Some first steps towards 2-representation theory of Coxeter groups

Or: The "next generation" of representation theory of Coxeter groups!?

Daniel Tubbenhauer

$$\begin{tikzcd}
\mathbf{Kar}(\boldsymbol{\mathcal{D}}_n) \arrow[r, "\text{full-grown 2-action}"] \arrow[d, "{[\cdot]_{\oplus}}"'] & \mathbf{End}(\mathbf{V}) \arrow[d, "{[\cdot]_{\oplus}}"] \\
\mathrm{W}_n \arrow[r, "\text{classical action}"'] \arrow[ur, dotted, "\text{categorical action}"] & \mathrm{End}(\mathrm{V})
\end{tikzcd}$$

Joint work with Marco Mackaay

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1. Categorical representation theory
   - Classical representation theory
   - Categorical representation theory

2. “Dihedral representation theory”
   - Dihedral groups as Coxeter groups
   - Dihedral groups and their representations

3. “Dihedral 2-representation theory”
   - Categorical representations of dihedral groups
   - Classification of dihedral 2-representations

Frobenius and Burnside: pioneers of representation theory

Let $\mathbb{C}[G]$ be the group ring of a (finite) group G.

Frobenius ($\sim$1895 onwards), Burnside ($\sim$1900 onwards): Representation theory is the (useful) study of linear group actions:

$$\mathrm{R}\colon \mathbb{C}[G] \longrightarrow \mathrm{End}(\mathrm{V}), \quad \mathrm{R}(g) = \text{a "matrix" in } \mathrm{End}(\mathrm{V}),$$

with V being some $\mathbb{C}$-vector space. We call V a G-module or a G-representation.

The "atoms" of such an action are called simple.

Maschke ($\sim$1899): All G-modules are built out of such atoms ("Jordan-Hölder").

The strategy

"Groups, as men, will be known by their actions." – Guillermo Moreno

The study of group actions is of fundamental importance in mathematics and related field. Sadly, it is also very hard.

Representation theory approach: the analogous linear problem of classifying G-modules has a satisfactory answer for many groups.

$$\begin{tikzcd}
\begin{matrix}\text{Problem involving}\\ \text{a group action}\\ G \circlearrowright X\end{matrix} \arrow[rr, dotted] & & \begin{matrix}\text{Problem involving}\\ \text{a linear group action}\\ \mathbb{C}[G] \circlearrowright \mathbb{C}[X]\end{matrix} \arrow[dl, dotted] \\
& \begin{matrix}\text{"Decomposition of}\\ \text{the problem"}\\ \mathbb{C}[G] \circlearrowright \bigoplus V_i\end{matrix} \arrow[ul, dotted] &
\end{tikzcd}$$

The basic theorems for finite groups

(a) All G-modules are built out of simple representations of G.

(b) The character of a simple G-module determines it.

(c) There is a one-to-one correspondence

$$|\{\text{simple } G\text{-modules}\}/\text{iso}| \xleftrightarrow{1:1} |\{\text{conjugacy classes in } G\}|.$$

(d) All simple G-modules can be constructed intrinsically using the regular G-module. For some groups these can be constructed explicitly.

We want to have a categorical version of this list! In this talk I discuss this for dihedral groups. (For most groups this is out of reach at the moment!)

But before let me explain what categorical representation theory is all about.

Categorified symmetries

Let A be some (group) algebra, V be an A-module and $\mathbf{V}$ be a (suitable) category.

$$\text{“Classical”} \rightsquigarrow \text{“Higher”}$$

$$a \mapsto \mathrm{R}(a) \in \mathrm{End}(\mathrm{V}) \rightsquigarrow a \mapsto \boldsymbol{\mathcal{R}}(a) \in \mathbf{End}(\mathbf{V})$$

$$(\mathrm{R}(a_1) \cdot \mathrm{R}(a_2))(v) = \mathrm{R}(a_1 a_2)(v) \rightsquigarrow (\boldsymbol{\mathcal{R}}(a_1) \circ \boldsymbol{\mathcal{R}}(a_2))\binom{X}{\alpha} \cong \boldsymbol{\mathcal{R}}(a_1 a_2)\binom{X}{\alpha}$$

