Di- and trihedral (2-)representation theory I

Or: Who colored my Dynkin diagrams?

Marco Mackaay & Daniel Tubbenhauer

s — ts — sts — tsts — ststs

1 w_0

t — st — tst — stst — tstst

Joint work with Volodymyr Mazorchuk and Vanessa Miemietz

July 2018

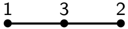
Let $A(\mathbf{\Gamma})$ be the adjacency matrix of a finite, connected, loopless graph $\mathbf{\Gamma}$. Let $\mathrm{U}_{e+1}(\mathtt{X})$ be the ▸ Chebyshev polynomial.

Classification problem (CP). Classify all $\mathbf{\Gamma}$ such that $\mathrm{U}_{e+1}(A(\mathbf{\Gamma})) = 0$.

Let $A(\boldsymbol{\Gamma})$ be the adjacency matrix of a finite, connected, loopless graph $\boldsymbol{\Gamma}$. Let $\mathrm{U}_{e+1}(\mathtt{X})$ be the Chebyshev polynomial.

Classification problem (CP). Classify all $\boldsymbol{\Gamma}$ such that $\mathrm{U}_{e+1}(A(\boldsymbol{\Gamma})) = 0$.

$$\mathrm{U}_3(\mathtt{X}) = (\mathtt{X} - 2\cos(\tfrac{\pi}{4}))\mathtt{X}(\mathtt{X} - 2\cos(\tfrac{3\pi}{4}))$$

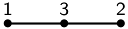
$\mathrm{A}_3 =$ 

$$\rightsquigarrow \quad A(\mathrm{A}_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \rightsquigarrow S_{\mathrm{A}_3} = \{2\cos(\tfrac{\pi}{4}), 0, 2\cos(\tfrac{3\pi}{4})\}$$

Let $A(\mathbf{\Gamma})$ be the adjacency matrix of a finite, connected, loopless graph $\mathbf{\Gamma}$. Let $\mathrm{U}_{e+1}(\mathtt{X})$ be the ▸ Chebyshev polynomial.

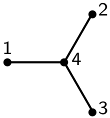
Classification problem (CP). Classify all $\mathbf{\Gamma}$ such that $\mathrm{U}_{e+1}(A(\mathbf{\Gamma})) = 0$.

$$\mathrm{U}_3(\mathtt{X}) = (\mathtt{X} - 2\cos(\tfrac{\pi}{4}))\mathtt{X}(\mathtt{X} - 2\cos(\tfrac{3\pi}{4}))$$

$\mathrm{A}_3 =$ 

$\rightsquigarrow \quad A(\mathrm{A}_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \rightsquigarrow S_{\mathrm{A}_3} = \{2\cos(\tfrac{\pi}{4}), 0, 2\cos(\tfrac{3\pi}{4})\}$

$$\mathrm{U}_5(\mathtt{X}) = (\mathtt{X} - 2\cos(\tfrac{\pi}{6}))(\mathtt{X} - 2\cos(\tfrac{2\pi}{6}))\mathtt{X}(\mathtt{X} - 2\cos(\tfrac{4\pi}{6}))(\mathtt{X} - 2\cos(\tfrac{5\pi}{6}))$$

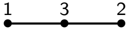
$\mathrm{D}_4 =$ 

$\rightsquigarrow \quad A(\mathrm{D}_4) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightsquigarrow S_{\mathrm{D}_4} = \{2\cos(\tfrac{\pi}{6}), 0^2, 2\cos(\tfrac{5\pi}{6})\}$

Let $A(\mathbf{\Gamma})$ be the adjacency matrix of a finite, connected, loopless graph $\mathbf{\Gamma}$. Let $U_{e+1}(X)$ be the ▸ Chebyshev polynomial.

Classification problem (CP). Classify all $\mathbf{\Gamma}$ such that $U_{e+1}(A(\mathbf{\Gamma})) = 0$.

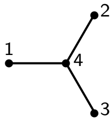
$$U_3(X) = (X - 2\cos(\tfrac{\pi}{4}))X(X - 2\cos(\tfrac{3\pi}{4}))$$

$A_3 =$ 

$\rightsquigarrow$ $A(A_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \rightsquigarrow S_{A_3} = \{2\cos(\tfrac{\pi}{4}), 0, 2\cos(\tfrac{3\pi}{4})\}$

✓ for $e = 2$

$$U_5(X) = (X - 2\cos(\tfrac{\pi}{6}))(X - 2\cos(\tfrac{2\pi}{6}))X(X - 2\cos(\tfrac{4\pi}{6}))(X - 2\cos(\tfrac{5\pi}{6}))$$

$D_4 =$ 

$\rightsquigarrow$ $A(D_4) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightsquigarrow S_{D_4} = \{2\cos(\tfrac{\pi}{6}), 0^2, 2\cos(\tfrac{5\pi}{6})\}$

✓ for $e = 4$

Let $A(\Gamma)$ … Γ. Let $U_{e+1}(X)$ …

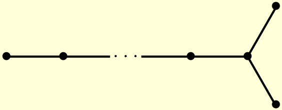
Cla… $= 0$.

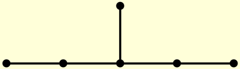
$A_3 =$ 1 … $\cos(\frac{3\pi}{4})\}$

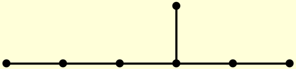
$D_4 =$ 1 … $\cos(\frac{5\pi}{6})\}$

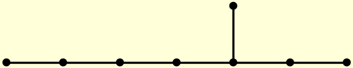
Smith ∼1969. The graphs solutions to (CP) are precisely ADE graphs for $e+2$ being (at most) the Coxeter number.

Type A_m:  for $e = m-1$

Type D_m:  for $e = 2m-4$

Type E_6:  for $e = 10$

Type E_7:  for $e = 16$

Type E_8:  for $e = 28$

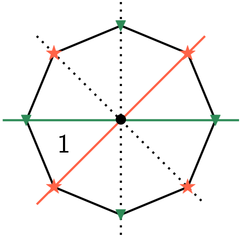
The main example today: dihedral groups

The dihedral groups are of Coxeter type $\mathrm{I}_2(e+2)$:

$$W_{e+2} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \overline{\mathtt{s}}_{e+2} = \underbrace{\dots\mathtt{sts}}_{e+2} = w_0 = \underbrace{\dots\mathtt{tst}}_{e+2} = \overline{\mathtt{t}}_{e+2} \rangle,$$

$$\text{e.g.: } W_4 = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \mathtt{tsts} = w_0 = \mathtt{stst} \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, e.g. for $e=2$ the Coxeter complex is:



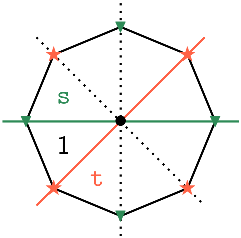
The main example today: dihedral groups

The dihedral groups are of Coxeter type $\mathrm{I}_2(e+2)$:

$$W_{e+2} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \overline{\mathtt{s}}_{e+2} = \underbrace{\ldots \mathtt{sts}}_{e+2} = w_0 = \underbrace{\ldots \mathtt{tst}}_{e+2} = \overline{\mathtt{t}}_{e+2} \rangle,$$

$$\text{e.g.: } W_4 = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \mathtt{tsts} = w_0 = \mathtt{stst} \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, e.g. for $e = 2$ the Coxeter complex is:



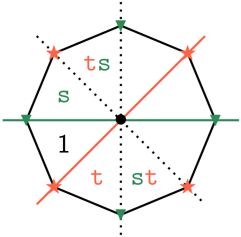
The main example today: dihedral groups

The dihedral groups are of Coxeter type $\mathrm{I}_2(e+2)$:

$$W_{e+2} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \overline{\mathtt{s}}_{e+2} = \underbrace{\ldots \mathtt{sts}}_{e+2} = w_0 = \underbrace{\ldots \mathtt{tst}}_{e+2} = \overline{\mathtt{t}}_{e+2} \rangle,$$

$$\text{e.g.: } W_4 = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \mathtt{tsts} = w_0 = \mathtt{stst} \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, e.g. for $e=2$ the Coxeter complex is:



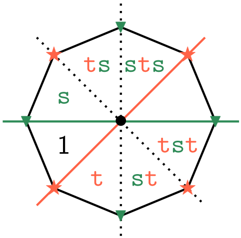
The main example today: dihedral groups

The dihedral groups are of Coxeter type $I_2(e+2)$:

$$W_{e+2} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \overline{\mathtt{s}}_{e+2} = \underbrace{\ldots \mathtt{sts}}_{e+2} = w_0 = \underbrace{\ldots \mathtt{tst}}_{e+2} = \overline{\mathtt{t}}_{e+2} \rangle,$$

$$\text{e.g.: } W_4 = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \mathtt{tsts} = w_0 = \mathtt{stst} \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, e.g. for $e = 2$ the Coxeter complex is:



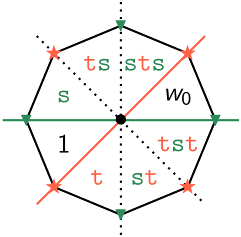
The main example today: dihedral groups

The dihedral groups are of Coxeter type $I_2(e+2)$:

$$W_{e+2} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \overline{\mathtt{s}}_{e+2} = \underbrace{\ldots \mathtt{sts}}_{e+2} = w_0 = \underbrace{\ldots \mathtt{tst}}_{e+2} = \overline{\mathtt{t}}_{e+2} \rangle,$$

$$\text{e.g.: } W_4 = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \mathtt{tsts} = w_0 = \mathtt{stst} \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, e.g. for $e = 2$ the Coxeter complex is:



The main example today: dihedral groups

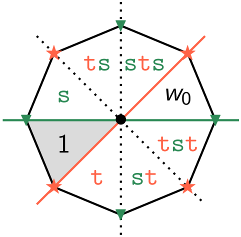
The dihedral groups are of Coxeter type $I_2(e+2)$:

$$W_{e+2} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \overline{\mathtt{s}}_{e+2} = \underbrace{\ldots \mathtt{sts}}_{e+2} = w_0 = \underbrace{\ldots \mathtt{tst}}_{e+2} = \overline{\mathtt{t}}_{e+2} \rangle,$$

$$\text{e.g.: } W_4 = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \mathtt{tsts} = w_0 = \mathtt{stst} \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, e.g. for $e = 2$ the Coxeter complex is:

I will explain in a few minutes what cells are.
For the moment: Never mind!



Lowest cell.

