Finitary 2-Representations and (co)algebra 1-morphisms

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Outline

• In this talk, I will explain the role of (co)algebra objects in the 2-representation theory of (graded) fiat monoidal categories (which are not necessarily abelian), generalizing results by Ostrik for tensor categories.

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• Time permitting, I will also recall the $\mathcal{H}$-reduction result, which Mazorchuk presented in his talk, giving a concrete example which is relevant for my second talk tomorrow.

2-Representation Theory in two slides

Let $\mathcal{C} = (\mathcal{C}, \oplus, \otimes, \mathbb{1}, \mathbb{0})$ be a finitary monoidal category, possibly with some additional nice properties (e.g. fiat, semisimple ...).

Definition

A **2-representation** of $\mathcal{C}$ is a finitary category $\mathcal{M}$ (possibly with some additional nice properties) together with a linear, monoidal functor

$$\mathcal{C} \to \mathrm{End}(\mathcal{M}) := \mathrm{Func}(\mathcal{M}, \mathcal{M}),$$

called the **2-action**.

There is a natural notion of 2-intertwiners and 2-equivalence of 2-representations.

Classification Problem

Mazorchuk and Miemietz proved a categorical Jordan-Hölder theorem for finitary 2-representations. The role of the simples is played by the so-called **simple transitive 2-representations**.

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Our goal

Solve the classification problem for the monoidal category of Soergel bimodules $\mathcal{S}$ of any finite Coxeter type.

Remark: $\mathcal{S}$ is not abelian, let alone semisimple.

- Using the principal/regular 2-representation (e.g. simple transitive 2-reps of $\mathrm{Rep}(G)$, cell 2-representations of arbitrary finitary 2-categories).

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• Using symmetries on a given simple transitive 2-representation (e.g. [M-Mazorchuk]).

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- Using symmetries on a given simple transitive 2-representation (e.g. [M-Mazorchuk]).
- Using a presentation of $\mathcal{C}$ by generating morphisms and relations to define a concrete monoidal functor to $\mathcal{C}_A := \mathrm{add}(A \oplus (A \otimes A))$ for some f.d. algebra A (e.g. for dihedral Soergel bimodules [M-Tubbenhauer]).

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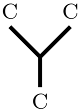
• Using (co)simple (co)algebra objects in $\mathcal{C}$. (All simple transitive 2-representations can be constructed in this way).

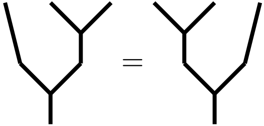
Coalgebra objects

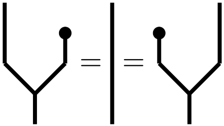
Definition (Coalgebra object)

A coalgebra object in a monoidal category $\mathcal{C}$ is an object $\mathrm{C} \in \mathcal{C}$ together with a comultiplication morphism $\delta\colon \mathrm{C} \to \mathrm{CC}$ and a counit morphism $\epsilon\colon \mathrm{C} \to \mathbb{1}$ satisfying coassociativity and counitality.

$\delta_{\mathrm{C}} =$ 

$\varepsilon_{\mathrm{C}} =$ 

 (coass)

 (counital)

Algebra objects

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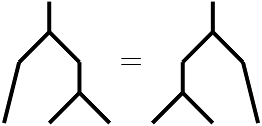
An algebra object in a monoidal category $\mathcal{C}$ is an object $\mathrm{A} \in \mathcal{C}$ together with a multiplication morphism $\mu_{\mathrm{A}} \colon \mathrm{AA} \to \mathrm{A}$ and a unit morphism $\iota_{\mathrm{A}} \colon \mathbb{1} \to \mathrm{A}$ satisfying associativity and unitality.

$\mu_{\mathrm{A}} =$ 

$\iota_{\mathrm{A}} =$ 



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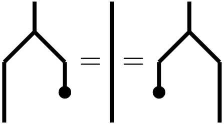
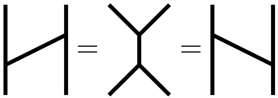


(unital)

Frobenius algebra objects

Definition (Frobenius algebra object)

A Frobenius algebra object in a monoidal category $\mathcal{C}$ is an algebra-and-coalgebra object satisfying an additional compatibility condition.

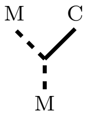


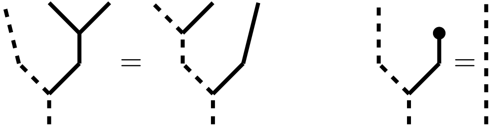
(Co)module objects

Let $\mathrm{C} = (\mathrm{C}, \delta_{\mathrm{C}}, \epsilon_{\mathrm{C}})$ be a coalgebra object in $\mathcal{C}$.