A (weak) categorification of the A-module V should be thought of a categorical action of A on $\mathbf{V}$ with an isomorphism ψ such that

$$\begin{tikzcd}
{[\mathbf{V}]_{\oplus}} \arrow[r, "{[\boldsymbol{\mathcal{R}}_a]}"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowleft"] & {[\mathbf{V}]_{\oplus}} \arrow[d, "\psi"] \\
\mathrm{V} \arrow[r, "\mathrm{R}(a)"'] & \mathrm{V}.
\end{tikzcd}$$

Categorification in a nutshell

The picture to keep in mind regarding categorification is:

$$\begin{tikzcd}
\text{categories} & \text{functors} \arrow[l, dotted] & \text{nat. trafos} \arrow[l, dotted] \\
\text{vector spaces} \arrow[from=u, "\text{“categorifies”}"] & \text{linear maps} \arrow[l, dotted] \arrow[from=u, "\text{“categorifies”}"] & \\
\text{numbers} \arrow[from=u, "\text{“categorifies”}"] & &
\end{tikzcd}$$

Mazorchuk-Miemietz ($\sim$ 2014): Notion of “2-atoms” (called simple transitive). All (suitable) 2-representations are built out of 2-atoms (“2-Jordan-Hölder”). These are “determined” on the level of the Grothendieck group.

These are the categorical analogs of (a)+(b) from our list.

The state of the arts

- **Chuang-Rouquier ($\sim$ 2004), Khovanov-Lauda ($\sim$ 2008):** Systematic study of 2-representations of Lie algebras.
- **Chuang-Rouquier ($\sim$ 2004), Khovanov-Lauda ($\sim$ 2008):** All (simple) representations have categorifications.
- **Rouquier ($\sim$ 2008), Losev-Webster ($\sim$ 2013):** These are "unique".
- **Mazorchuk-Miemietz ($\sim$ 2014):** These are all 2-atoms (morally).
- Plenty of applications are known, e.g. in low-dimensional topology: Lauda-Queffelec-Rose ($\sim$ 2012) realized Khovanov-Rozansky homology via such 2-representations.

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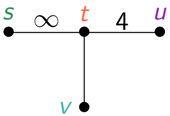
- Classification results are rare at the moment.
- Applications: Work in progress!

Coxeter groups and reflections

Given a finite set $S = \{s, t, \dots\}$, then the group

$$\mathrm{W} = \langle S, s^2 = t^2 = \cdots = 1, \underbrace{\dots sts}_{m_{st}} = \underbrace{\dots tst}_{m_{ts}}, \text{etc.}\rangle$$

is called a Coxeter group. They correspond to Coxeter graphs or matrices e.g.:



$$\rightsquigarrow \quad \begin{gathered} m_{st} = \infty \text{ (aka no relation)}, \\ m_{tu} = 4, m_{tv} = 3, \\ \text{rest commute.} \end{gathered} \quad \rightsquigarrow \begin{pmatrix} 2 & \infty & 2 & 2 \\ \infty & 2 & 4 & 3 \\ 2 & 4 & 2 & 2 \\ 2 & 3 & 2 & 2 \end{pmatrix}$$

Coxeter (∼ 1935), Tits (∼ 1961): Coxeter groups are abstract groups giving a generator-relation presentation of reflection groups.

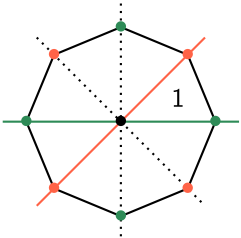
The main example today: dihedral groups

The dihedral groups are of Coxeter type $\mathrm{I}_2(n)$:

$$\mathrm{W}_n = \langle s, t | s^2 = t^2 = 1, s_n = \underbrace{\ldots sts}_{n} = w_0 = \underbrace{\ldots tst}_{n} = t_n \rangle,$$

$$\text{e.g.: } \mathrm{W}_4 = \langle s, t | s^2 = t^2 = 1, tsts = w_0 = stst \rangle$$

These are the symmetry groups of regular n-gons. For example take $n = 4$, the Coxeter complex is:



