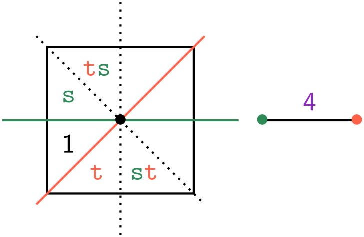

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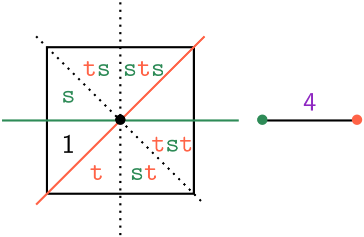

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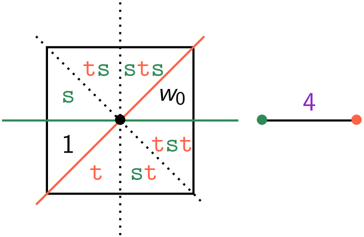

Dihedral representation theory on one slide.

The Bott–Samelson (BS) generators $b_{\mathtt{s}} = \mathtt{s} + 1$, $b_{\mathtt{t}} = \mathtt{t} + 1$.
There is also a Kazhdan–Lusztig (KL) basis. We will nail it down later.

One-dimensional modules. $\mathrm{M}_{\lambda_{\mathtt{s}},\lambda_{\mathtt{t}}}, \lambda_{\mathtt{s}}, \lambda_{\mathtt{t}} \in \mathbb{C}, b_{\mathtt{s}} \mapsto \lambda_{\mathtt{s}}, b_{\mathtt{t}} \mapsto \lambda_{\mathtt{t}}$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_{0,0}, \mathrm{M}_{2,0}, \mathrm{M}_{0,2}, \mathrm{M}_{2,2}$	$\mathrm{M}_{0,0}, \mathrm{M}_{2,2}$

Two-dimensional modules. $\mathrm{M}_z, z \in \mathbb{C}, b_{\mathtt{s}} \mapsto \left(\begin{smallmatrix} 2 & z \\ 0 & 0 \end{smallmatrix}\right), b_{\mathtt{t}} \mapsto \left(\begin{smallmatrix} 0 & 0 \\ \overline{z} & 2 \end{smallmatrix}\right)$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm} - \{0\}$	$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm}$

$\mathrm{V}_e = \mathrm{roots}(\mathrm{U}_{e+1}(\mathtt{X}))$ and $\mathrm{V}_e^{\pm}$ the $\mathbb{Z}/2\mathbb{Z}$-orbits under $z \mapsto -z$.

Dihedral representation theory on one slide.

One-dimension $_{\mathtt{t}}$.

Proposition (Lusztig?).

The list of one- and two-dimensional W_{e+2}-modules is a complete, irredundant list of simple modules.

$\mathrm{M}_{0,0}$, $\mathrm{M}_{2,0}$, $\mathrm{M}_{0,2}$, $\mathrm{M}_{2,2}$	$\mathrm{M}_{0,0}$, $\mathrm{M}_{2,2}$

I learned this construction in 2017.

Two-dimensional modules. $\mathrm{M}_z, z \in \mathbb{C}, b_{\mathtt{s}} \mapsto \left(\begin{smallmatrix} 2 & z \\ 0 & 0 \end{smallmatrix}\right), b_{\mathtt{t}} \mapsto \left(\begin{smallmatrix} 0 & 0 \\ \overline{z} & 2 \end{smallmatrix}\right)$.

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Example.

$\mathrm{M}_{0,0}$ is the sign representation and $\mathrm{M}_{2,2}$ is the trivial representation.

In case e is odd, $\mathrm{U}_{e+1}(\mathtt{X})$ has a constant term, so $\mathrm{M}_{2,0}, \mathrm{M}_{0,2}$ are not representations.

$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm} - \{0\}$	$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm}$

$\mathrm{V}_e = \mathrm{roots}(\mathrm{U}_{e+1}(\mathtt{X}))$ and $\mathrm{V}_e^{\pm}$ the $\mathbb{Z}/2\mathbb{Z}$-orbits under $z \mapsto -z$.

Dihedral representation theory on one slide

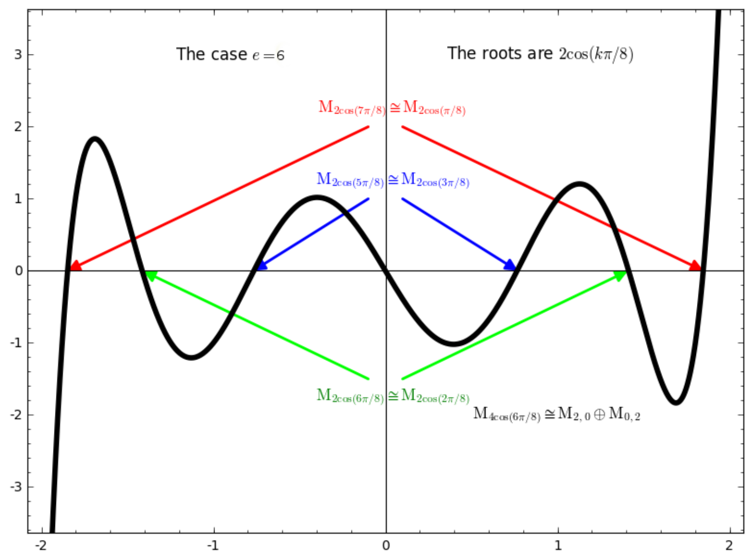
One-

Two-

$V_e =$

Example.

These representations are indexed by $\mathbb{Z}/2\mathbb{Z}$-orbits of the Chebyshev roots:



An algebra A with a fixed basis B^{A} is called a (multi) $\mathbb{N}$-algebra if

$$\mathrm{xy} \in \mathbb{N}\mathrm{B}^{\mathrm{A}} \quad (\mathrm{x}, \mathrm{y} \in \mathrm{B}^{\mathrm{A}}).$$

A A-module M with a fixed basis B^{M} is called a $\mathbb{N}$-module if

$$\mathrm{xm} \in \mathbb{N}\mathrm{B}^{\mathrm{M}} \quad (\mathrm{x} \in \mathrm{B}^{\mathrm{A}}, \mathrm{m} \in \mathrm{B}^{\mathrm{M}}).$$

These are $\mathbb{N}$-equivalent if there is a $\mathbb{N}$-valued change of basis matrix.

Example. $\mathbb{N}$-algebras and $\mathbb{N}$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}$-equivalence comes from 2-equivalence.