The main example today: dihedral groups

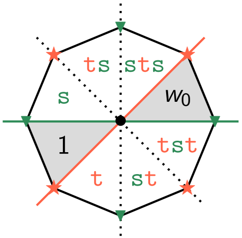
The dihedral groups are of Coxeter type $\mathrm{I}_2(e+2)$:

$$W_{e+2} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \overline{\mathtt{s}}_{e+2} = \underbrace{\ldots \mathtt{sts}}_{e+2} = w_0 = \underbrace{\ldots \mathtt{tst}}_{e+2} = \overline{\mathtt{t}}_{e+2} \rangle,$$

$$\text{e.g.: } W_4 = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \mathtt{tsts} = w_0 = \mathtt{stst} \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, e.g. for $e = 2$ the Coxeter complex is:

I will explain in a few minutes what cells are.
For the moment: Never mind!



Lowest cell.

Highest cell.

The main example today: dihedral groups

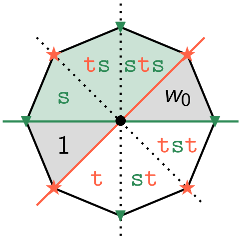
The dihedral groups are of Coxeter type $I_2(e+2)$:

$$W_{e+2} = \langle s, t \mid s^2 = t^2 = 1,\ \overline{s}_{e+2} = \underbrace{\dots sts}_{e+2} = w_0 = \underbrace{\dots tst}_{e+2} = \overline{t}_{e+2} \rangle,$$

$$\text{e.g.: } W_4 = \langle s, t \mid s^2 = t^2 = 1,\ tsts = w_0 = stst \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, e.g. for $e = 2$ the Coxeter complex is:

I will explain in a few minutes what cells are.
For the moment: Never mind!



Lowest cell.

Highest cell.

s-cell.

The main example today: dihedral groups

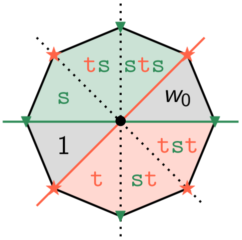
The dihedral groups are of Coxeter type $\mathrm{I}_2(e+2)$:

$$W_{e+2} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \overline{\mathtt{s}}_{e+2} = \underbrace{\dots\mathtt{sts}}_{e+2} = w_0 = \underbrace{\dots\mathtt{tst}}_{e+2} = \overline{\mathtt{t}}_{e+2} \rangle,$$

$$\text{e.g.: } W_4 = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \mathtt{tsts} = w_0 = \mathtt{stst} \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, e.g. for $e=2$ the Coxeter complex is:

I will explain in a few minutes what cells are.
For the moment: Never mind!



Lowest cell.

Highest cell.

s-cell.

t-cell.

Kazhdan–Lusztig combinatorics of dihedral groups

Consider $\mathrm{W}_{e+2} = \mathbb{C}[W_{e+2}]$ for $e \in \mathbb{Z}_{>0} \cup \{\infty\}$.

The Bott–Samelson (BS) basis is

$$\theta_{\mathtt{s}} = \mathtt{s} + 1, \qquad \theta_{\mathtt{t}} = \mathtt{t} + 1,$$
$$\{\theta_{\overline{w}} = \theta_{w_r} \cdots \theta_{w_1} \mid w = w_r \cdots w_1 \text{ reduced word}\}$$

The Kazhdan–Lusztig (KL) basis is

$$\{\theta_w = w + \textstyle\sum_{w' < w} w' \mid w, w' \text{ reduced words}\}.$$

Relations for the BS generators:

$$\theta_{\mathtt{s}}\theta_{\mathtt{s}} = 2\theta_{\mathtt{s}}, \qquad \theta_{\mathtt{t}}\theta_{\mathtt{t}} = 2\theta_{\mathtt{t}},$$

$$\text{some relation for } \underbrace{\ldots \mathtt{sts}}_{e+2} = w_0 = \underbrace{\ldots \mathtt{tst}}_{e+2}.$$

Example ($e > 2$).

	1	s	ts	sts	$tsts$
BS	1	$s+1$	$ts+s+t+1$	sts $+ts+2s+t+2$	$tsts+sts+tst$ $+3ts+st+3s+3t+3$
KL	1	$s+1$	$ts+s+t+1$	sts $ts+st+s+t+1$	$tsts+sts+tst$ $+ts+st+s+t+1$

etc.

$\{\theta_w = w + \sum_{w'<w} w' \mid w, w' \text{ reduced words}\}.$

Relations for the BS generators:

$$\theta_s\theta_s = 2\theta_s, \qquad \theta_t\theta_t = 2\theta_t,$$

$$\text{some relation for } \underbrace{\ldots sts}_{e+2} = w_0 = \underbrace{\ldots tst}_{e+2}.$$

Example ($e > 2$).

	1	s	ts	sts	tsts
BS	1	s + 1	ts + s + t + 1	sts + ts + 2s + t + 2	tsts + sts + tst + 3ts + st + 3s + 3t + 3
KL	1	s + 1	ts + s + t + 1	sts + ts + st + s + t + 1	tsts + sts + tst + ts + st + s + t + 1

etc.

$\{\theta_w = w + \sum_{w'<w} w' \mid w, w' \text{ reduced words}\}$.

Relations

The magic formulas.

$$\theta_s\theta_{ts\dots} = \theta_{sts\dots} + \theta_{s\dots} \quad \text{and} \quad \theta_t\theta_{st\dots} = \theta_{tst\dots} + \theta_{t\dots}. \qquad .$$

Example ($e = 2$).

$$\theta_s\theta_{tst}$$

Example ($e > 2$).

	1	s	ts	sts	tsts
BS	1	s + 1	ts + s + t + 1	sts + ts + 2s + t + 2	tsts + sts + tst + 3ts + st + 3s + 3t + 3
KL	1	s + 1	ts + s + t + 1	sts ts + st + s + t + 1	tsts + sts + tst + ts + st + s + t + 1

etc.

$\{\theta_w = w + \sum_{w'<w} w' \mid w, w' \text{ reduced words}\}.$

Relations

The magic formulas.

$$\theta_{s}\theta_{ts\dots} = \theta_{sts\dots} + \theta_{s\dots} \quad \text{and} \quad \theta_{t}\theta_{st\dots} = \theta_{tst\dots} + \theta_{t\dots}.$$

Example ($e = 2$).

$$\theta_{s}\theta_{tst} = (s + 1)(tst + st + ts + t + s + 1)$$

Example ($e > 2$).

	1	s	ts	sts	tsts
BS	1	s + 1	ts + s + t + 1	sts + ts + 2s + t + 2	tsts + sts + tst + 3ts + st + 3s + 3t + 3
KL	1	s + 1	ts + s + t + 1	sts + ts + st + s + t + 1	tsts + sts + tst + ts + st + s + t + 1

etc.

$\{\theta_w = w + \sum_{w'<w} w' \mid w, w'$ reduced words$\}$.

Relations

The magic formulas.

$$\theta_{s}\theta_{ts\dots} = \theta_{sts\dots} + \theta_{s\dots} \quad \text{and} \quad \theta_{t}\theta_{st\dots} = \theta_{tst\dots} + \theta_{t\dots}.$$

Example ($e = 2$).

$$\theta_{s}\theta_{tst} = \begin{matrix} w_0 + t + sts + st + 1 + s \\ tst + st + ts + t + s + 1 \end{matrix}$$

Example ($e > 2$).

	1	s	ts	sts	tsts
BS	1	s + 1	ts + s + t + 1	sts +ts + 2s + t + 2	tsts + sts + tst +3ts + st + 3s + 3t + 3
KL	1	s + 1	ts + s + t + 1	sts ts + st + s + t + 1	tsts + sts + tst +ts + st + s + t + 1

etc.

$\{\theta_w = w + \sum_{w'<w} w' \mid w, w'$ reduced words$\}$.

Relations

The magic formulas.

$$\theta_s\theta_{ts\dots} = \theta_{sts\dots} + \theta_{s\dots} \quad \text{and} \quad \theta_t\theta_{st\dots} = \theta_{tst\dots} + \theta_{t\dots}.$$

Example ($e = 2$).

$$\theta_s\theta_{tst} = \begin{matrix} w_0 + tst + sts + st + ts + t + s + 1 \\ st + t + s + 1 \end{matrix}$$

Example ($e > 2$).

	1	s	ts	sts	tsts
BS	1	s + 1	ts + s + t + 1	sts +ts + 2s + t + 2	tsts + sts + tst +3ts + st + 3s + 3t + 3
KL	1	s + 1	ts + s + t + 1	sts ts + st + s + t + 1	tsts + sts + tst +ts + st + s + t + 1

etc.

Relations

The magic formulas.

$$\theta_{s}\theta_{ts\dots} = \theta_{sts\dots} + \theta_{s\dots} \quad \text{and} \quad \theta_{t}\theta_{st\dots} = \theta_{tst\dots} + \theta_{t\dots}.$$

Example ($e = 2$).

$$\theta_{s}\theta_{tst} = \begin{matrix}\theta_{stst}\\ \theta_{st}\end{matrix}$$

Example ($e > 2$).

	1	s	ts	sts	tsts
BS	1	s + 1	ts + s + t + 1	sts + ts + 2s + t + 2	tsts + sts + tst + 3ts + st + 3s + 3t + 3
KL	1	s + 1	ts + s + t + 1	sts ts + st + s + t + 1	tsts + sts + tst + ts + st + s + t + 1

etc.

$\{\theta_w = w + \sum_{w'<w} w' \mid w, w' \text{ reduced words}\}$.

Relations

The magic formulas.

$$\begin{array}{c}\theta_{s}\theta_{ts\cdots} = \theta_{sts\cdots} + \theta_{s\cdots}\\ \mathtt{X}\,\mathrm{U}_{e+1}(\mathtt{X}) = \mathrm{U}_{e+2}(\mathtt{X}) + \mathrm{U}_{e}(\mathtt{X})\end{array} \quad\text{and}\quad \begin{array}{c}\theta_{t}\theta_{st\cdots} = \theta_{tst\cdots} + \theta_{t\cdots}\\ \mathtt{X}\,\mathrm{U}_{e+1}(\mathtt{X}) = \mathrm{U}_{e+2}(\mathtt{X}) + \mathrm{U}_{e}(\mathtt{X})\end{array}.$$

Example ($e = 2$).

$$\theta_{s}\theta_{tst} = \begin{array}{c}\theta_{stst}\\ \theta_{st}\end{array}$$

Kazhdan–Lusztig combinatorics of dihedral groups

Consider $\mathrm{W}_{e+2} = \mathbb{C}[W_{e+2}]$ for $e \in \mathbb{Z}_{>0} \cup \{\infty\}$.