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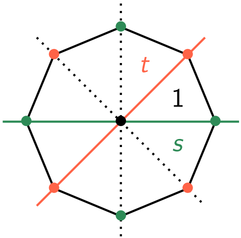

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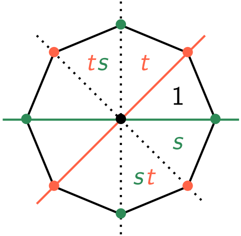

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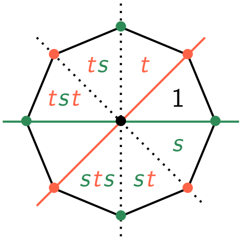

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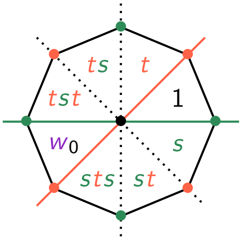


But there are also more fancy examples.

Kazhdan-Lusztig combinatorics of dihedral groups

Let $\mathrm{W} = \mathbb{C}[\mathrm{W}_n]$ for $n \in \mathbb{Z}_{>2} \cup \{\infty\}$.

For any word $w \in \mathrm{W}_n$ define (hereby $\leq$ denotes the Bruhat order)

$$\theta_w = \sum_{w' \leq w} w', \quad w, w' \in \mathrm{W}_n.$$

The set $\{\theta_w \mid w \in \mathrm{W}_n\}$ forms the Kazhdan-Lusztig basis. For example:

$$\theta_s = s + 1, \qquad \theta_t = t + 1, \qquad \theta_{sts} = sts + ts + st + s + t + 1, \quad \text{etc.}$$

These basis elements have positive structure constants, e.g.:

$$\theta_s\theta_s = 2 \cdot \theta_s, \qquad \theta_t\theta_t = 2 \cdot \theta_t, \qquad \theta_s\theta_t\theta_s = \theta_{sts} + \theta_s,$$
$$n = 4\colon\ \theta_s\theta_t\theta_s\theta_t + 2 \cdot \theta_t\theta_s = \theta_{w_0} = \theta_t\theta_s\theta_t\theta_s + 2 \cdot \theta_s\theta_t.$$

Thus, we have a good chance for categorification.

Most representations are "not nice"

Let n be even. (The odd case is similar.) Then the simple W_n-modules are either one-dimensional or two-dimensional (for $k = 1, \dots, \frac{n-2}{2}$):

$$\mathrm{V}_{\pm\pm} = \mathbb{C}; \begin{cases} s \rightsquigarrow +1, -1; t \rightsquigarrow +1, -1, \\ \theta_s \rightsquigarrow 2, 0; \theta_t \rightsquigarrow 2, 0, \end{cases}$$

$$\mathrm{V}_k = \mathbb{C}^2; \begin{cases} s \rightsquigarrow \begin{pmatrix} \cos(\frac{2\pi k}{n}) & \sin(\frac{2\pi k}{n}) \\ \sin(\frac{2\pi k}{n}) & -\cos(\frac{2\pi k}{n}) \end{pmatrix}; t \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \\ \theta_s \rightsquigarrow \begin{pmatrix} 2 \cdot \cos^2(\frac{\pi k}{n}) & \sin(\frac{2\pi k}{n}) \\ \sin(\frac{2\pi k}{n}) & 2 \cdot \sin^2(\frac{\pi k}{n}) \end{pmatrix}; \theta_t \rightsquigarrow \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \end{cases} \cong \mathrm{V}_k.$$

Most of these do not look suitable for categorification...

“Zig-zag algebras” provide categories

Fix a bipartite graph G. The double quiver Q_G associated to G is

$$G = \underline{1} = \overline{2} — \underline{3} \rightsquigarrow Q_G = \underline{1} \leftrightarrows\leftrightarrows \overline{2} \leftrightarrows \underline{3}.$$

Let QG denote the quotient algebra obtained from PG by some (today not-super-important) “zig-zag-relations”. Consider $\mathbf{G} = \mathrm{QG}\text{-}\mathrm{p}\mathbf{Mod}$.