Definition

A right comodule object of C is an object $\mathrm{M} \in \mathcal{C}$ together with a coaction morphism $\delta_{\mathrm{M}} \colon \mathrm{M} \to \mathrm{MC}$ satisfying the usual axioms for comodules. (Module objects of algebra objects are defined similarly.)

$\delta_{\mathrm{M}} =$ 



The category of comodule objects

Let C be a coalgebra object and A an algebra object in $\mathcal{C}$.

Definition

Let $\mathrm{comod}_{\mathcal{C}}(\mathrm{C})$ be the category of comodule objects of C and intertwiners between them. (Similarly, let $\mathrm{mod}_{\mathcal{C}}(\mathrm{A})$ be the category of module objects of A and intertwiners between them.)

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Fact

Left multiplication defines a left 2-action of $\mathcal{C}$ on $\mathrm{comod}_{\mathcal{C}}(\mathrm{C})$ (resp. $\mathrm{mod}_{\mathcal{C}}(\mathrm{A})$).

Fiatness

Let $\mathcal{C}$ be **fiat**.

- The involution sends coalgebra/algebra objects to algebra/coalgebra objects, sending left/right comodule/module objects to right/left module/comodule objects.

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- The involution sends coalgebra/algebra objects to algebra/coalgebra objects, sending left/right comodule/module objects to right/left module/comodule objects.
- The involution sends Frobenius objects to Frobenius objects.

Abelianizations

• If $\mathcal{C}$ is not abelian, we can consider its injective abelianization $\underline{\mathcal{C}}$ or its projective abelianization $\overline{\mathcal{C}}$, which contains $\mathcal{C}$ as the full subcategory of injective/projective objects.

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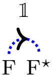
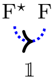
- If $\mathcal{C}$ is fiat, then the objects of $\mathcal{C}$ act by exact endofunctors on $\underline{\mathcal{M}}$ and $\overline{\mathcal{M}}$.
- There is a natural notion of Morita(-Takeuchi) equivalence between (co)algebra objects in $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$.

Framing (co)algebra objects

Let $\mathcal{C}$ be fiat and C a coalgebra object in $\underline{\mathcal{C}}$. Let $\mathrm{F} \in \mathcal{C}$ be such that $\mathrm{FCF}^{\star} \neq 0$ and draw the (co)unit of the adjunction $(\mathrm{F}, \mathrm{F}^{\star})$ as follows

$\epsilon_{\mathrm{F}} =$ 

, $\eta_{\mathrm{F}} =$ 

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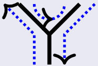
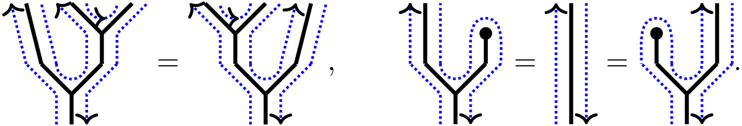
Lemma

$\mathrm{FCF}^{\star} \in \underline{\mathcal{C}}$ *has a coalgebra structure with*

$\delta_{\mathrm{FCF}^{\star}} :=$  , $\varepsilon_{\mathrm{FCF}^{\star}} :=$  .

Coassociativity and counitality:



Simple transitive 2-representations

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Theorem (MMMT, MMMZ, MMMTZ)

Let $\mathcal{M}$ be a simple transitive 2-representation of $\mathcal{C}$ with apex $\mathcal{J}$ and $0 \neq \mathrm{X} \in \mathcal{M}$. There is an absolutely cosimple coalgebra object C_{X} in $\mathrm{add}(\mathcal{J}) \subseteq \mathcal{C}$ such that

$$\mathrm{inj}_{\underline{\mathcal{C}}}(\mathrm{C}_{\mathrm{X}}) \simeq \mathcal{M}.$$

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- $$\{\text{simple transitive 2-reps of } \mathcal{C} \text{ with apex } \mathcal{J}\}/\simeq$$
  $$\overset{1:1}{\longleftrightarrow}$$
  $$\{\text{Absolutely cosimple coalgebra objects in } \mathrm{add}(\mathcal{J})\}/\simeq_{\mathrm{MT}}.$$

Sketch of the proof of the Theorem

Choose $\mathrm{X} \in \mathcal{M}$.