The Bott–Samelson (BS) basis is

$$\theta_{s} = s + 1, \qquad \theta_{t} = t + 1.$$

Lusztig $\leq$ 2003.

The change of basis matrix between the BS and the KL basis is given by the coefficients d_e^k of the Chebyshev polynomials.

The Kazhda

$$\{\theta_w = w + \sum w' \mid w, w' \text{ reduced words}\}.$$

Example.

$$\mathrm{U}_7(\mathtt{X}) = 1 \cdot \mathtt{X}^7 - 6 \cdot \mathtt{X}^5 + 10 \cdot \mathtt{X}^3 - 4 \cdot \mathtt{X}$$

&

$$\theta_{tststststs} = 1 \cdot \theta_t\theta_s\theta_t\theta_s\theta_t\theta_s\theta_t\theta_s - 6 \cdot \theta_t\theta_s\theta_t\theta_s\theta_t\theta_s + 10 \cdot \theta_t\theta_s\theta_t\theta_s - 4 \cdot \theta_t\theta_s.$$

Relati

$$\text{some relation for } \underbrace{\dots sts}_{e+2} = w_0 = \underbrace{\dots tst}_{e+2}.$$

Kazhdan–Lusztig combinatorics of dihedral groups

Consider $\mathrm{W}_{e+2} = \mathbb{C}[W_{e+2}]$ for $e \in \mathbb{Z}_{>0} \cup \{\infty\}$.

The Bott–Samelson (BS) basis is

$$\theta_{s} = s + 1, \qquad \theta_{t} = t + 1.$$

Lusztig ≤ 2003.

The change of basis matrix between the BS and the KL basis is given by the coefficients d_e^k of the Chebyshev polynomials.

The Kazhda

$$\{\theta_w = w + \sum \ldots w' \mid w, w' \text{ reduced words}\}.$$

Example.

$$\mathrm{U}_7(\mathtt{X}) = 1 \cdot \mathtt{X}^7 - 6 \cdot \mathtt{X}^5 + 10 \cdot \mathtt{X}^3 - 4 \cdot \mathtt{X}$$

&

$$\theta_{tststststs} = 1 \cdot \theta_t\theta_s\theta_t\theta_s\theta_t\theta_s\theta_t\theta_s - 6 \cdot \theta_t\theta_s\theta_t\theta_s\theta_t\theta_s + 10 \cdot \theta_t\theta_s\theta_t\theta_s - 4 \cdot \theta_t\theta_s.$$

Relati

$$\sum_k d_e^k \theta_{\bar{s}_k} = \sum_k d_e^k \theta_{\bar{t}_k}.$$

"Chebyshev–braid-like"

Dihedral representation theory on one slide

One-dimensional representations. $\mathrm{M}_{\lambda_s,\lambda_t}, \lambda_s, \lambda_t \in \mathbb{C}, \theta_s \mapsto \lambda_s, \theta_t \mapsto \lambda_t$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_{0,0}, \mathrm{M}_{2,0}, \mathrm{M}_{0,2}, \mathrm{M}_{2,2}$	$\mathrm{M}_{0,0}, \mathrm{M}_{2,2}$

Two-dimensional representations. $\mathrm{M}_z, z \in \mathbb{C}, \theta_s \mapsto \begin{pmatrix} 2 & z \\ 0 & 0 \end{pmatrix}, \theta_t \mapsto \begin{pmatrix} 0 & 0 \\ \bar{z} & 2 \end{pmatrix}$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm} - \{0\}$	$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm}$

$\mathrm{V}_e = \mathrm{roots}(\mathsf{U}_{e+1}(\mathtt{X}))$ and $\mathrm{V}_e^{\pm}$ the $\mathbb{Z}/2\mathbb{Z}$-orbits under $z \mapsto -z$.

Dihedral representation theory on one slide

One-dimens $\mapsto \lambda_{t}$.

Proposition (Lusztig?).

The list of one- and two-dimensional W_{e+2}-representations is a complete, irredundant list of simple representations.

$\mathrm{M}_{0,0}, \mathrm{M}_{2,0}, \mathrm{M}_{0,2}, \mathrm{M}_{2,2}$	$\mathrm{M}_{0,0}, \mathrm{M}_{2,2}$

I learned this construction from Mackaay in 2017.

Two-dimensional representations. $\mathrm{M}_z, z \in \mathbb{C}, \theta_{s} \mapsto \left(\begin{smallmatrix} 2 & z \\ 0 & 0 \end{smallmatrix}\right), \theta_{t} \mapsto \left(\begin{smallmatrix} 0 & 0 \\ \bar{z} & 2 \end{smallmatrix}\right)$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm} - \{0\}$	$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm}$

$\mathrm{V}_e = \mathrm{roots}(\mathrm{U}_{e+1}(\mathtt{X}))$ and $\mathrm{V}_e^{\pm}$ the $\mathbb{Z}/2\mathbb{Z}$-orbits under $z \mapsto -z$.

Dihedral representation theory on one slide

One-dimensional representations. $\mathrm{M}_{\lambda_s,\lambda_t}, \lambda_s, \lambda_t \in \mathbb{C}, \theta_s \mapsto \lambda_s, \theta_t \mapsto \lambda_t$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_{0,0}, \mathrm{M}_{2,0}, \mathrm{M}_{0,2}, \mathrm{M}_{2,2}$	$\mathrm{M}_{0,0}, \mathrm{M}_{2,2}$

Example.

$\mathrm{M}_{0,0}$ is the sign representation and $\mathrm{M}_{2,2}$ is the trivial representation.

In case e is odd, $\mathrm{U}_{e+1}(\mathtt{X})$ has a constant term, so $\mathrm{M}_{2,0}, \mathrm{M}_{0,2}$ are not representations.

$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm} - \{0\}$	$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm}$

$\mathrm{V}_e = \mathrm{roots}(\mathrm{U}_{e+1}(\mathtt{X}))$ and $\mathrm{V}_e^{\pm}$ the $\mathbb{Z}/2\mathbb{Z}$-orbits under $z \mapsto -z$.

Dihedral representation theory on one slide

One-dimensional representations. $\mathrm{M}_{\lambda_{s},\lambda_{t}}, \lambda_{s}, \lambda_{t} \in \mathbb{C}, \theta_{s} \mapsto \lambda_{s}, \theta_{t} \mapsto \lambda_{t}$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_{0,0}, \mathrm{M}_{2,0}, \mathrm{M}_{0,2}, \mathrm{M}_{2,2}$	$\mathrm{M}_{0,0}, \mathrm{M}_{2,2}$

Example.

M_z for z being a root of the Chebyshev polynomial is a representation because then $\sum_k d_e^k \theta_{\bar{s}_k} = 0 = \sum_k d_e^k \theta_{\bar{t}_k}$.

Two-dimensio ... $\begin{pmatrix} 0 & 0 \\ z & 2 \end{pmatrix}$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm} - \{0\}$	$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm}$

$\mathrm{V}_e = \mathrm{roots}(\mathsf{U}_{e+1}(\mathtt{X}))$ and $\mathrm{V}_e^{\pm}$ the $\mathbb{Z}/2\mathbb{Z}$-orbits under $z \mapsto -z$.

Dihe

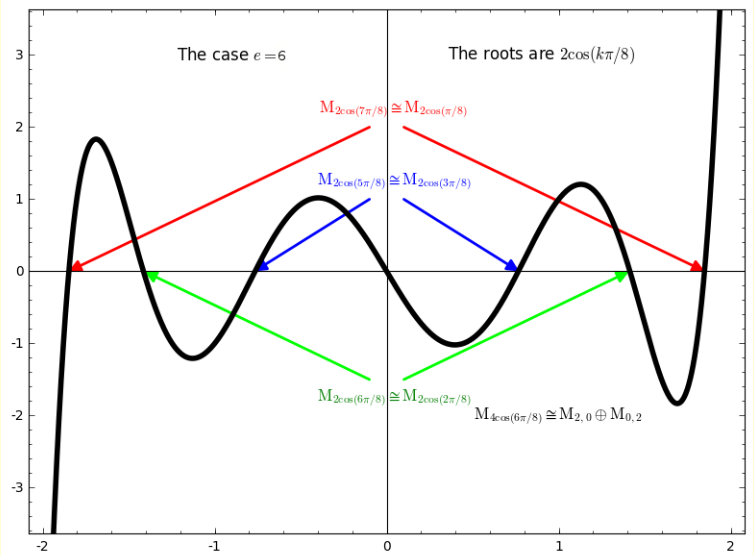
One-d

Two-d

$\mathrm{V}_e =$

Example.

These representations are indexed by $\mathbb{Z}/2\mathbb{Z}$-orbits of the Chebyshev roots:



$\mathbb{N}_0$-algebras and their representations

An algebra P with a basis B^{P} with $1 \in \mathrm{B}^{\mathrm{P}}$ is called a $\mathbb{N}_0$-algebra if

$$\mathrm{xy} \in \mathbb{N}_0\mathrm{B}^{\mathrm{P}} \quad (\mathrm{x}, \mathrm{y} \in \mathrm{B}^{\mathrm{P}}).$$

A P-representation M with a basis B^{M} is called a $\mathbb{N}_0$-representation if

$$\mathrm{xm} \in \mathbb{N}_0\mathrm{B}^{\mathrm{M}} \quad (\mathrm{x} \in \mathrm{B}^{\mathrm{P}}, \mathrm{m} \in \mathrm{B}^{\mathrm{M}}).$$

These are $\mathbb{N}_0$-equivalent if there is a $\mathbb{N}_0$-valued change of basis matrix.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-representations arise naturally as the decategorification of 2-categories and 2-representations, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

$\mathbb{N}_0$-algebras and their representations

Example.

Group algebras of finite groups with basis given by group elements are $\mathbb{N}_0$-algebras.

The regular representation is a $\mathbb{N}_0$-representation.

A P-representation M with a basis B^{M} is called a $\mathbb{N}_0$-representation if

$$\mathrm{xm} \in \mathbb{N}_0\mathrm{B}^{\mathrm{M}} \quad (\mathrm{x} \in \mathrm{B}^{\mathrm{P}}, \mathrm{m} \in \mathrm{B}^{\mathrm{M}}).$$

These are $\mathbb{N}_0$-equivalent if there is a $\mathbb{N}_0$-valued change of basis matrix.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-representations arise naturally as the decategorification of 2-categories and 2-representations, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

$\mathbb{N}_0$-algebras and their representations

Example.