Example à la Khovanov-Seidel ($\sim$ 2000) (for $n-1$ vertices):

$$\underline{1} \leftrightarrows \overline{2} \leftrightarrows \underline{3} \leftrightarrows \overline{4} \leftrightarrows \underline{5} \leftrightarrows \overline{6} \leftrightarrows \underline{7}$$

two steps in one direction are zero, e.g.: $\underline{5}|\overline{4}|\underline{3} = 0$,

returning to a vertex is “unique”, e.g.: $\underline{3}|\overline{2}|\underline{3} = \underline{3}|\underline{3} = \underline{3}|\overline{4}|\underline{3}$.

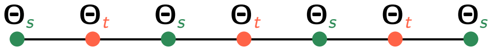
Looks promising: $[\mathbf{G}]_{\oplus} \cong \mathbb{C}^{\text{vertices}}$. We need an action!

A functorial action

There is a QG-bimodule ${}_{\mathrm{i}}P_{\mathrm{i}}$ for each $i \in G$ given by "path that start in i tensor path that end in i". (Formally it is $\mathrm{QG}\bullet\mathrm{i} \otimes \mathrm{i}\bullet\mathrm{QG}$.) Define endofunctors of $\mathbf{G}$ via

$$\mathbf{\Theta}_s = \bigoplus_{\underline{\mathrm{i}}\in G} {}_{\underline{\mathrm{i}}}P_{\underline{\mathrm{i}}} \otimes_{\mathrm{QG}} -, \qquad \mathbf{\Theta}_t = \bigoplus_{\overline{\mathrm{j}}\in G} {}_{\overline{\mathrm{j}}}P_{\overline{\mathrm{j}}} \otimes_{\mathrm{QG}} -.$$

Example: We sum over the graph of type A_7 as



Lemma: One checks that (for simplicity in the case of the example)

$$\mathbf{\Theta}_s(P_{\mathrm{i}}) \cong \begin{cases} P_{\underline{\mathrm{i}}} \oplus P_{\underline{\mathrm{i}}}, & \text{if } i \in \underline{\mathrm{S}}, \\ P_{\underline{\mathrm{j}-1}} \oplus P_{\underline{\mathrm{j}+1}}, & \text{if } i \in \overline{\mathrm{T}}, \end{cases} \qquad \mathbf{\Theta}_t(P_{\mathrm{i}}) \cong \begin{cases} P_{\overline{\mathrm{i}}} \oplus P_{\overline{\mathrm{i}}}, & \text{if } i \in \overline{\mathrm{T}}, \\ P_{\overline{\mathrm{j}-1}} \oplus P_{\overline{\mathrm{j}+1}}, & \text{if } i \in \underline{\mathrm{S}}. \end{cases}$$

Note: $\mathbf{\Theta}_s\mathbf{\Theta}_s \cong \mathbf{\Theta}_s \oplus \mathbf{\Theta}_s$ and $\mathbf{\Theta}_t\mathbf{\Theta}_t \cong \mathbf{\Theta}_t \oplus \mathbf{\Theta}_t$. Looks very promising.

A completely explicit example

$[\mathbf{\Theta}_t]$ act on $[\mathbf{A}(3)]_\oplus$ and $[\tilde{\mathbf{A}}(3)]_\oplus$ via

$$[\mathbf{\Theta}_s] = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad [\mathbf{\Theta}_t] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{(in type } \mathrm{A}_3\text{)}$$

$$[\mathbf{\Theta}_s] = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad [\mathbf{\Theta}_t] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix} \quad \text{(in type } \tilde{\mathrm{A}}_3\text{)}$$

(These are written on the bases $\{[P_{\underline{1}}], [P_{\underline{3}}], [P_{\overline{2}}]\}$ and $\{[P_{\underline{0}}], [P_{\underline{2}}], [P_{\overline{1}}], [P_{\overline{3}}]\}$.)