Example (group like).

Group algebras of finite groups with basis given by group elements are $\mathbb{N}$-algebras.

The regular module is a $\mathbb{N}$-module.

An

A A-module M with a fixed basis B^{M} is called a $\mathbb{N}$-module if

$$\mathrm{xm} \in \mathbb{N}\mathrm{B}^{\mathrm{M}} \quad (\mathrm{x} \in \mathrm{B}^{\mathrm{A}}, \mathrm{m} \in \mathrm{B}^{\mathrm{M}}).$$

These are $\mathbb{N}$-equivalent if there is a $\mathbb{N}$-valued change of basis matrix.

Example. $\mathbb{N}$-algebras and $\mathbb{N}$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}$-equivalence comes from 2-equivalence.

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Fusion rings are with basis given by classes of simples are $\mathbb{N}$-algebras.

Key example: $K_0(\mathcal{R}\mathrm{ep}(G,\mathbb{C}))$ (easy $\mathbb{N}$-representation theory).

Key example: $K_0(\mathcal{R}\mathrm{ep}_q^{ss}(\mathrm{U}_q(\mathfrak{g}))=\mathrm{G}_q)$ (intricate $\mathbb{N}$-representation theory).

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Exam $\mathbb{N}$-algebras and $\mathbb{N}$-modules arise naturally as the decategorification of 2-categories and 2 quivalence.

Example (semigroup like).

Hecke algebras of (finite) Coxeter groups with their KL basis are $\mathbb{N}$-algebras.

Their $\mathbb{N}$-representation theory is non-semisimple.

Clifford, Munn, Ponizovskiĭ, Green $\sim$1942++, Kazhdan–Lusztig $\sim$1979.
$\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ if y appears in zx with non-zero coefficient for $\mathrm{z} \in \mathrm{B}^{\mathrm{A}}$. $\mathrm{x} \sim_{\mathsf{L}} \mathrm{y}$ if $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ and $\mathrm{y} \leq_{\mathsf{L}} \mathrm{x}$.
$\sim_{\mathsf{L}}$ partitions A into left cells L. Similarly for right R, two-sided cells LR or $\mathbb{N}$-modules.

A $\mathbb{N}$-module M is transitive if all basis elements belong to the same $\sim_{\mathsf{L}}$ equivalence class. An apex of M is a maximal two-sided cell not killing it.

Fact. Each transitive $\mathbb{N}$-module has a unique apex.

Hence, one can study them cell-wise.

Example. Transitive $\mathbb{N}$-modules arise naturally as the decategorification of simple transitive 2-modules.

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Example (group like).

Group algebras with the group element basis have only one cell, G itself.

Transitive ℕ-modules are $\mathbb{C}[G/H]$ for $H \subset G$ subgroup/conjugacy. The apex is G.

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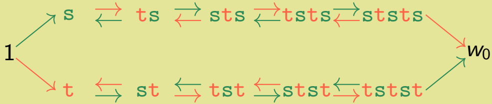
Example (Lusztig ≤2003; semigroup like).

Hecke algebras for the dihedral group with KL basis have the following cells:

Left cells.



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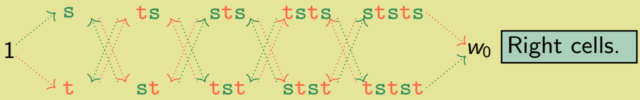

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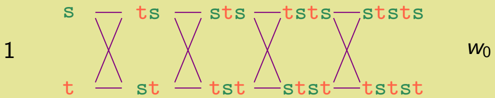


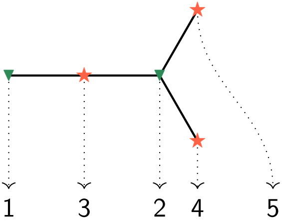
Two-sided cells.

We will see the transitive ℕ-modules in a second.

$\mathbb{N}$-modules via graphs.

Construct a W_∞-module M associated to a bipartite graph **Γ**:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5\rangle$$



$$b_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad b_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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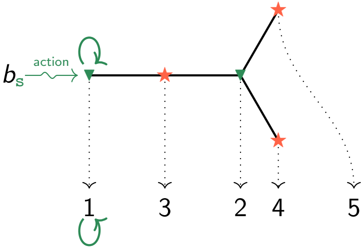

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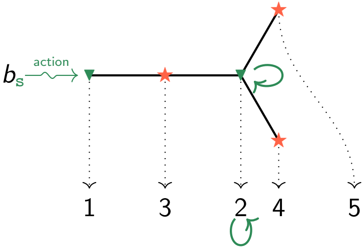

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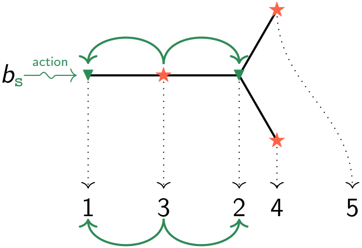

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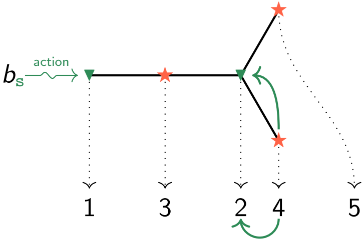

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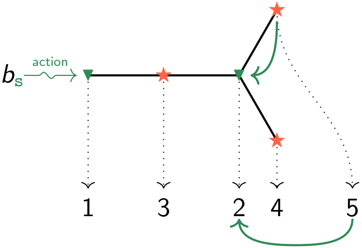

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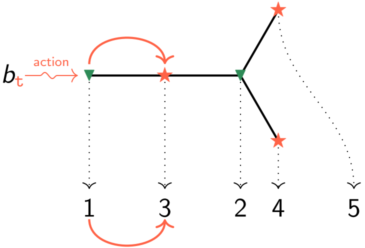

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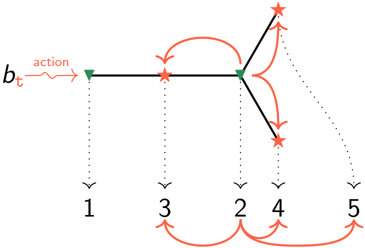

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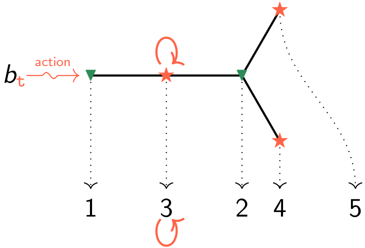

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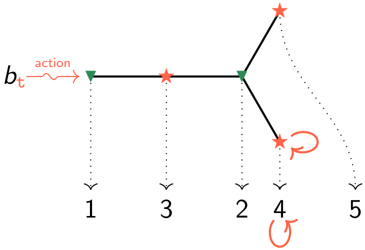

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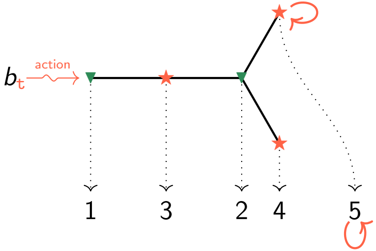

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$\mathbb{N}$-modules via graphs.