Group algebras of finite groups with basis given by group elements are $\mathbb{N}_0$-algebras.

The regular representation is a $\mathbb{N}_0$-representation.

An

$xy \in \mathbb{N}_0 B$ $(x, y \in B)$.

Example.

The regular representation of group algebras decomposes over $\mathbb{C}$ into simples.

However, this decomposition is almost never an $\mathbb{N}_0$-equivalence.

A P-

These are $\mathbb{N}_0$-equivalent if there is a $\mathbb{N}_0$-valued change of basis matrix.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-representations arise naturally as the decategorification of 2-categories and 2-representations, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

$\mathbb{N}_0$-algebras and their representations

An

$xy \in \mathbb{N}_0 B$ ($x, y \in B$).

Example.

Group algebras of finite groups with basis given by group elements are $\mathbb{N}_0$-algebras.

The regular representation is a $\mathbb{N}_0$-representation.

A P-

Example.

The regular representation of group algebras decomposes over $\mathbb{C}$ into simples.

However, this decomposition is almost never an $\mathbb{N}_0$-equivalence.

These are $\mathbb{N}_0$-equivalent if there is a $\mathbb{N}_0$-valued change of basis matrix.

Exa

deca ... omes

fron

Example.

Hecke algebras of (finite) Coxeter groups with their KL basis are $\mathbb{N}_0$-algebras.

For the symmetric group a ▸ miracle happens: all simples are $\mathbb{N}_0$-representations.

Cells of $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-representations

Kazhdan–Lusztig ∼1979. $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ if x appears in zy with non-zero coefficient for some $\mathrm{z} \in \mathrm{B}^{\mathrm{P}}$. $\mathrm{x} \sim_{\mathsf{L}} \mathrm{y}$ if $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ and $\mathrm{y} \leq_{\mathsf{L}} \mathrm{x}$.
$\sim_{\mathsf{L}}$ partitions P into left cells L. Similarly for right R, two-sided cells J or $\mathbb{N}_0$-representations.

A $\mathbb{N}_0$-representation M is transitive if all basis elements belong to the same $\sim_{\mathsf{L}}$ equivalence class. An apex of M is a maximal two-sided cell not killing it.

Fact. Each transitive $\mathbb{N}_0$-representation has a unique apex.

Example. Transitive $\mathbb{N}_0$-representations arise naturally as the decategorification of simple 2-representations.

Example.

Group algebras with the group element basis have only one cell, G itself.

Transitive $\mathbb{N}_0$-representations are $\mathbb{C}[G/H]$ for H being a subgroup. The apex is G.

C

K ... for

so ...

$\sim_{\mathrm{L}}$ partitions P into left cells L. Similarly for right R, two-sided cells J or $\mathbb{N}_0$-representations.

A $\mathbb{N}_0$-representation M is transitive if all basis elements belong to the same $\sim_{\mathrm{L}}$ equivalence class. An apex of M is a maximal two-sided cell not killing it.

Fact. Each transitive $\mathbb{N}_0$-representation has a unique apex.

Example. Transitive $\mathbb{N}_0$-representations arise naturally as the decategorification of simple 2-representations.

Example.

Group algebras with the group element basis have only one cell, G itself.

Transitive $\mathbb{N}_0$-representations are $\mathbb{C}[G/H]$ for H being a subgroup. The apex is G.

Example (Kazhdan–Lusztig $\sim$1979).

Hecke algebras for the symmetric group with KL basis have cells coming from the Robinson–Schensted correspondence.

The transitive $\mathbb{N}_0$-representations are the simples with apex given by elements for the same shape of Young tableaux.

Ka… for so… $\sim_L$ partiti… $\mathbb{N}_0$-repres…

A $\mathbb{N}_0$-rep… me $\sim_L$ equivalen… t.

Example. Transitive $\mathbb{N}_0$-representations arise naturally as the decategorification of simple 2-representations.

Example.

Group algebras with the group element basis have only one cell, G itself.

Transitive $\mathbb{N}_0$-representations are $\mathbb{C}[G/H]$ for H being a subgroup. The apex is G.

Example (Kazhdan–Lusztig $\sim$1979).

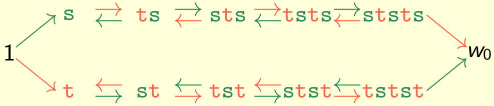
Hecke algebras for the symmetric group with KL basis have ▸ cells coming from the Robinson–Schensted correspondence.

The transitive $\mathbb{N}_0$-representations are the simples with apex given by elements for the same shape of Young tableaux.

Example (Lusztig $\leq$2003).

Left cells

Hecke algebras for the dihedral group with KL basis have the following cells:



We will see the transitive $\mathbb{N}_0$-representations in a second.

C

Example.

Group algebras with the group element basis have only one cell, G itself.

Transitive $\mathbb{N}_0$-representations are $\mathbb{C}[G/H]$ for H being a subgroup. The apex is G.

Ka ... for

so ...

$\sim_L$ partiti...

$\mathbb{N}_0$-repres...

Example (Kazhdan–Lusztig $\sim$1979).

Hecke algebras for the symmetric group with KL basis have cells coming from the Robinson–Schensted correspondence.

The transitive $\mathbb{N}_0$-representations are the simples with apex given by elements for the same shape of Young tableaux.

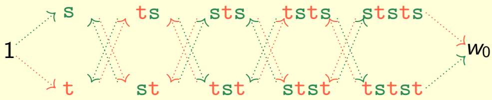
A $\mathbb{N}_0$-rep ... me $\sim_L$

equivalen ... t.

Example (Lusztig $\leq$2003).

Right cells

Hecke algebras for the dihedral group with KL basis have the following cells:



We will see the transitive $\mathbb{N}_0$-representations in a second.

Exan ... ion of

simpl...

Example.

Group algebras with the group element basis have only one cell, G itself.

Transitive $\mathbb{N}_0$-representations are $\mathbb{C}[G/H]$ for H being a subgroup. The apex is G.

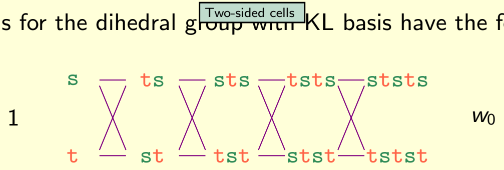
Example (Kazhdan–Lusztig ∼1979).

Hecke algebras for the symmetric group with KL basis have cells coming from the Robinson–Schensted correspondence.

The transitive $\mathbb{N}_0$-representations are the simples with apex given by elements for the same shape of Young tableaux.

Example (Lusztig ≤2003).

Hecke algebras for the dihedral group with KL basis have the following cells:

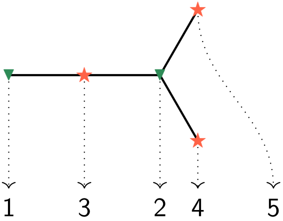


We will see the transitive $\mathbb{N}_0$-representations in a second.

$\mathbb{N}_0$-representations via graphs

Construct a W_∞-representation M associated to a bipartite graph $\mathbf{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1,2,3,4,5\rangle$$

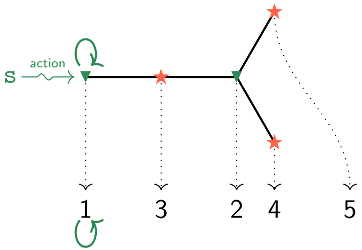


$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}_0$-representations via graphs

Construct a W_∞-representation M associated to a bipartite graph $\boldsymbol{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5\rangle$$

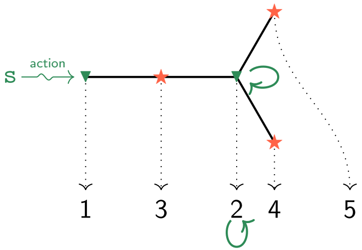


$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}_0$-representations via graphs

Construct a W_∞-representation M associated to a bipartite graph $\boldsymbol{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5 \rangle$$

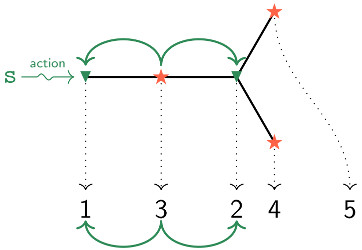


$$\theta_{\mathrm{s}} \rightsquigarrow \mathrm{M}_{\mathrm{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathrm{t}} \rightsquigarrow \mathrm{M}_{\mathrm{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}_0$-representations via graphs

Construct a W_∞-representation M associated to a bipartite graph $\boldsymbol{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5 \rangle$$

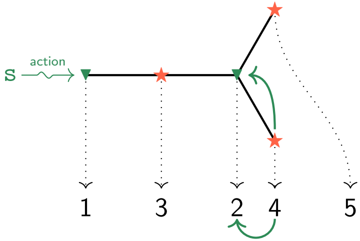


$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}_0$-representations via graphs

Construct a W_∞-representation M associated to a bipartite graph $\mathbf{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5\rangle$$

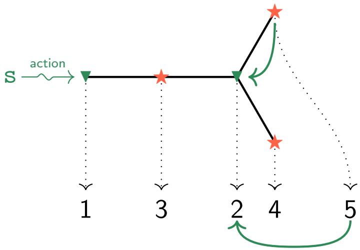


$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}_0$-representations via graphs

Construct a W_∞-representation M associated to a bipartite graph $\boldsymbol{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5\rangle$$

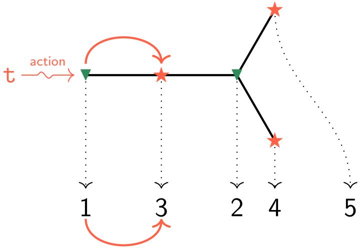


$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}_0$-representations via graphs

Construct a W_∞-representation M associated to a bipartite graph $\boldsymbol{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5 \rangle$$

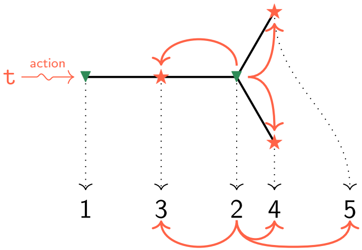


$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}_0$-representations via graphs

Construct a W_∞-representation M associated to a bipartite graph $\mathbf{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5\rangle$$

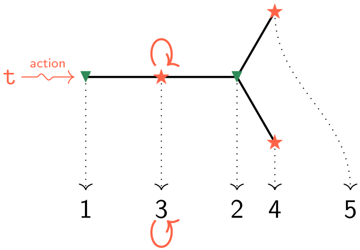


$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}_0$-representations via graphs

Construct a W_∞-representation M associated to a bipartite graph **Γ**:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5\rangle$$

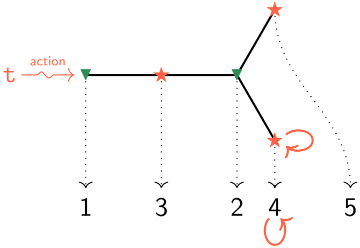


$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}_0$-representations via graphs

Construct a W_∞-representation M associated to a bipartite graph $\mathbf{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5\rangle$$

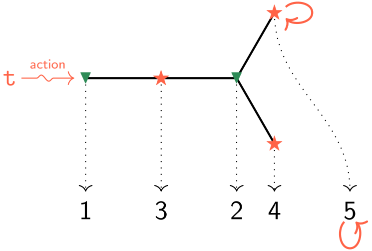


$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}_0$-representations via graphs

Construct a W_∞-representation M associated to a bipartite graph $\mathbf{\Gamma}$:

$$M = \mathbb{C}\langle 1, 2, 3, 4, 5\rangle$$



$$\theta_{s} \rightsquigarrow M_{s} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{t} \rightsquigarrow M_{t} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}_0$-representations via graphs

Construc

The adjacency matrix $A(\mathbf{\Gamma})$ of $\mathbf{\Gamma}$ is

$$A(\mathbf{\Gamma}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

These are W_{e+2}-representations for some e
only if $A(\mathbf{\Gamma})$ is killed by the Chebyshev polynomial $\mathrm{U}_{e+1}(\mathtt{X})$.