$$[\mathbf{\Theta}_s][\mathbf{\Theta}_t][\mathbf{\Theta}_s][\mathbf{\Theta}_t] + 2 \cdot [\mathbf{\Theta}_t][\mathbf{\Theta}_s] = [\mathbf{\Theta}_t][\mathbf{\Theta}_s][\mathbf{\Theta}_t][\mathbf{\Theta}_s] + 2 \cdot [\mathbf{\Theta}_s][\mathbf{\Theta}_t], \quad \text{(in type } \mathrm{A}_3\text{)}$$

$$[\mathbf{\Theta}_s][\mathbf{\Theta}_t][\mathbf{\Theta}_s][\mathbf{\Theta}_t] + 2 \cdot [\mathbf{\Theta}_t][\mathbf{\Theta}_s] \neq [\mathbf{\Theta}_t][\mathbf{\Theta}_s][\mathbf{\Theta}_t][\mathbf{\Theta}_s] + 2 \cdot [\mathbf{\Theta}_s][\mathbf{\Theta}_t], \quad \text{(in type } \tilde{\mathrm{A}}_3\text{)}$$

Thus, $[\mathbf{A}(3)]_\oplus$ has the structure of an W_4-module, but $[\tilde{\mathbf{A}}(3)]_\oplus$ does not.

The "categorical list"

For fixed n, we say G is of ADE type if either G is of type A_{n-1}, of type $\mathrm{D}_{n/2+1}$ (if n is even) or of type E_6, E_7, E_8 (if $n = 12, 18, 30$). (Example)

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(d) All simple W_n-modules can be constructed explicitly (remember: I gave the matrices a few slides back).

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(d) For each G of ADE type the corresponding simple transitive 2-representations of W_n can be constructed explicitly via "zig-zag-quivers".

This is the categorical version of our list! And we have a new one:

(e) All (suitable) categorifications of W_n-modules arise in this way (in particular, most W_n-modules are not "categorifiable").

Sorry, I have to bore you a bit more

- ★ Works graded as well, giving the same for the associated Hecke algebras H_n.
- ★ We also have the higher structure, i.e. we have a strong 2-action:

$$\begin{tikzcd}
\mathbf{Kar}(\mathcal{D}_n) \arrow[rr, "\text{full-grown 2-action}"] \arrow[d, "{[\cdot]_{\oplus}}"'] & & \mathbf{End}(\mathbf{G}) \arrow[d, "{[\cdot]_{\oplus}}"] \\
\mathrm{H}_n \arrow[rr, "\text{classical action}"'] \arrow[urr, dotted, "\text{categorical action}"] & & {[\mathbf{End}(\mathbf{G})]_{\oplus}}
\end{tikzcd}$$

Here $\mathcal{D}_n$ is the Hecke 2-category ("Soergel bimodules") categorifying H_n.

- ★ Everything (should) work for more general Coxeter groups (using "rank-colored" graphs instead of 2-colored graphs), e.g. for W_∞. But the classification story is way more complicated and open at the moment.
- ★ "Application": There is a reason for the classification in the dihedral case, i.e. the categories acted on are essentially the fusion subcategories of $\mathrm{U}_q(\mathfrak{sl}_2)$ (which are thus, naturally graded).

But most questions still remain mysterious!

There is still much to do...

Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside – top: first edition (1897); bottom: second edition (1911).

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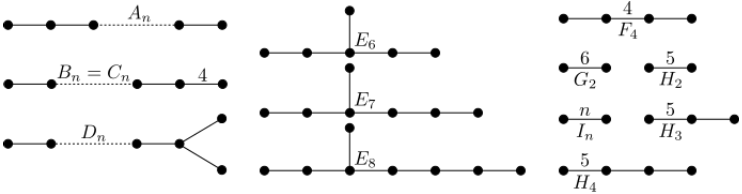

Figure: The Coxeter graphs of finite type.

Example: The type A family is given by the symmetric groups using the simple transpositions as generators.

(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

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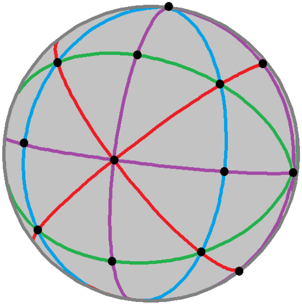

Figure: The Coxeter complex of type B_3.

(Pictures from https://en.wikipedia.org/wiki/Coxeter_notation.)