Construct a W_{∞}-module M associated to a bipartite graph $\mathbf{\Gamma}$:

The adjacency matrix $A(\mathbf{\Gamma})$ of $\mathbf{\Gamma}$ is

$$A(\mathbf{\Gamma}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

These are W_{e+2}-modules for some e
only if $A(\mathbf{\Gamma})$ is killed by the Chebyshev polynomial $\mathrm{U}_{e+1}(\mathtt{X})$.

Morally speaking: These are constructed as the simples
but with integral matrices having the Chebyshev-roots as eigenvalues.

It is not hard to see that the Chebyshev–braid-like relation can not hold otherwise.

$$b_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad b_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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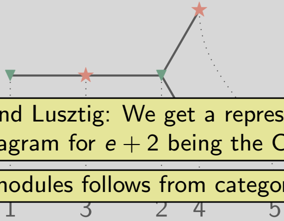

Hence, by Smith's (CP) and Lusztig: We get a representation of W_{e+2} if $\mathbf{\Gamma}$ is a ADE Dynkin diagram for $e+2$ being the Coxeter number.

That these are $\mathbb{N}$-modules follows from categorification.

'Smaller solutions' are never $\mathbb{N}$-modules.

$$b_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad b_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}$-modules via graphs.

Construct a W_∞-module M associated to a bipartite graph Γ:

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Classification.

Complete, irredundant ▸ list of transitive $\mathbb{N}$-modules of W_{e+2}:

apex	① cell	s – t cell	w_0 cell
$\mathbb{N}$-reps.	$\mathrm{M}_{0,0}$	$\mathrm{M}_{\text{ADE+bicolering}}$ for $e+2=$ Cox. num.	$\mathrm{M}_{2,2}$

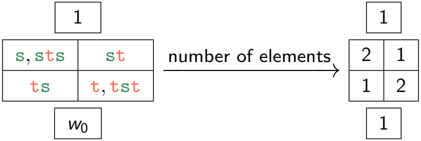
I learned this from Kildetoft–Mackaay–Mazorchuk–Zimmermann ∼2016.

Fun fact about associated simples: ▸ Click.

$$b_s \rightsquigarrow \mathrm{M}_s = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad b_t \rightsquigarrow \mathrm{M}_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Example ($\mathrm{I}_2(4), e = 2$).

Cell structure:



1

s, sts | st

ts | t, tst

w_0

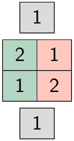
number of elements

1

2 | 1

1 | 2

1



left cells

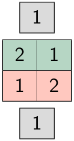
1

2 | 1

1 | 2

1

“left modules”



right cells

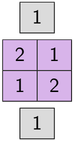
1

2 | 1

1 | 2

1

“right modules”



two-sided cells

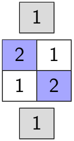
1

2 | 1

1 | 2

1

“bimodules”



$\mathcal{H}$-cells

1

2 | 1

1 | 2

1

“subalgebras”

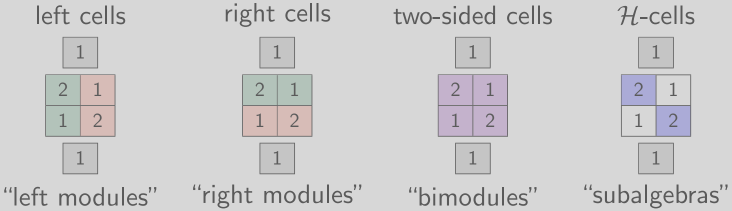
Example ($\mathrm{I}_2(4), e = 2$)**.**

Cell structure:

Example.

$$1 \cdot 1 = \mathrm{v}^0 1.$$

(v is the Hecke parameter deforming *e.g.* $\mathrm{s}^2 = 1$ to $T_{\mathrm{s}}^2 = (\mathrm{v}^{-1} - \mathrm{v})T_{\mathrm{s}} + 1$.)



Example ($I_2(4)$, $e = 2$).

Cell structure:

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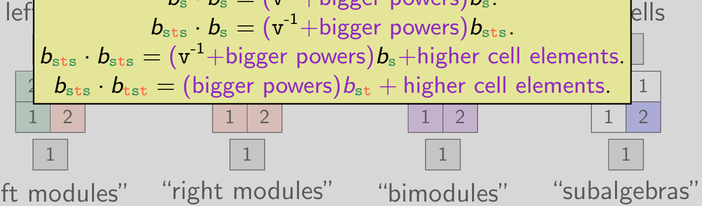
$$b_{\mathrm{sts}} \cdot b_{\mathrm{sts}} = (\mathrm{v}^{-1} + \text{bigger powers})b_{\mathrm{s}} + \text{higher cell elements}.$$
$$b_{\mathrm{sts}} \cdot b_{\mathrm{tst}} = (\text{bigger powers})b_{\mathrm{st}} + \text{higher cell elements}.$$



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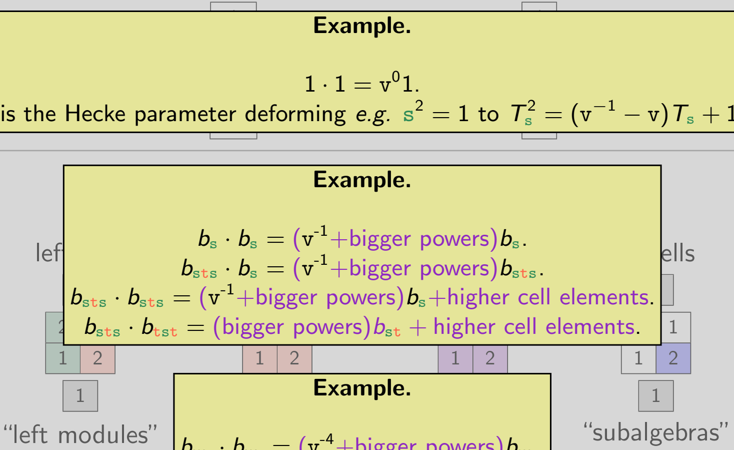

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Example.