Morally speaking: These are constructed as the simples
but with integral matrices having the Chebyshev-roots as eigenvalues.

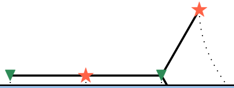
It is not hard to see that the Chebyshev–braid-like relation can not hold otherwise.

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}_0$-representations via graphs

Construct a W_∞-representation M associated to a bipartite graph $\boldsymbol{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5\rangle$$



Hence, by Smith's (CP) and Lusztig: We get a representation of W_{e+2} if $\boldsymbol{\Gamma}$ is a ADE Dynkin diagram for $e+2$ being the Coxeter number.

That these are $\mathbb{N}_0$-representations follows from categorification.

'Smaller solutions' are never $\mathbb{N}_0$-representations.

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}_0$-representations via graphs

Construct a W_∞-representation M associated to a bipartite graph $\boldsymbol{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5 \rangle$$

Classification.

Complete, irredundant list of transitive $\mathbb{N}_0$-representations of W_{e+2}:

Apex	(1) cell	(s) − (t) cell	(w_0) cell
$\mathbb{N}_0$-reps.	$\mathrm{M}_{0,0}$	$\mathrm{M}_{\text{ADE+bicolering}}$ for $e+2 =$ Cox. num.	$\mathrm{M}_{2,2}$

1 3 2 4 5

$$\theta_{s} \rightsquigarrow \mathrm{M}_{s} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{t} \rightsquigarrow \mathrm{M}_{t} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

"Lifting" $\mathbb{N}_0$-representation theory

An additive, $\mathbb{K}$-linear, idempotent complete, Krull–Schmidt 2-category $\mathscr{C}$ is called finitary if some finiteness conditions hold.

A simple transitive 2-representation (2-simple) of $\mathscr{C}$ is an additive, $\mathbb{K}$-linear 2-functor

$$\mathscr{M}\colon \mathscr{C} \to \mathscr{A}^{\mathrm{f}} (= \text{2-cat of finitary cats}),$$

such that there are no non-zero proper $\mathscr{C}$-stable ideals.

There is also the notion of 2-equivalence.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-representations arise naturally as the decategorification of 2-categories and 2-representations, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

"Lifting" $\mathbb{N}_0$-representation theory

Mazorchuk–Miemietz ∼2014.

2-Simples $\leftrightsquigarrow$ simples (e.g. 2-Jordan–Hölder theorem),

but their decategorifications are transitive $\mathbb{N}_0$-representations and usually not simple.

A ... ed

fi...

A ...

2-functor

$$\mathscr{M}\colon \mathscr{C} \to \mathscr{A}^{\mathrm{f}} (= \text{2-cat of finitary cats}),$$

such that there are no non-zero proper $\mathscr{C}$-stable ideals.

There is also the notion of 2-equivalence.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-representations arise naturally as the decategorification of 2-categories and 2-representations, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

"Lifting" $\mathbb{N}_0$-representation theory

Mazorchuk–Miemietz ∼2014.

2-Simples ⇝ simples (e.g. 2-Jordan–Hölder theorem),

but their decategorifications are transitive $\mathbb{N}_0$-representations and usually not simple.

A ... ed
fi ...

A ...
2-functor

Mazorchuk–Miemietz ∼2011.

suc ...

Define cell theory similarly as for $\mathbb{N}_0$-algebras and -representations.

The ...

2-simple ⇒ transitive, and transitive 2-representations have a 2-simple quotient.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-representations arise naturally as the decategorification of 2-categories and 2-representations, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

"Lifting" $\mathbb{N}_0$-representation theory

Mazorchuk–Miemietz ∼2014.

2-Simples ⇝ simples (e.g. 2-Jordan–Hölder theorem),

but their decategorifications are transitive $\mathbb{N}_0$-representations and usually not simple.

A ... ed fi...

A ...

2-functor

Mazorchuk–Miemietz ∼2011.

Define cell theory similarly as for $\mathbb{N}_0$-algebras and -representations.

2-simple ⇒ transitive, and transitive 2-representations have a 2-simple quotient.

suc...

The...

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-representations arise naturally as the decategorific... alence comes from 2-equiv...

Chan–Mazorchuk ∼2016.

Every 2-simple has an associated apex not killing it.

Thus, we can again study them separately for different cells.

"Lifting" $\mathbb{N}_0$-representation theory

An [...] lled
fini[...]

Example.

$\mathrm{B}\text{-}\mathcal{M}\mathrm{od}$ (+fc=some finiteness condition) is a prototypical object of $\mathscr{A}^{\mathrm{f}}$.

A 2-representation for us is very often on the category of quiver representations.

A s[...]
2-functor

$$\mathscr{M}\colon \mathscr{C} \to \mathscr{A}^{\mathrm{f}} (= \text{2-cat of finitary cats}),$$

such that there are no non-zero proper $\mathscr{C}$-stable ideals.

There is also the notion of 2-equivalence.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-representations arise naturally as the decategorification of 2-categories and 2-representations, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

"Lifting" $\mathbb{N}_0$-representation theory

An ... lled
fini...

Example.

$\mathrm{B}\text{-}\mathcal{M}\mathrm{od}$ (+fc=some finiteness condition) is a prototypical object of $\mathscr{A}^{\mathrm{f}}$.

A 2-representation for us is very often on the category of quiver representations.

A s...
2-functor

$$\mathscr{M}\colon \mathscr{C} \to \mathscr{A}^{\mathrm{f}} (= \text{2-cat of finitary cats}),$$

such that there are no non-zero proper $\mathscr{C}$-stable ideals.

There is also the notion of 2-equivalence.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-representations arise naturally as the
c... es
f...

Example (Mazorchuk–Miemietz & Chuang–Rouquier & Khovanov–Lauda & ...).

2-Kac–Moody algebras (+fc) are finitary 2-categories.

Their 2-simples are categorifications of the simples.

"Lifting" $\mathbb{N}_0$-representation theory

Example (Mazorchuk–Miemietz & Soergel & Khovanov–Mazorchuk–Stroppel & ...).

Soergel bimodules for finite Coxeter groups are finitary 2-categories.
(Coxeter=Weyl: 'Indecomposable projective functors on $\mathcal{O}_0$.')

Symmetric group: the 2-simples are categorifications of the simples.

2-functor

$$\mathscr{M}\colon \mathscr{C} \to \mathscr{A}^{\mathrm{f}} (= \text{2-cat of finitary cats}),$$

such that there are no non-zero proper $\mathscr{C}$-stable ideals.

There is also the notion of 2-equivalence.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-representations arise naturally as the decategorification of 2-categories and 2-representations, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

“Lifting” $\mathbb{N}_0$-representation theory

Example (Mazorchuk–Miemietz & Soergel & Khovanov–Mazorchuk–Stroppel & ...).

Soergel bimodules for finite Coxeter groups are finitary 2-categories.
(Coxeter=Weyl: ‘Indecomposable projective functors on $\mathcal{O}_0$.’)

Symmetric group: the 2-simples are categorifications of the simples.

2-functor

su

T

Example (Kildetoft–Mackaay–Mazorchuk–Miemietz–Zhang & ...).

Quotients of Soergel bimodules (+fc), e.g. small quotients, are finitary 2-categories.

Except for the small quotients+ϵ the classification is widely open.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-representations arise naturally as the decategorification of 2-categories and 2-representations, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

"Lifting" $\mathbb{N}_0$-representation theory

Example (Mazorchuk–Miemietz & Soergel & Khovanov–Mazorchuk–Stroppel & ...).

Soergel bimodules for finite Coxeter groups are finitary 2-categories.
(Coxeter=Weyl: 'Indecomposable projective functors on $\mathcal{O}_0$.')

Symmetric group: the 2-simples are categorifications of the simples.

2-functor

su

T

Example (Kildetoft–Mackaay–Mazorchuk–Miemietz–Zhang & ...).

Quotients of Soergel bimodules (+fc), e.g. small quotients, are finitary 2-categories.

Except for the small quotients+ϵ the classification is widely open.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-representations arise naturally as the

Example (Mackaay–Mazorchuk–Miemietz & Kirillov–Ostrik & Elias & ...).

Singular Soergel bimodules and various 2-subcategories (+fc) are finitary 2-categories.
(Coxeter=Weyl: 'Indecomposable projective functors between singular blocks of $\mathcal{O}$.')

For a quotient of maximal singular type $\tilde{\mathrm{A}}_1$ non-trivial 2-simples are ADE classified.

Excuse me?

"Lifting" $\mathbb{N}_0$-representation theory

An additive, $\mathbb{K}$-linear, idempotent complete, Krull–Schmidt 2-category $\mathscr{C}$ is called finitary if some finiteness conditions hold.

A simple transitiv ve, $\mathbb{K}$-linear 2-functor

$\mathscr{M}: \mathscr{C} \to \mathscr{A}^{\mathrm{f}}$ (= 2-cat of finitary cats)

such that there

There is also th

Example. $\mathbb{N}_0$-a ne decategorification of 2-categories and 2-representations, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

Question ("2-representation theory").

Classify all 2-simples of a fixed finitary 2-category.