- Internal hom [Ostrik]: Take $\mathrm{C}_{\mathrm{X}} := [\mathrm{X}, \mathrm{X}] \in \underline{\mathrm{add}(\mathcal{J})}$, defined such that for all $\mathrm{F} \in \underline{\mathcal{C}}$:

$$\mathrm{Hom}_{\underline{\mathcal{S}}}([\mathrm{X}, \mathrm{X}], \mathrm{F}) \cong \mathrm{Hom}_{\underline{\mathcal{M}}}(\mathrm{X}, \mathrm{FX}).$$

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- Since $\mathcal{M}$ is simple transitive and $\mathcal{C}$ is $\mathcal{J}$-simple, $[\mathrm{X}, \mathrm{X}]$ is absolutely cosimple.
- For any $\mathrm{G} \in \mathrm{add}(\mathcal{J})$, the coalgebra object $\mathrm{G}[\mathrm{X}, \mathrm{X}]\mathrm{G}^*$ is isomorphic to $[\mathrm{GX}, \mathrm{GX}]$, whence MT-equivalent to $\mathrm{C_X}$.

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- For any $\mathrm{G} \in \mathrm{add}(\mathcal{J})$, $\mathrm{GC_X G}^\star \in \mathrm{add}(\mathcal{J}) = \mathrm{inj}(\underline{\mathrm{add}(\mathcal{J})})$.

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- For any $\mathrm{G} \in \mathrm{add}(\mathcal{J})$, $\mathrm{GC_XG}^\star \in \mathrm{add}(\mathcal{J}) = \mathrm{inj}(\underline{\mathrm{add}(\mathcal{J})})$.
- Choose G such that $\mathrm{X} \subseteq_\oplus \mathrm{GX}$. Then $[\mathrm{X}, \mathrm{X}] \subseteq_\oplus [\mathrm{GX}, \mathrm{GX}]$, so $\mathrm{C_X} = [\mathrm{X}, \mathrm{X}] \in \mathrm{add}(\mathcal{J})$.

Grouplike examples

Let G be a finite group and $\mathcal{C} := \mathrm{Vect}_G$ (semisimple). For every subgroup $H \subseteq G$ and every normalized $\omega \in \mathrm{Z}^2(H, \mathbb{C}^*)$, the group algebra $\mathbb{C}[H]$ is a Frobenius algebra object in Vect_G with

$$\mu_H^\omega(x,y) := \omega(x,y)^{-1}xy \qquad \iota(1) := e;$$
$$\delta_H^\omega(h) := \sum_{xy=h} \omega(x,y) x \otimes y, \quad \epsilon_H(h) := \delta_{h,e}.$$

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- For any $g \in G$ and $\omega \in \mathrm{Z}^2(H, \mathbb{C}^*)$, define

$$\omega^g(x,y) := \omega(gxg^{-1}, gyg^{-1}) \in \mathrm{Z}^2(gHg^{-1}, \mathbb{C}^*).$$

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- We have

$$(\mathbb{C}[H], \delta_H, \epsilon_H) \simeq_{MT} (\mathbb{C}[H'], \delta_{H'}, \epsilon_{H'})$$

$$\Leftrightarrow$$

$$\exists g \in G : \quad H' = gHg^{-1} \;\; \wedge \;\; [\omega'] = [\omega^g] \in \mathrm{H}^2(H', \mathbb{C}^*).$$

Comodule objects

Let $H \subseteq G$ and $\omega \in \mathrm{Z}^2(H, \mathbb{C}^*)$. The simple comodule objects of $\mathbb{C}[H] = (\mathbb{C}[H], \delta_H^\omega, \epsilon_H)$ are indexed by G/H: Let $\overline{g} = gH$ and define

$$\mathrm{L}_{\overline{g}} := \bigoplus_{h \in H} \mathbb{C}_{gh} \qquad \delta_{\overline{g}}(1_{gh}) := \sum_{xy=h} \omega(x,y) 1_{gx} \otimes y$$

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The Vect_G 2-action on $\mathrm{mod}_{\mathrm{Vect}_G}(\mathbb{C}[H])$ is given by

$$\mathbb{C}_{g_1} \boxtimes \mathrm{L}_{\overline{g_2}} \mapsto \mathrm{L}_{\overline{g_1 g_2}}$$

Let $q^{2n} = 1$ and $\mathcal{C} := \mathrm{U}_q(\mathfrak{sl}_2)\text{-mod}_{\mathrm{ss}}$. There is a complete and irredundant set of simples $L_0, \ldots, L_{n-2}$ ($\dim_q(L_i) = [i+1]_q$).

Theorem (Kirillov-Ostrik)

Up to Morita equivalence, simple algebra objects in $\mathrm{U}_q(\mathfrak{sl}_2)\text{-mod}_{\mathrm{ss}}$ are classified by ADE Dynkin diagrams with $h = n$. For each such diagram Γ

a) the isoclasses of the simple module objects correspond to the vertices of Γ;

b) the 2-action of L_1 on the category of module objects decategorifies to $2I - A(\Gamma)$, where $A(\Gamma)$ is the Cartan matrix.