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Example ($I_2(4)$, $e = 2$).

Cell str

Fact (Lusztig ∼1980++).

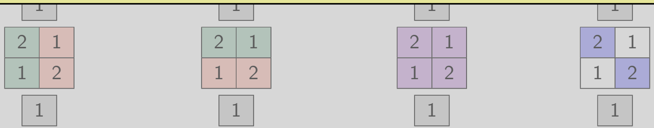
For any Coxeter group W
there is a well-defined function

$$\mathbf{a} \colon \mathrm{W} \to \mathbb{N}$$

which is constant on two-sided cells.

Asymptotic limit $\mathrm{v} \to 0$ "=" kill non-leading terms of $c_w = \mathrm{v}^{\mathbf{a}} b_w$,
e.g. $c_{\mathrm{s}} = \mathrm{v}^1 b_{\mathrm{s}}$ and $c_{\mathrm{s}}^2 = (1+\mathrm{v}^2) c_{\mathrm{s}}$.

Think: Positively graded, and asymptotic limit is taking degree 0 part.



"left modules" "right modules" "bimodules" "subalgebras"

Compare multiplication tables. Example ($e = 2$).

a=asymptotic element and $[2] = 1 + \mathrm{v}^2$. (Note the “subalgebras”.)

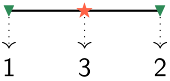
	a_{s}	a_{sts}	a_{st}	a_{t}	a_{tst}	a_{ts}
a_{s}	a_{s}	a_{sts}	a_{st}			
a_{sts}	a_{sts}	a_{s}	a_{st}			
a_{ts}	a_{ts}	a_{ts}	$a_{t} + a_{tst}$			
a_{t}				a_{t}	a_{tst}	a_{ts}
a_{tst}				a_{tst}	a_{t}	a_{ts}
a_{st}				a_{st}	a_{st}	$a_{s} + a_{sts}$

	c_{s}	c_{sts}	c_{st}	c_{t}	c_{tst}	c_{ts}
c_{s}	$[2]c_{s}$	$[2]c_{sts}$	$[2]c_{st}$	c_{st}	$c_{st} + c_{w_0}$	$c_{s} + c_{sts}$
c_{sts}	$[2]c_{sts}$	$[2]c_{s} + [2]^2c_{w_0}$	$[2]c_{st} + [2]c_{w_0}$	$c_{s} + c_{sts}$	$c_{s} + [2]^2c_{w_0}$	$c_{s} + c_{sts} + [2]c_{w_0}$
c_{ts}	$[2]c_{ts}$	$[2]c_{ts} + [2]c_{w_0}$	$[2]c_{t} + [2]c_{tst}$	$c_{t} + c_{tst}$	$c_{t} + c_{tst} + [2]c_{w_0}$	$2c_{ts} + c_{w_0}$
c_{t}	c_{ts}	$c_{ts} + c_{w_0}$	$c_{t} + c_{tst}$	$[2]c_{t}$	$[2]c_{tst}$	$[2]c_{ts}$
c_{tst}	$c_{t} + c_{tst}$	$c_{t} + [2]^2c_{w_0}$	$c_{t} + c_{tst} + [2]c_{w_0}$	$[2]c_{tst}$	$[2]c_{t} + [2]^2c_{w_0}$	$[2]c_{ts} + [2]c_{w_0}$
c_{st}	$c_{s} + c_{sts}$	$c_{s} + c_{sts} + [2]c_{w_0}$	$2c_{st} + c_{w_0}$	$[2]c_{st}$	$[2]c_{st} + [2]c_{w_0}$	$[2]c_{s} + [2]c_{sts}$

The limit $\mathrm{v} \to 0$ is much simpler! Have you seen this ▸ before?

Back to graphs. Example ($e = 2$).

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3\rangle$$



$$c_{\mathtt{s}} \rightsquigarrow \begin{pmatrix} 1+\mathrm{v}^2 & 0 & \mathrm{v} \\ 0 & 1+\mathrm{v}^2 & \mathrm{v} \\ 0 & 0 & 0 \end{pmatrix} \qquad c_{\mathtt{t}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{v} & \mathrm{v} & 1+\mathrm{v}^2 \end{pmatrix}$$

$$c_{\mathtt{sts}} \rightsquigarrow \begin{pmatrix} 0 & 1+\mathrm{v}^2 & \mathrm{v} \\ 1+\mathrm{v}^2 & 0 & \mathrm{v} \\ 0 & 0 & 0 \end{pmatrix} \qquad c_{\mathtt{tst}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{v} & \mathrm{v} & 1+\mathrm{v}^2 \end{pmatrix}$$

$$c_{\mathtt{ts}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1+\mathrm{v}^2 & 1+\mathrm{v}^2 & \mathrm{v} \end{pmatrix} \qquad c_{\mathtt{st}} \rightsquigarrow \begin{pmatrix} \mathrm{v} & \mathrm{v} & 1+\mathrm{v}^2 \\ \mathrm{v} & \mathrm{v} & 1+\mathrm{v}^2 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$a_{s} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad a_{t} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Back

Example.

$$a_{\mathtt{st}}a_{\mathtt{ts}} = a_{\mathtt{s}} + a_{\mathtt{sts}}$$

$$\leftrightsquigarrow$$

$$[L_1][L_1] = [L_0] + [L_2]$$

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$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

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This works in general and recovers the transitive $\mathbb{N}$-modules of $K_0(\mathrm{SL}(2)_q)$ found by Etingof–Khovanov ∼1995, Kirillov–Ostrik ∼2001 and Ostrik ∼2003, which are also ADE classified.

(For the experts: the bicoloring kills the tadpole solutions.)

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a_{ts}

However, at this point this was just an observation and it took a while until we understood its meaning.

(Cliffhanger: Wait for Marco's talk.)

Back to graphs. Example ($e = 2$).

Classification.

Complete, irredundant list of graded
simple transitive 2-modules of dihedral Soergel bimodules:

apex	1 cell	s – t cell	w_0 cell
2-reps.	$\mathbf{M}_{0,0}$	$\mathbf{M}_{\text{ADE+bicolering}}$ for $e + 2 =$ Cox. num.	$\mathbf{M}_{2,2}$

▸ Construction

$a_{s} \rightsquigarrow$ 0 1 0

$a_{t} \rightsquigarrow$ 0 0 0

I learned this from Kildetoft–Mackaay–Mazorchuk–Zimmermann ∼2016.