This is the categorification of

'Classify all simples a fixed finite-dimensional algebra',

but much harder, e.g. it is unknown whether there are always only finitely many 2-simples.

2-representations of dihedral Soergel bimodules

Theorem (Soergel ∼1992 & Williamson ∼2010 & Elias ∼2013 & ...).
Dihedral singular Soergel bimodules $\boldsymbol{s}\mathscr{W}_{e+2}$ categorify the dihedral algebroid with indecomposables categorifying the KL basis.

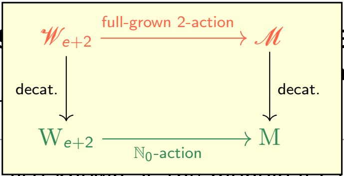
The regular part $\mathscr{W}_{e+2}$ is also known as the monoidal category of dihedral Soergel bimodules.

There is also the maximally singular part $\boldsymbol{m}\mathscr{W}_{e+2}$, which actually is semisimple.

Note that $\boldsymbol{s}\mathscr{W}_{e+2}$ has a ▸ diagrammatic incarnation.

2-representations of dihedral Soergel bimodules

Theorem (Soergel ∼199



Elias ∼2013 & ...).
Dihedral singular Soergel ... he dihedral algebroid with indecomposables categori...

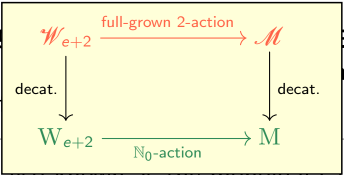
The regular part $\mathscr{W}_{e+2}$ is also known as the monoidal category of dihedral Soergel bimodules.

There is also the maximally singular part $\boldsymbol{m}\mathscr{W}_{e+2}$, which actually is semisimple.

Note that $\boldsymbol{s}\mathscr{W}_{e+2}$ has a diagrammatic incarnation.

2-representations of dihedral Soergel bimodules

Theorem (Soergel ∼199 ... **Elias ∼2013 & ...).**
Dihedral singular Soergel ... he dihedral algebroid with indecomposables categori...



The regular part $\mathscr{W}_{e+2}$ is also known as the monoidal category of dihedral Soergel bimodules.

Classification (Kildetoft–Mackaay–Mazorchuk–Miemietz–Zimmermann ∼ 2016).

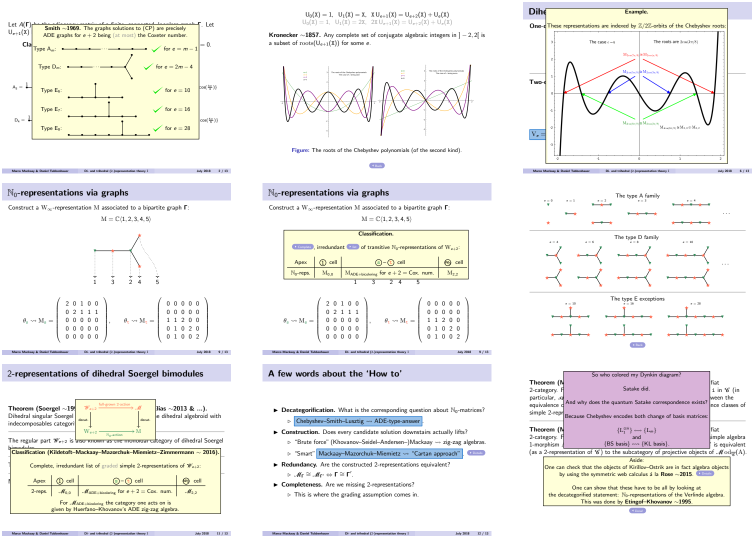
Complete, irredundant list of graded simple 2-representations of $\mathscr{W}_{e+2}$:

Apex	① cell	ⓢ – ⓣ cell	w_0 cell
2-reps.	$\mathscr{M}_{0,0}$	$\mathscr{M}_{\text{ADE+bicolering}}$ for $e+2 =$ Cox. num.	$\mathscr{M}_{2,2}$

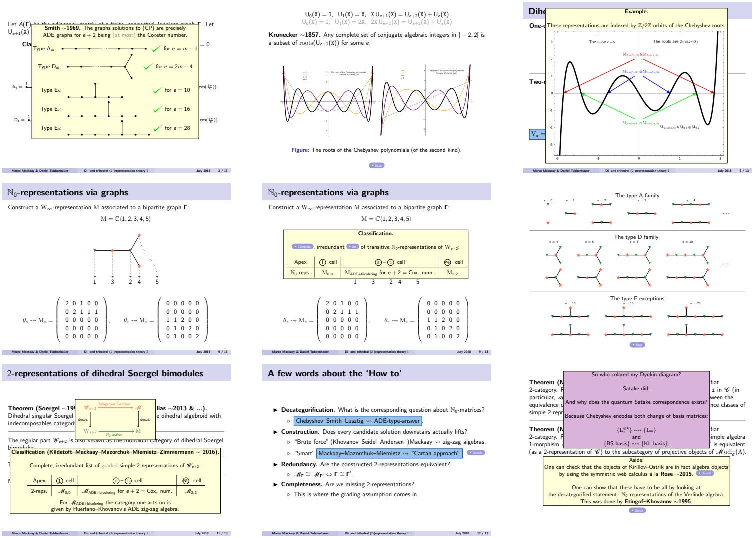
For $\mathscr{M}_{\text{ADE+bicolering}}$ the category one acts on is given by Huerfano–Khovanov's ADE zig-zag algebra.

A few words about the ‘How to’

- **Decategorification.** What is the corresponding question about $\mathbb{N}_0$-matrices?
  - ▷ Chebyshev–Smith–Lusztig ⇝ ADE-type-answer.
- **Construction.** Does every candidate solution downstairs actually lifts?
  - ▷ “Brute force” (Khovanov–Seidel–Andersen–)Mackaay ⇝ zig-zag algebras.
  - ▷ “Smart” Mackaay–Mazorchuk–Miemietz ⇝ “Cartan approach”. ▸ Details
- **Redundancy.** Are the constructed 2-representations equivalent?
  - ▷ $\mathscr{M}_{\mathbf{\Gamma}} \cong \mathscr{M}_{\mathbf{\Gamma}'} \Leftrightarrow \mathbf{\Gamma} \cong \mathbf{\Gamma}'$.
- **Completeness.** Are we missing 2-representations?
  - ▷ This is where the grading assumption comes in.



There is still much to do...

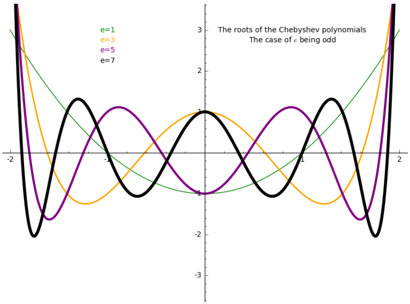


Thanks for your attention!

$$\mathtt{U}_0(\mathtt{X}) = 1, \quad \mathtt{U}_1(\mathtt{X}) = \mathtt{X}, \quad \mathtt{X}\,\mathtt{U}_{e+1}(\mathtt{X}) = \mathtt{U}_{e+2}(\mathtt{X}) + \mathtt{U}_e(\mathtt{X})$$

$$\mathtt{U}_0(\mathtt{X}) = 1, \quad \mathtt{U}_1(\mathtt{X}) = 2\mathtt{X}, \quad 2\mathtt{X}\,\mathtt{U}_{e+1}(\mathtt{X}) = \mathtt{U}_{e+2}(\mathtt{X}) + \mathtt{U}_e(\mathtt{X})$$

Kronecker ∼1857. Any complete set of conjugate algebraic integers in $]-2,2[$ is a subset of $\mathrm{roots}(\mathtt{U}_{e+1}(\mathtt{X}))$ for some e.



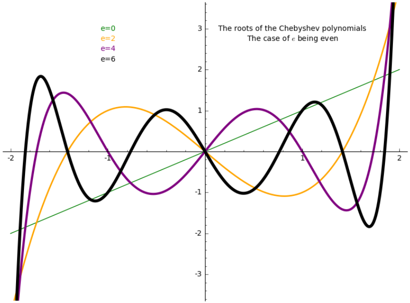


Figure: The roots of the Chebyshev polynomials (of the second kind).

Back

The KL basis elements for $\mathrm{S}_3 \cong \mathrm{W}_3$ with $\mathtt{sts} = w_0 = \mathtt{tst}$ are:

$$\theta_1 = 1, \quad \theta_{\mathtt{s}} = \mathtt{s} + 1, \quad \theta_{\mathtt{t}} = \mathtt{t} + 1, \quad \theta_{\mathtt{ts}} = \mathtt{ts} + \mathtt{s} + \mathtt{t} + 1,$$
$$\theta_{\mathtt{st}} = \mathtt{st} + \mathtt{s} + \mathtt{t} + 1, \quad \theta_{w_0} = w_0 + \mathtt{ts} + \mathtt{st} + \mathtt{s} + \mathtt{t} + 1.$$

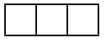
	1	s	t	ts	st	w_0
	1	1	1	1	1	1
	2	0	0	−1	−1	0
	1	−1	−1	1	1	−1

Figure: The character table of $\mathrm{S}_3 \cong \mathrm{W}_3$.

The KL basis elements for $\mathrm{S}_3 \cong \mathrm{W}_3$ with $\mathtt{sts} = w_0 = \mathtt{tst}$ are:

$$\theta_1 = 1, \quad \theta_{\mathtt{s}} = \mathtt{s} + 1, \quad \theta_{\mathtt{t}} = \mathtt{t} + 1, \quad \theta_{\mathtt{ts}} = \mathtt{ts} + \mathtt{s} + \mathtt{t} + 1,$$
$$\theta_{\mathtt{st}} = \mathtt{st} + \mathtt{s} + \mathtt{t} + 1, \quad \theta_{w_0} = w_0 + \mathtt{ts} + \mathtt{st} + \mathtt{s} + \mathtt{t} + 1.$$

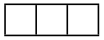
	θ_1	$\theta_{\mathtt{s}}$	$\theta_{\mathtt{t}}$	$\theta_{\mathtt{ts}}$	$\theta_{\mathtt{st}}$	θ_{w_0}
	1	2	2	4	4	6
	2	2	2	1	1	0
	1	0	0	0	0	0

Figure: The character table of $\mathrm{S}_3 \cong \mathrm{W}_3$.