Quantum $\mathfrak{sl}_2$ examples. Type A_{n-1}

• $\mathrm{A}_{A_{n-1}} := L_0$ is a simple algebra object with μ the canonical isomorphism $L_0 L_0 \cong L_0$ and ι the identity on L_0.

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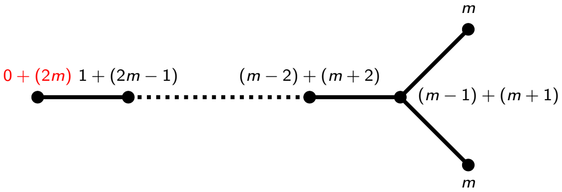
• When $n = 2m+2$, there is an interesting $\mathbb{Z}/2\mathbb{Z}$ symmetry on $\mathrm{mod}_{\mathcal{C}}(\mathrm{A}_{A_{n-1}})$ given by $L_i \leftrightarrow L_{m-i}$. It has one fixed point: L_m.

Quantum $\mathfrak{sl}_2$ examples: Type D_{m+1} $(n = 2m + 2)$



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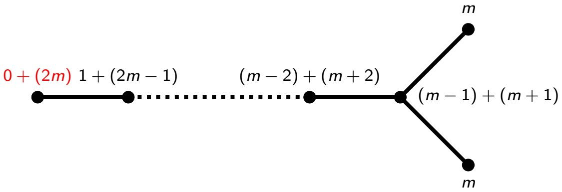

- $\mathrm{A}_{D_{m+1}} := L_0 \oplus L_{2m}$ is a simple algebra object, with μ given by suitably normalized isomorphisms $L_0 L_0 \to L_0$, $L_0 L_{2m} \to L_{2m}$, $L_{2m} L_0 \to L_{2m}$ and $L_{2m} L_{2m} \to L_0$, and ι by the canonical embedding $L_0 \hookrightarrow L_0 \oplus L_{2m}$.

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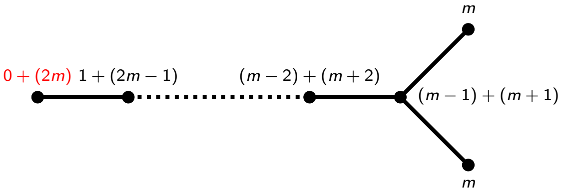

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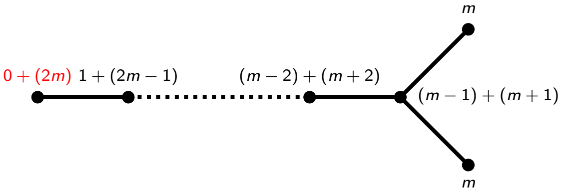

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- For each vertex, the decomposition in $\mathcal{C}$ of the corresponding simple module object is given.
- $\mathrm{mod}_{\mathcal{C}}(\mathrm{A}_{D_{m+1}}) \simeq \Omega_{\mathbb{Z}/2\mathbb{Z}}\left(\mathrm{mod}_{\mathcal{C}}(\mathrm{A}_{A_{n-1}})\right)$ (orbit category).

Projective bimodules

Let A be a finite-dimensional, complex, connected, basic algebra and let $e_1, \ldots, e_n \in A$ be a complete set of orthogonal primitive idempotents. Let

$$\mathcal{C}_A := \mathrm{add}\,(A \oplus (A \otimes A)) \subseteq \mathrm{bim}(A, A)$$

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- $\otimes_A$ with a projective A-A bimodule is an exact endofunctor on A-mod which sends any object to a projective object.

Projective bimodules

- $Ae_i \otimes e_iA$ is a coalgebra object in $\mathcal{C}_A$, for $i = 1, \ldots, n$, with

$$\delta\colon Ae_i \otimes e_iA \to Ae_i \otimes e_iAe_i \otimes e_iA, \quad \delta(a \otimes b) := a \otimes e_i \otimes b;$$
$$\epsilon\colon Ae_i \otimes e_iA \to A, \quad \epsilon(a \otimes b) := ab.$$