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Proof?

The first proof was “brute force”.
Now we have a much better way of doing this.
(Again: cliffhanger.)

▸ Please stop!

$a_{sts} \rightsquigarrow$

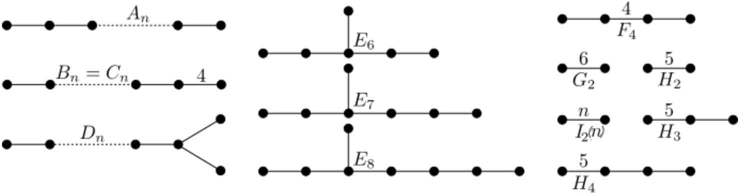
$$\begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{matrix}\Bigg)$$

$a_{ts} \rightsquigarrow$ (1 1 0)

$$\begin{matrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{matrix}\Bigg)$$

Where to find $\mathrm{SL}(m)_q$?

First try: What are the asymptotic limits of finite types?



- ▶ No luck in finite Weyl type: $\mathrm{v} \to 0$ is (almost always) $\mathcal{R}\mathrm{ep}((\mathbb{Z}/2\mathbb{Z})^k)$.
- ▶ No luck in dihedral type: $\mathrm{v} \to 0$ is $\mathrm{SL}(2)_q$ ($q^{2(n-2)} = 1$).
- ▶ No luck for the pentagon types H_3 and ▸ H_4.
- ▷ Maybe generalize the dihedral case?

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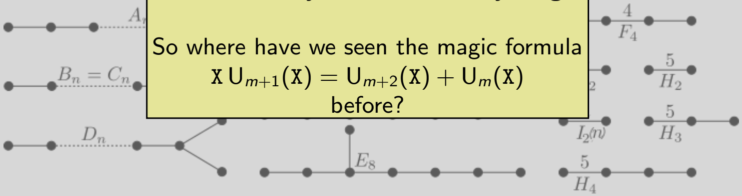

Idea 1: Chebyshev knows everything!

So where have we seen the magic formula

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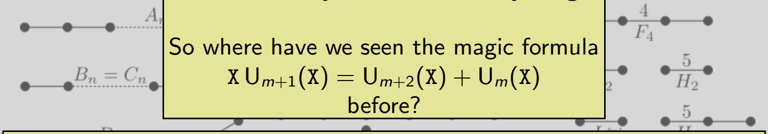

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$$[2]\cdot[e+1] = [e+2] + [e]$$

$$\mathrm{L}_1 \otimes \mathrm{L}_{e+1} \cong \mathrm{L}_{e+2} \oplus \mathrm{L}_e$$

$\mathrm{L}_e = e^{\mathrm{th}}$ symmetric power of the vector representation of (quantum) $\mathfrak{sl}_2$.

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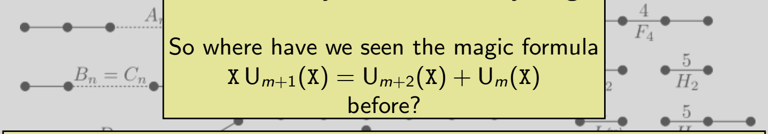

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- No l
- Mayb

Idea 2: The dihedral type is
a quotient of affine type A_1.
Very vague philosophy I want to sell:
Fusion categories appear as degree 0 parts of Soergel bimodules.

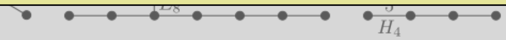
Whe

Quantum Satake (Elias ∼2013, Mackaay–Mazorchuk–Miemietz ∼2018)
– rough version.

$\mathrm{SL}(m)_q$ is the semisimple version of
a subquotient of Soergel bimodules for affine type A_{m-1}.
The KL basis correspond to the images of L_e.

Beware: Only the cases $m = 2$ (dihedral) and $m = 3$ (trihedral) are proven,
as everything gets combinatorially more complicated.

First



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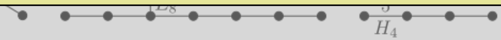


Summary of Nhedral.

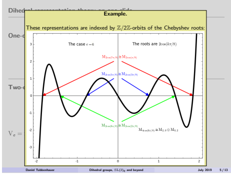
Most questions are still open, but nice ▸ patterns appear.

Leaves the realm of groups. (No associated Coxeter group; only a subquotient.)

Generalized zigzag algebras, Chebyshev polynomials and ADE diagrams appear.

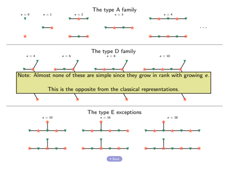
ADE-type classification(?) of 2-representations.

Fusion: $\mathrm{SL}(m)_q$ appears.


Example.

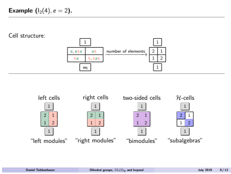
These representations are indexed by $\mathbb{Z}/2\mathbb{Z}$-orbits of the Chebyshev roots:

Daniel Tubbenhauer Dihedral groups, $\mathrm{SL}(2)_q$ and beyond July 2019 5 / 13



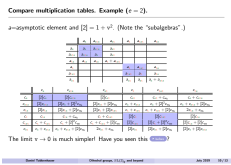
Note: Almost none of these are simple since they grow in rank with growing e.

This is the opposite from the classical representations.


Example ($I_2(4), e=2$).

Cell structure:

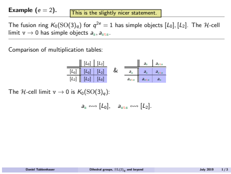
Daniel Tubbenhauer Dihedral groups, $\mathrm{SL}(2)_q$ and beyond July 2019 9 / 13


Compare multiplication tables. Example ($e=2$).

a=asymptotic element and $[2]=1+v^2$. (Note the "subalgebras".)

The limit $v\to 0$ is much simpler! Have you seen this before?

Daniel Tubbenhauer Dihedral groups, $\mathrm{SL}(2)_q$ and beyond July 2019 10 / 13


Example ($e=2$). This is the slightly nicer statement.

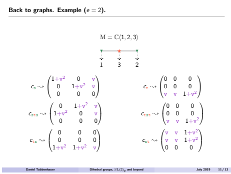
The fusion ring $K_0(\mathrm{SO}(3)_q)$ for $q^{2e}=1$ has simple objects $[L_0],[L_2]$. The $\mathcal{H}$-cell limit $v\to 0$ has simple objects a_s, a_{sts}.