Back

The KL basis elements for $\mathrm{S}_3 \cong \mathrm{W}_3$ with $\mathtt{sts} = w_0 = \mathtt{tst}$ are:

$$\theta_1 = 1, \quad \theta_{\mathtt{s}} = \mathtt{s} + 1, \quad \theta_{\mathtt{t}} = \mathtt{t} + 1, \quad \theta_{\mathtt{ts}} = \mathtt{ts} + \mathtt{s} + \mathtt{t} + 1,$$
$$\theta_{\mathtt{st}} = \mathtt{st} + \mathtt{s} + \mathtt{t} + 1, \quad \theta_{w_0} = w_0 + \mathtt{ts} + \mathtt{st} + \mathtt{s} + \mathtt{t} + 1.$$

						θ_{w_0}
						6
						0
	1	0	0	0	0	0

Remark.

This non-negativity of the KL basis
is true for all symmetric groups,
but not for most other groups (as we will see).

The case $e = 1$ is the last case
where the Chebyshev polynomial has only integer roots.

Figure: The character table of $\mathrm{S}_3 \cong \mathrm{W}_3$.

The KL basis elements for $\mathrm{S}_3 \cong \mathrm{W}_3$ with $\mathtt{sts} = w_0 = \mathtt{tst}$ are:

$$\theta_1 = 1, \quad \theta_{\mathtt{s}} = \mathtt{s} + 1, \quad \theta_{\mathtt{t}} = \mathtt{t} + 1, \quad \theta_{\mathtt{ts}} = \mathtt{ts} + \mathtt{s} + \mathtt{t} + 1,$$

The first ever published character table ($\sim$1896) by Frobenius.
Note the root of unity ρ.

[1011] Frobenius: Über Gruppencharaktere. 27

samen Factor f abgesehen) einen relativen Charakter von $\mathfrak{H}$, und umgekehrt lässt sich jeder relative Charakter von $\mathfrak{H}$, $\chi_0, \cdots \chi_{k-1}$, auf eine oder mehrere Arten durch Hinzufügung passender Werthe $\chi_k, \cdots \chi_{k'-1}$ zu einem Charakter von $\mathfrak{H}'$ ergänzen.

§ 8.

Ich will nun die Theorie der Gruppencharaktere an einigen Beispielen erläutern. Die geraden Permutationen von 4 Symbolen bilden eine Gruppe $\mathfrak{H}$ der Ordnung $h=12$. Ihre Elemente zerfallen in 4 Classen, die Elemente der Ordnung 2 bilden eine zweiseitige Classe (1), die der Ordnung 3 zwei inverse Classen (2) und (3) = (2′). Sei ρ eine primitive cubische Wurzel der Einheit.

Tetraeder. $h=12$.

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	h_α
χ_0	1	3	1	1	1
χ_1	1	-1	1	1	3
χ_2	1	0	ρ	ρ^2	4
χ_3	1	0	ρ^2	ρ	4

(Robinson ∼1938 &)Schensted ∼1961 & Kazhdan–Lusztig ∼1979.

Elements of $\mathrm{S}_n \xleftrightarrow{1:1} (P, Q)$ standard Young tableaux of the same shape. Left, right and two-sided cells of S_n:

- $s \sim_{\mathrm{L}} t$ if and only if $Q(s) = Q(t)$.
- $s \sim_{\mathrm{R}} t$ if and only if $P(s) = P(t)$.
- $s \sim_{\mathrm{J}} t$ if and only if $P(s)$ and $P(t)$ have the same shape.

Example ($n = 3$).

$$1 \leftrightsquigarrow \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$$

$$\mathtt{s} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \qquad \mathtt{ts} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$$

$$\mathtt{t} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \qquad \mathtt{st} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

$$w_0 \leftrightsquigarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

Back

(Robinson ∼1938 &)Schensted ∼1961 & Kazhdan–Lusztig ∼1979.

Elements of $\mathrm{S}_n \xleftrightarrow{1:1} (P, Q)$ standard Young tableaux of the same shape. Left, right and two-sided cells of S_n:

- $s \sim_{\mathrm{L}} t$ if and only if $Q(s) = Q(t)$.
- $s \sim_{\mathrm{R}} t$ if and only if $P(s) = P(t)$.
- $s \sim_{\mathrm{J}} t$ if and only if $P(s)$ and $P(t)$ have the same shape.

Example ($n = 3$).

Left cells

$$1 \leftrightsquigarrow \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$$

$$\mathtt{s} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \qquad \mathtt{ts} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$$

$$\mathtt{t} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \qquad \mathtt{st} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

$$w_0 \leftrightsquigarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

(Robinson $\sim$1938 &)Schensted $\sim$1961 & Kazhdan–Lusztig $\sim$1979.

Elements of $\mathrm{S}_n \xleftrightarrow{1:1} (P, Q)$ standard Young tableaux of the same shape. Left, right and two-sided cells of S_n:

- $s \sim_{\mathrm{L}} t$ if and only if $Q(s) = Q(t)$.
- $s \sim_{\mathrm{R}} t$ if and only if $P(s) = P(t)$.
- $s \sim_{\mathrm{J}} t$ if and only if $P(s)$ and $P(t)$ have the same shape.

Example ($n = 3$).

Right cells

$1 \leftrightsquigarrow \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$

$\mathtt{s} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ $\qquad \mathtt{ts} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$

$\mathtt{t} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ $\qquad \mathtt{st} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$

$w_0 \leftrightsquigarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$

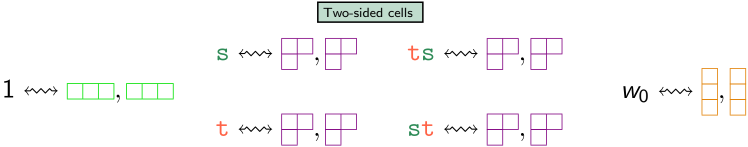
Back

(Robinson ∼1938 &)Schensted ∼1961 & Kazhdan–Lusztig ∼1979.

Elements of $\mathrm{S}_n \xleftrightarrow{1:1} (P, Q)$ standard Young tableaux of the same shape. Left, right and two-sided cells of S_n:

- $s \sim_{\mathrm{L}} t$ if and only if $Q(s) = Q(t)$.
- $s \sim_{\mathrm{R}} t$ if and only if $P(s) = P(t)$.
- $s \sim_{\mathrm{J}} t$ if and only if $P(s)$ and $P(t)$ have the same shape.

Example ($n = 3$).



Back

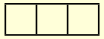
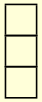
(Robinson ∼1938 &)Schensted ∼1961 & Kazhdan–Lusztig ∼1979.

Elements of $\mathrm{S}_n \overset{1:1}{\longleftrightarrow} (P, Q)$ standard Young tableaux of the same shape. Left, right and two-

- $s \sim$
- $s \sim$
- $s \sim$

Exampl

Apexes:

	θ_1	θ_{s}	θ_{t}	θ_{ts}	θ_{st}	θ_{w_0}
	1	2	2	4	4	6
	2	2	2	1	1	0
	1	0	0	0	0	0

The $\mathbb{N}_0$-representations are the simples.

Back

In case you are wondering why this is supposed to be true, here is the main observation of **Smith ∼1969**:

$$\mathsf{U}_{e+1}(\mathtt{X},\mathtt{Y}) = \pm\det(\mathtt{X}\mathrm{Id} - A(\mathsf{A}_{e+1}))$$

Chebyshev poly. = char. poly. of the type A_{e+1} graph

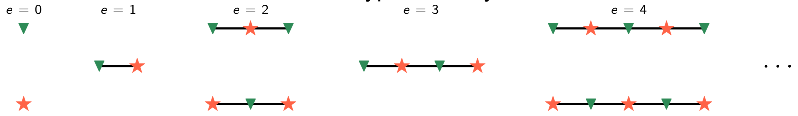
and

$$\mathtt{X}\mathsf{T}_{n-1}(\mathtt{X}) = \pm\det(\mathtt{X}\mathrm{Id} - A(\mathsf{D}_n)) \pm (-1)^{n \bmod 4}$$

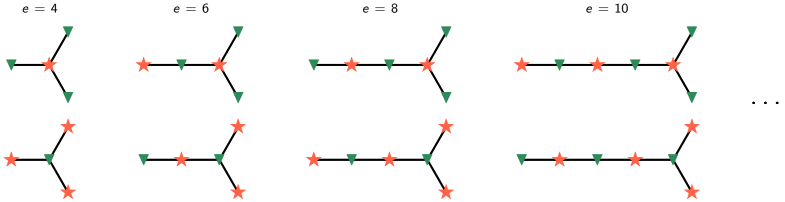
first kind Chebyshev poly. '=' char. poly. of the type D_n graph ($n = \frac{e+4}{2}$).

Back

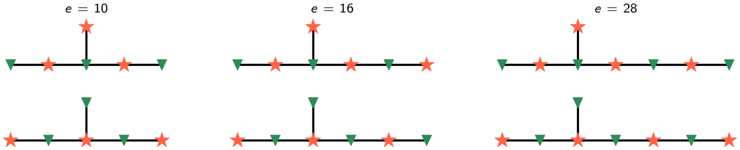
The type A family



The type D family

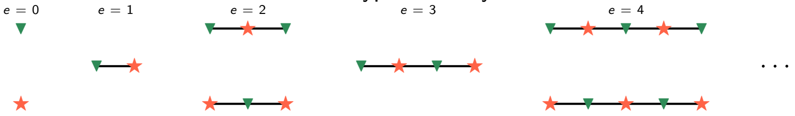


The type E exceptions

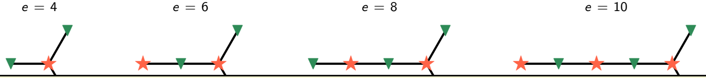


Back

The type A family

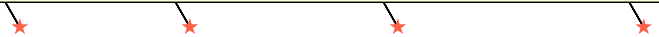


The type D family

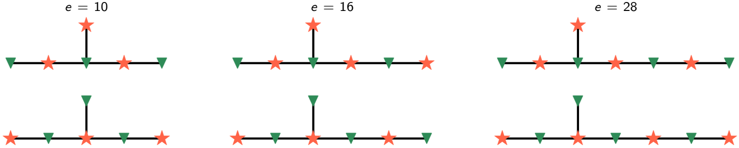


Note: Almost none of these are simple since they grow in rank with growing e.

This is the opposite from the symmetric group case.