- If A is a weakly symmetric Frobenius algebra with trace $\mathrm{tr}\colon A \to \mathbb{C}$, then $Ae_i \otimes e_iA$ is a Frobenius object. Let $\{a_1, \ldots, a_n\}$ be a basis of e_iA and $\{a^1, \ldots, a^n\}$ a basis of Ae_i such that $\mathrm{tr}(a_ia^j) = \delta_{i,j}$, then

$$\mu\colon Ae_i \otimes e_iAe_i \otimes e_iA \to Ae_i \otimes e_iA, \quad \mu(a \otimes b \otimes c) := \mathrm{tr}(b)a \otimes c;$$
$$\iota\colon A \to\colon Ae_i \otimes e_iA, \quad \iota(1) := \sum_{j=1}^n a^j \otimes a_j.$$

Let Γ be any bipartite graph, e.g. the type A_{n-1} Dynkin diagram. Define the **double quiver**:

$$Q(\Gamma) := \; 0 \rightleftarrows 1 \;\; \rightleftarrows \;\; n-3 \rightleftarrows n-2 \; .$$

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Let $A(\Gamma)$ denote the quotient of the path algebra of $Q(\Gamma)$ by the following relations:

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$A(\Gamma)$ is a finite-dimensional, positively graded, symmetric algebra. (grading=path length and $\mathrm{tr}(i|j) = \delta_{i,j}$, $\mathrm{tr}(e_i) = 0$.)

Zigzag algebras

The graded, projective $A(\Gamma)$-$A(\Gamma)$ bimodule

$$\mathrm{F} := \bigoplus_{i \text{ even}} A(\Gamma)e_i \otimes e_i A(\Gamma)$$

is a Frobenius algebra object in $\mathcal{C}_{A(\Gamma)}$, with structural morphisms given by

$$\delta_\Gamma(a \otimes b) := a \otimes e_i \otimes b;$$
$$\epsilon_\Gamma(a \otimes b) := ab;$$
$$\mu_\Gamma(a \otimes b \otimes c) = \delta_{b,i|i}\, a \otimes c;$$
$$\iota_\Gamma(e_i) := \begin{cases} i|i \otimes e_i + e_i \otimes i|i, & i \text{ even} \\ \sum_{j:\ i \neq j} i|j \otimes j|i, & i \text{ odd.} \end{cases}$$

Simple transitive 2-representations

Suppose that $\mathcal{C}$ is fiat and $\mathcal{J}$-simple for a certain two-sided cell $\mathcal{J}$. Let $\mathcal{M}$ be a simple transitive 2-representation of $\mathcal{C}$ with apex $\mathcal{J}$ and underlying algebra A (i.e. $\mathcal{M} \simeq A\text{-proj}$).

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In the 2-representation on A-proj, objects in $\mathrm{add}(\mathcal{J})$ are mapped to endofunctors given by tensoring$_{/A}$ with projective A-A bimodules and morphisms in $\mathrm{add}(\mathcal{J})$ are mapped to natural transformations given by A-A bimodule maps.

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Example: The $A(\Gamma)$-$A(\Gamma)$ bimodule F is the image of the dihedral Soergel bimodule B_s in the cell 2-representation associated to the left cell containing s in D_{2n}.

Reduction to $\mathcal{H}$-cells

- Maintaining the assumptions from the previous slide, let $\mathcal{L} \subseteq \mathcal{J}$ be any left cell and define $\mathcal{H} := \mathcal{L} \cap \mathcal{L}^* \subseteq \mathcal{J}$.

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- By Mazorchuk's talk and the theorem from some slides ago:

$$\{\text{Simple transitive 2-reps of } \mathcal{C} \text{ with apex } \mathcal{J}\}/\simeq$$

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$$\{\text{Absolutely cosimple coalgebra objects in } \mathrm{add}(\mathcal{H})\}/\simeq_{\mathrm{MT}} .$$

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- Let $\mathcal{S}$ be the category of Soergel bimodules of type $I_2(n)$. For the left cell $\mathcal{L}_s := \{s, ts, sts, \ldots\}$, we have $\mathcal{H}_s = \{s, sts, \ldots\}$.

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$$A(\Gamma)_s := \bigoplus_{i\colon \text{even}} e_i A(\Gamma) e_i.$$

Note that $A(\Gamma)_s$ is still postively graded and symmetric, but much simpler than $A(\Gamma)$:

$$\mathrm{Hom}(A(\Gamma)_s e_i, A(\Gamma)_s e_j) \cong \begin{cases} \mathbb{C}[x]/(x^2) & i = j; \\ \{0\} & i \neq j. \end{cases}$$

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- On $A(\Gamma)_s\text{-mod}_{\mathrm{gr}}$, the object B_s acts by tensoring$_{/A(\Gamma)_s}$ with

$$\mathrm{F}_s := \bigoplus_{i\text{: even}} A(\Gamma)_s e_i \otimes e_i A(\Gamma)_s.$$

Tomorrow: Applications to Soergel bimodules!!!