Comparison of multiplication tables:

The $\mathcal{H}$-cell limit $v\to 0$ is $K_0(\mathrm{SO}(3)_q)$:

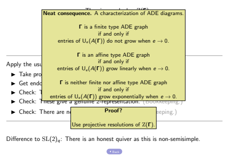
$$a_s \rightsquigarrow [L_0], \quad a_{sts} \rightsquigarrow [L_2].$$

Daniel Tubbenhauer Dihedral groups, $\mathrm{SL}(2)_q$ and beyond July 2019 1 / 3


Back to graphs. Example ($e=2$).

$$M=\mathbb{C}\langle 1,2,3\rangle$$

Daniel Tubbenhauer Dihedral groups, $\mathrm{SL}(2)_q$ and beyond July 2019 11 / 13


Neat consequence. A characterization of ADE diagrams.

Γ is a finite type ADE graph if and only if entries of $U_e(A(\Gamma))$ do not grow when $e\to 0$.

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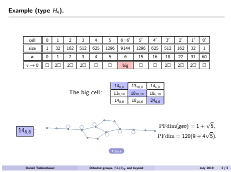
Proof?

Use projective resolutions of $\mathcal{Z}(\Gamma)$.

Difference to $\mathrm{SL}(2)_q$: There is an honest quiver as this is non-semisimple.


Example (type H_4).

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$\mathrm{PFdim}(gen)=1+\sqrt{5}$,

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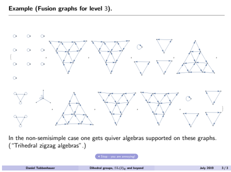
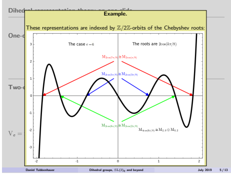
Daniel Tubbenhauer Dihedral groups, $\mathrm{SL}(2)_q$ and beyond July 2019 2 / 3


Example (Fusion graphs for level 3).

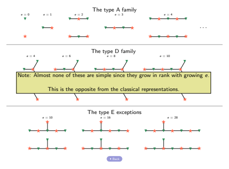
In the non-semisimple case one gets quiver algebras supported on these graphs. ("Trihedral zigzag algebras".)

Daniel Tubbenhauer Dihedral groups, $\mathrm{SL}(2)_q$ and beyond July 2019 3 / 3

There is still much to do...


Example.

These representations are indexed by $\mathbb{Z}/2\mathbb{Z}$-orbits of the Chebyshev roots:

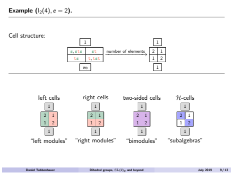

The type A family

The type D family

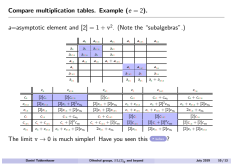
Note: Almost none of these are simple since they grow in rank with growing e.

This is the opposite from the classical representations.

The type E exceptions

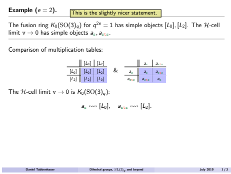

Example ($I_2(4)$, $e=2$).

Cell structure:


Compare multiplication tables. Example ($e=2$).

a=asymptotic element and $[2]=1+v^2$. (Note the "subalgebras".)

The limit $v\to 0$ is much simpler! Have you seen this before?

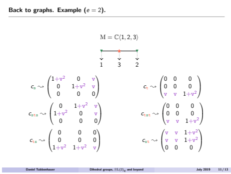

Example ($e=2$). This is the slightly nicer statement.

The fusion ring $K_0(\mathrm{SO}(3)_q)$ for $q^{2e}=1$ has simple objects $[L_0]$, $[L_2]$. The $\mathcal{H}$-cell limit $v\to 0$ has simple objects a_s, a_{sts}.

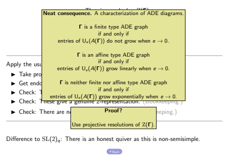
Comparison of multiplication tables:

The $\mathcal{H}$-cell limit $v\to 0$ is $K_0(\mathrm{SO}(3)_q)$:

$$a_s \rightsquigarrow [L_0],\quad a_{sts}\rightsquigarrow [L_2].$$


Back to graphs. Example ($e=2$).

$$M=\mathbb{C}\langle 1,2,3\rangle$$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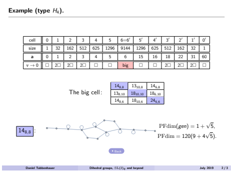
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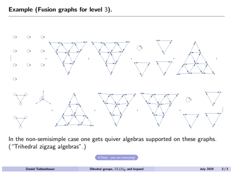

Example (Fusion graphs for level 3).

In the non-semisimple case one gets quiver algebras supported on these graphs. ("Trihedral zigzag algebras".)

Thanks for your attention!

$$\mathrm{U}_0(\mathtt{X}) = 1, \quad \mathrm{U}_1(\mathtt{X}) = \mathtt{X}, \quad \mathtt{X}\,\mathrm{U}_{e+1}(\mathtt{X}) = \mathrm{U}_{e+2}(\mathtt{X}) + \mathrm{U}_e(\mathtt{X})$$

$$\mathrm{U}_0(\mathtt{X}) = 1, \quad \mathrm{U}_1(\mathtt{X}) = 2\mathtt{X}, \quad 2\mathtt{X}\,\mathrm{U}_{e+1}(\mathtt{X}) = \mathrm{U}_{e+2}(\mathtt{X}) + \mathrm{U}_e(\mathtt{X})$$

Kronecker ∼1857. Any complete set of conjugate algebraic integers in $]-2,2[$ is a subset of $\mathrm{roots}(\mathrm{U}_{e+1}(\mathtt{X}))$ for some e.