More or Back

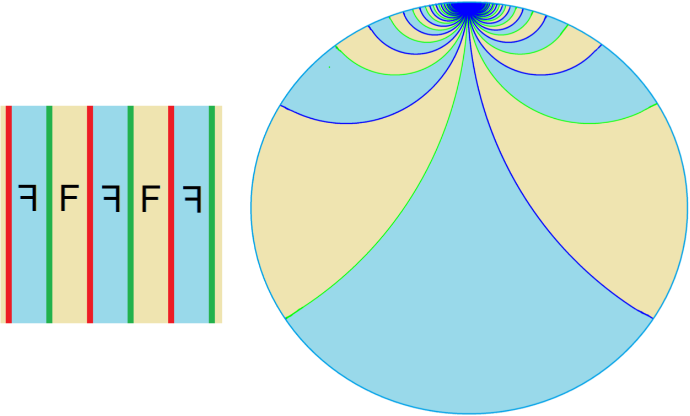

Figure: The action and the Coxeter complex of type $\mathrm{I}_2(\infty)$.

(Pictures from https://en.wikipedia.org/wiki/Coxeter-Dynkin_diagram.)

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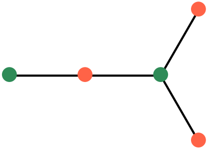
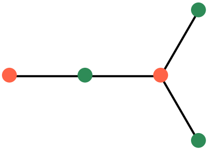
For $n = 8$ there are four ADE-G's which are non-isomorphic as bipartite graphs:

$|\underline{\mathrm{S}}| = 4, |\overline{\mathrm{T}}| = 3$:  ,

$|\underline{\mathrm{S}}| = 3, |\overline{\mathrm{T}}| = 4$:  ,

$|\underline{\mathrm{S}}| = 2, |\overline{\mathrm{T}}| = 3$: 

$|\underline{\mathrm{S}}| = 3, |\overline{\mathrm{T}}| = 2$: 

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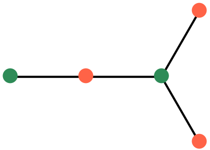
$G =$  $, \ [\mathbf{G}]_{\oplus} \cong \mathrm{V}_1 \oplus \mathrm{V}_3 \oplus \mathrm{V}_{-+}$

$$\mathrm{V}_1 = \mathbb{C}^2; \quad \theta_s \rightsquigarrow \frac{1}{2} \cdot \begin{pmatrix} 2+\sqrt{2} & \sqrt{2} \\ \sqrt{2} & 2-\sqrt{2} \end{pmatrix}; \quad \theta_t \rightsquigarrow \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathrm{V}_3 = \mathbb{C}^2; \quad \theta_s \rightsquigarrow \frac{1}{2} \cdot \begin{pmatrix} 2-\sqrt{2} & \sqrt{2} \\ \sqrt{2} & 2+\sqrt{2} \end{pmatrix}; \quad \theta_t \rightsquigarrow \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathrm{V}_{-+} = \mathbb{C}; \quad \theta_s \rightsquigarrow 0; \ \theta_t \rightsquigarrow 2.$$

Hence, there is a bases change such that all matrices with positive integer entries.

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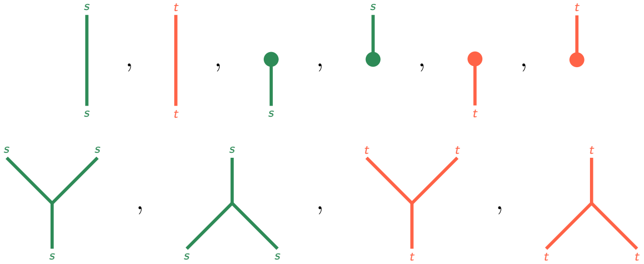
Elias-Khovanov ($\sim$ 2009), Elias-Williamson ($\sim$ 2013): For any Coxeter group W the Hecke 2-category $\mathcal{D}_{\mathrm{W}}$ is given by diagrammtic generators and relations, e.g.:



Soergel ($\sim$ 1992): If W is a Weyl group, $\mathcal{D}_{\mathrm{W}}$ is equivalent to the 2-category of projective endofunctors on $\mathcal{O}_0$ attached to the Lie algebra $\mathfrak{g}$ for W.

Morally: $\mathcal{D}_{\mathrm{W}}$ is a combinatorial way to analyze infinite-dimensional modules of $\mathfrak{g}$.

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