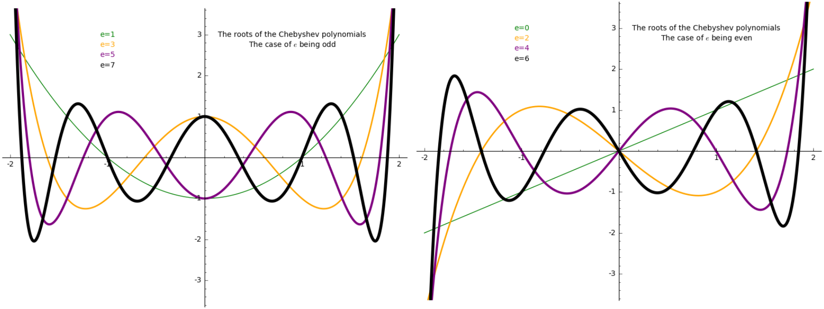
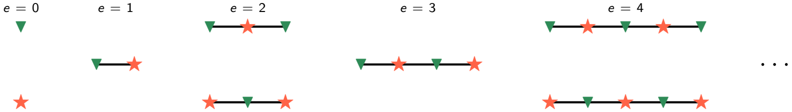


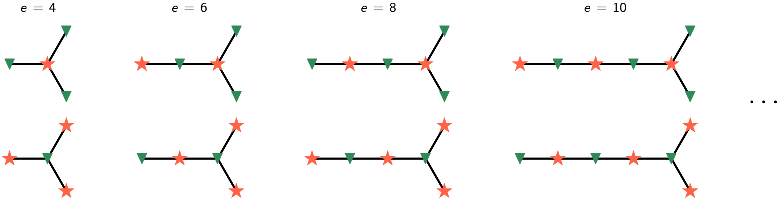
Figure: The roots of the Chebyshev polynomials (of the second kind).

◂ Back

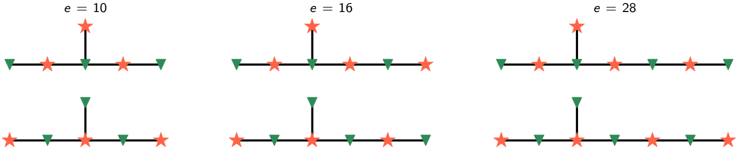
The type A family



The type D family



The type E exceptions



Back

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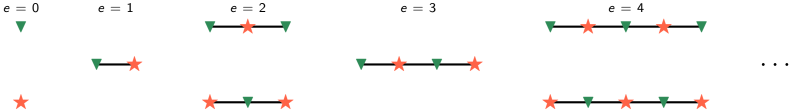

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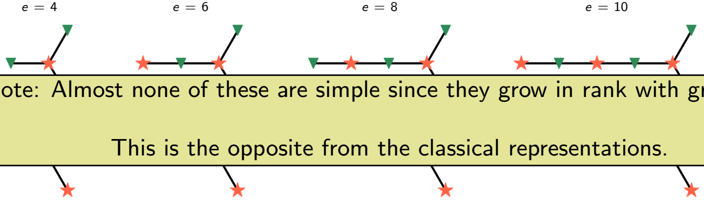

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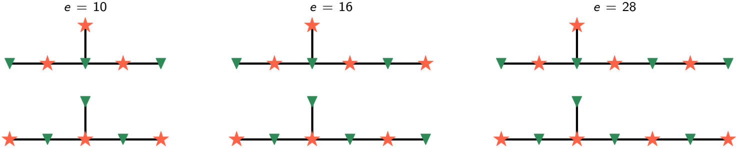

Back

Example ($e = 2$). Simples associated to cells.

Classical representation theory. The simples from before.

	$\mathrm{M}_{0,0}$	$\mathrm{M}_{2,0}$	$\mathrm{M}_{\sqrt{2}}$	$\mathrm{M}_{0,2}$	$\mathrm{M}_{2,2}$
atom	sign	trivial-sign	rotation	sign-trivial	trivial
rank	1	1	2	1	1
apex(KL)	①	s – t	s – t	s – t	w_0

KL basis. ADE diagrams and ranks of transitive $\mathbb{N}$-modules.

	bottom cell	▼—★—▼	★—▼—★	top cell
atom	sign	$\mathrm{M}_{2,0}\oplus\mathrm{M}_{\sqrt{2}}$	$\mathrm{M}_{0,2}\oplus\mathrm{M}_{\sqrt{2}}$	trivial
rank	1	3	3	1
apex(KL)	①	s – t	s – t	w_0

The simples are arranged according to cells. However, one cell might have more than one associated simple.

(For the experts: This means that the Hecke algebra with the KL basis is in general not cellular in the sense of Graham–Lehrer.)

◂ Back

Example ($e = 2$).

The fusion ring $K_0(\mathrm{SL}(2)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_1], [L_2]$. The limit $\mathrm{v} \to 0$ has simple objects $a_{\mathtt{s}}, a_{\mathtt{sts}}, a_{\mathtt{st}}, a_{\mathtt{t}}, a_{\mathtt{tst}}, a_{\mathtt{ts}}$.

Comparison of multiplication tables:

	$[L_0]$	$[L_2]$	$[L_1]$
$[L_0]$	$[L_0]$	$[L_2]$	$[L_1]$
$[L_2]$	$[L_2]$	$[L_0]$	$[L_1]$
$[L_1]$	$[L_1]$	$[L_1]$	$[L_0] + [L_2]$

&

	$a_{\mathtt{s}}$	$a_{\mathtt{sts}}$	$a_{\mathtt{st}}$	$a_{\mathtt{t}}$	$a_{\mathtt{tst}}$	$a_{\mathtt{ts}}$
$a_{\mathtt{s}}$	$a_{\mathtt{s}}$	$a_{\mathtt{sts}}$	$a_{\mathtt{st}}$			
$a_{\mathtt{sts}}$	$a_{\mathtt{sts}}$	$a_{\mathtt{s}}$	$a_{\mathtt{st}}$			
$a_{\mathtt{ts}}$	$a_{\mathtt{ts}}$	$a_{\mathtt{ts}}$	$a_{\mathtt{t}} + a_{\mathtt{tst}}$			
$a_{\mathtt{t}}$				$a_{\mathtt{t}}$	$a_{\mathtt{tst}}$	$a_{\mathtt{ts}}$
$a_{\mathtt{tst}}$				$a_{\mathtt{tst}}$	$a_{\mathtt{t}}$	$a_{\mathtt{ts}}$
$a_{\mathtt{st}}$				$a_{\mathtt{st}}$	$a_{\mathtt{st}}$	$a_{\mathtt{s}} + a_{\mathtt{sts}}$

The limit $\mathrm{v} \to 0$ is a bicolored version of $K_0(\mathrm{SL}(2)_q)$:

$$a_{\mathtt{s}} \& a_{\mathtt{t}} \leftrightsquigarrow [L_0], \quad a_{\mathtt{sts}} \& a_{\mathtt{tst}} \leftrightsquigarrow [L_2], \quad a_{\mathtt{st}} \& a_{\mathtt{ts}} \leftrightsquigarrow [L_1].$$

Example ($e = 2$).

This is the slightly nicer statement.

The fusion ring $K_0(\mathrm{SO}(3)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_2]$. The $\mathcal{H}$-cell limit $\mathrm{v} \to 0$ has simple objects $a_{\mathtt{s}}, a_{\mathtt{sts}}$.

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$[L_2]$	$[L_2]$	$[L_0]$

&

	$a_{\mathtt{s}}$	$a_{\mathtt{sts}}$
$a_{\mathtt{s}}$	$a_{\mathtt{s}}$	$a_{\mathtt{sts}}$
$a_{\mathtt{sts}}$	$a_{\mathtt{sts}}$	$a_{\mathtt{s}}$

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&

	$a_{\mathtt{s}}$	$a_{\mathtt{sts}}$
$a_{\mathtt{s}}$	$a_{\mathtt{s}}$	$a_{\mathtt{sts}}$
$a_{\mathtt{sts}}$	$a_{\mathtt{sts}}$	$a_{\mathtt{s}}$

The $\mathcal{H}$-cell limit v

$$a_{\mathtt{s}} \leftrightsquigarrow [L_0], \quad a_{\mathtt{sts}} \leftrightsquigarrow [L_2].$$

Fact.

Both connections are always true (*i.e.* for any e).

Example ($e = 2$).

The fusion ring $K_0(\mathrm{SO}(3)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_2]$. The $\mathcal{H}$-cell

li

C

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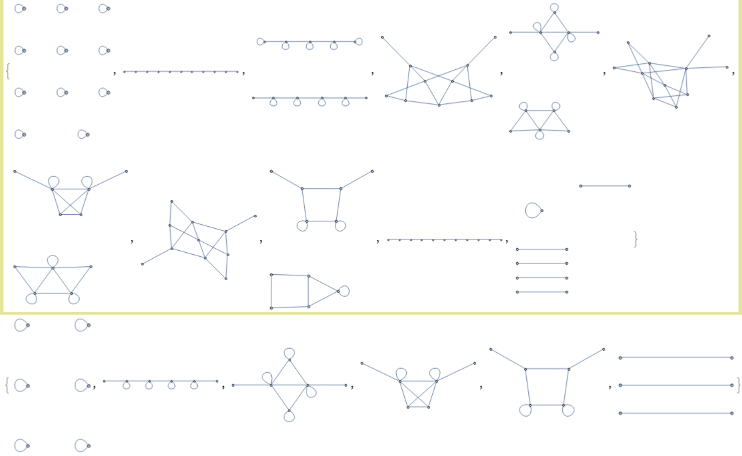
The bicoloring is basically coming from slightly different fusion graphs *e.g.* for $e = 6$:



◂ Back

The zigzag algebra $\mathrm{Z}(\boldsymbol{\Gamma})$

$$\begin{tikzcd}
\blacktriangledown \arrow[r, shift left, "\mathrm{u}"] & \bigstar \arrow[l, shift left, "\mathrm{d}"] \arrow[r, shift left, "\mathrm{u}"] & \blacktriangledown \arrow[l, shift left, "\mathrm{d}"]
\end{tikzcd}$$

$$uu = 0 = dd, ud = du$$

Apply the usual philosophy:

- Take projectives $\mathrm{P}_{\mathrm{s}} = \bigoplus_{\blacktriangledown} P_i$ and $\mathrm{P}_{\mathrm{t}} = \bigoplus_{\bigstar} P_i$.
- Get endofunctors $\mathrm{B}_{\mathrm{s}} = \mathrm{P}_{\mathrm{s}} \otimes_{\mathrm{Z}(\boldsymbol{\Gamma})} -$ and $\mathrm{B}_{\mathrm{t}} = \mathrm{P}_{\mathrm{t}} \otimes_{\mathrm{Z}(\boldsymbol{\Gamma})} -$.
- Check: These decategorify to b_{s} and b_{t}. (Easy.)
- Check: These give a genuine 2-representation. (Bookkeeping.)
- Check: There are no graded deformations. (Bookkeeping.)

Difference to $\mathrm{SL}(2)_q$: There is an honest quiver as this is non-semisimple.

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Apply the usu

- Take pro
- Get endo
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Proof?

Use projective resolutions of $\mathrm{Z}(\mathbf{\Gamma})$.

Difference to $\mathrm{SL}(2)_q$: There is an honest quiver as this is non-semisimple.

Back

Example (type H_4).

cell	0	1	2	3	4	5	6=6′	5′	4′	3′	2′	1′	0′
size	1	32	162	512	625	1296	9144	1296	625	512	162	32	1
a	0	1	2	3	4	5	6	15	16	18	22	31	60
$v \to 0$	□	2□	2□	2□	□	□	big	□	□	2□	2□	2□	□

The big cell:

$14_{8,8}$	$13_{10,8}$	$14_{6,8}$
$13_{8,10}$	$18_{10,10}$	$18_{6,10}$
$14_{8,6}$	$18_{10,6}$	$24_{6,6}$

$\boxed{14_{8,8}}$:

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$$\mathrm{PFdim} = 120(9 + 4\sqrt{5}).$$

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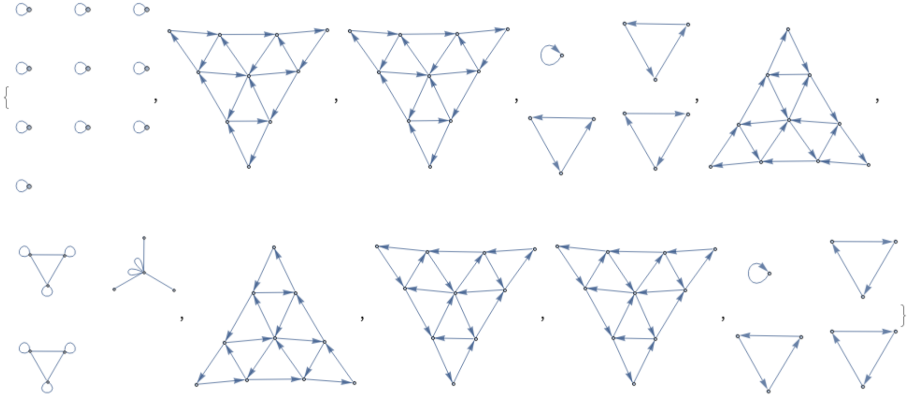

In the non-semisimple case one gets quiver algebras supported on these graphs. ("Trihedral zigzag algebras".)

◂ Stop - you are annoying!