The type E exceptions

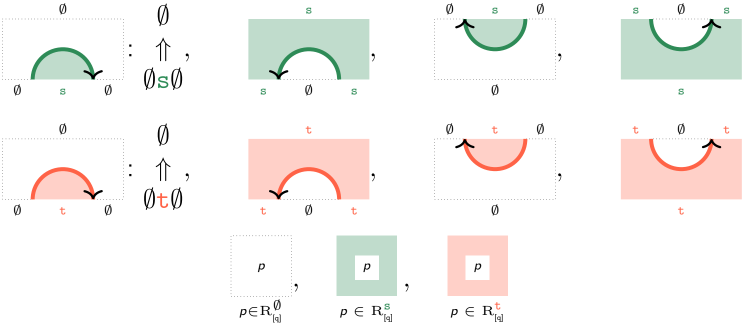


Back

Objects. Parabolic subsets $\emptyset, \mathtt{s}, \mathtt{t}$.

1-**morphism generators.** Color changes $\emptyset\mathtt{s}$ or $\mathtt{s}\emptyset$ or $\emptyset\mathtt{t}$ or $\mathtt{t}\emptyset$.

2-**morphism generators.** Diagrams and polynomials.



Relations. Some relations coming from Frobenius extensions.

Back

Theorem (Mackaay–Mazorchuk–Miemietz ∼2016). Let $\mathscr{C}$ be a fiat 2-category. For $\mathtt{i} \in \mathscr{C}$, consider the endomorphism 2-category $\mathscr{A}$ of $\mathtt{i}$ in $\mathscr{C}$ (in particular, $\mathscr{A}(\mathtt{i},\mathtt{i}) = \mathscr{C}(\mathtt{i},\mathtt{i})$). Then there is a natural bijection between the equivalence classes of simple 2-representations of $\mathscr{A}$ and the equivalence classes of simple 2-representations of $\mathscr{C}$ having a non-trivial value at $\mathtt{i}$.

Theorem (Mackaay–Mazorchuk–Miemietz ∼2016). Let $\mathscr{C}$ be a fiat 2-category. For any simple 2-representation $\mathscr{M}$ of $\mathscr{C}$, there exists a simple algebra 1-morphism A in $\overline{\mathscr{C}}$ (the projective abelianization of $\mathscr{C}$) such that $\mathscr{M}$ is equivalent (as a 2-representation of $\mathscr{C}$) to the subcategory of projective objects of $\mathscr{M}\mathrm{od}_{\overline{\mathscr{C}}}(\mathrm{A})$.

◂ Done!

Theorem (M [partially obscured] fiat
2-category. F [obscured] f $\mathtt{i}$ in $\mathscr{C}$ (in
particular, $\mathscr{A}$ [obscured] tween the
equivalence cl [obscured] ence classes of
simple 2-repre [obscured]

"Cartan approach".

This means for us that it suffices to find
algebra 1-morphisms in the semisimple 2-category $\boldsymbol{m}\mathscr{W}_{e+2}$
which we can then 'induce up' to $\mathscr{W}_{e+2}$.

So it remains to study 2-representations of $\boldsymbol{m}\mathscr{W}_{e+2}$.
But how to do that?

Theorem (Mac [obscured] e a fiat
2-category. For any simple 2-representation $\mathscr{M}$ of $\mathscr{C}$, there exists a simple algebra 1-morphism A in $\overline{\mathscr{C}}$ (the projective abelianization of $\mathscr{C}$) such that $\mathscr{M}$ is equivalent (as a 2-representation of $\mathscr{C}$) to the subcategory of projective objects of $\mathscr{M}\mathrm{od}_{\overline{\mathscr{C}}}(\mathrm{A})$.

Done!

Theorem (M ... fiat
2-category. F ... f $\mathtt{i}$ in $\mathscr{C}$ (in
particular, $\mathscr{A}$ ... tween the
equivalence cl ... ence classes of
simple 2-repre ...

"Cartan approach".

This means for us that it suffices to find algebra 1-morphisms in the semisimple 2-category $\boldsymbol{m}\mathscr{W}_{e+2}$ which we can then 'induce up' to $\mathscr{W}_{e+2}$.

So it remains to study 2-representations of $\boldsymbol{m}\mathscr{W}_{e+2}$.
But how to do that?

Theorem (Mac ... e a fiat
2-category. For any s ... exists a simple algebra
1-morphism A in $\overline{\mathscr{C}}$ (... that $\mathscr{M}$ is equivalent
(as a 2-representation ... objects of $\mathscr{M}\mathrm{od}_{\overline{\mathscr{C}}}(\mathrm{A})$.

Idea: Chebyshev knows everything!

So where have we seen the magic formula
$$\mathtt{X}\,\mathrm{U}_{m+1}(\mathtt{X}) = \mathrm{U}_{m+2}(\mathtt{X}) + \mathrm{U}_{m}(\mathtt{X})$$
before?

Done!

Theorem (M ... fiat
2-category. F... f $\mathtt{i}$ in $\mathscr{C}$ (in
particular, $\mathscr{A}$ ... tween the
equivalence cl... ence classes of
simple 2-repre...

"Cartan approach".

This means for us that it suffices to find
algebra 1-morphisms in the semisimple 2-category $\boldsymbol{m}\mathscr{W}_{e+2}$
which we can then 'induce up' to $\mathscr{W}_{e+2}$.

So it remains to study 2-representations of $\boldsymbol{m}\mathscr{W}_{e+2}$.
But how to do that?

Theorem (Mac... e a fiat
2-category. For any s... exists a simple algebra
1-morphism A in $\overline{\mathscr{C}}$ (... that $\mathscr{M}$ is equivalent
(as a 2-representatio... objects of $\mathscr{M}\mathrm{od}_{\overline{\mathscr{C}}}(\mathrm{A})$.

Idea: Chebyshev knows everything!

So where have we seen the magic formula
$$\mathtt{X}\,\mathrm{U}_{m+1}(\mathtt{X}) = \mathrm{U}_{m+2}(\mathtt{X}) + \mathrm{U}_{m}(\mathtt{X})$$
before?

Here:
$$[2]_{\mathrm{q}} \cdot [m+1]_{\mathrm{q}} = [m+2]_{\mathrm{q}} + [m]_{\mathrm{q}}$$
$$\mathrm{L}_1 \otimes \mathrm{L}_{m+1} \cong \mathrm{L}_{m+2} \oplus \mathrm{L}_m$$
$\mathrm{L}_m = m^{\mathrm{th}}$ symmetric power of the vector representation of (quantum) $\mathfrak{sl}_2$.

Done!

Theorem (Mackaay–Mazorchuk–Miemietz ∼2016). Let $\mathscr{C}$ be a fiat 2-category. For $\mathtt{i} \in \mathscr{C}$, consider the endomorphism 2-category $\mathscr{A}$ of $\mathtt{i}$ in $\mathscr{C}$ (in particular, $\mathscr{A}(\mathtt{i},\mathtt{i}) = \mathscr{C}(\mathtt{i},\mathtt{i})$). Then there is a natural bijection between the equivalence classes of simple 2-representations of $\mathscr{A}$ and the equivalence classes of
si

Th
2-c ... bra
1-r ... ent
(as ... A).

Quantum Satake (Elias ∼2013).

Let $\mathcal{Q}_e$ be the semisimplyfied quotient of the category of (quantum) $\mathfrak{sl}_2$-modules for η being a $2(e+2)^{\mathrm{th}}$ primitive, complex root of unity. There are two degree-zero equivalences, depending on a choice of a starting color,

$$\mathsf{S}_e^{\mathsf{s}} \colon \mathcal{Q}_e \to \boldsymbol{m}\mathscr{W}_{e+2}$$

and

$$\mathsf{S}_e^{\mathsf{t}} \colon \mathcal{Q}_e \to \boldsymbol{m}\mathscr{W}_{e+2}.$$

The point: it suffices to find algebra objects in $\mathcal{Q}_e$.

Done!

Theorem (Mackaay–Mazorchuk–Miemietz ∼2016). Let $\mathscr{C}$ be a fiat 2-category. For $\mathtt{i} \in \mathscr{C}$, consider the endomorphism 2-category $\mathscr{A}$ of $\mathtt{i}$ in $\mathscr{C}$ (in particular, $\mathscr{A}(\mathtt{i},\mathtt{i}) = \mathscr{C}(\mathtt{i},\mathtt{i})$). Then there is a natural bijection between the equivalence classes of simple 2-representations of $\mathscr{A}$ and the equivalence classes of simple 2-representa

Theorem (Kirillov–Ostrik ∼2003).

The algebra objects in $\mathcal{Q}_e$ are ADE classified.

Theorem (Mackaa be a fiat 2-category. For any simple 2-representation $\mathscr{M}$ of $\mathscr{C}$, there exists a simple algebra 1-morphism A in $\overline{\mathscr{C}}$ (the projective abelianization of $\mathscr{C}$) such that $\mathscr{M}$ is equivalent (as a 2-representation of $\mathscr{C}$) to the subcategory of projective objects of $\mathscr{M}\mathrm{od}_{\overline{\mathscr{C}}}(\mathrm{A})$.

Done!

Theorem (M fiat
2-category. F i in $\mathscr{C}$ (in
particular, $\mathscr{A}$ ween the
equivalence c nce classes of
simple 2-repr

Theorem (M fiat
2-category. F simple algebra
1-morphism A is equivalent
(as a 2-representation of $\mathscr{C}$) to the subcategory of projective objects of $\mathscr{M}\mathrm{od}_{\overline{\mathscr{C}}}(\mathrm{A})$.

So who colored my Dynkin diagram?

Satake did.

And why does the quantum Satake correspondence exists?

Because Chebyshev encodes both change of basis matrices:

$$\{\mathrm{L}_1^{\otimes k}\} \leftrightsquigarrow \{\mathrm{L}_m\}$$

and

{BS basis} $\leftrightsquigarrow$ {KL basis}.

Aside:

One can check that the objects of Kirillov–Ostrik are in fact algebra objects by using the symmetric web calculus á la **Rose** $\sim$**2015**. Details

One can show that these have to be all by looking at the decategorified statement: $\mathbb{N}_0$-representations of the Verlinde algebra. This was done by **Etingof–Khovanov** $\sim$**1995**.

Done!

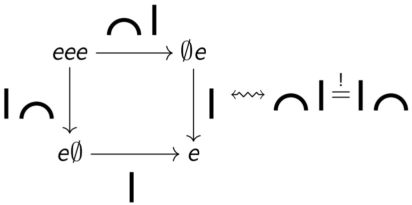
The algebra object in type D:

$$\mathtt{A}^{\mathsf{D}} \cong \mathrm{L}_0 \oplus \mathrm{L}_e \cong \mathbb{C}_{\mathtt{v}} \oplus \mathrm{Sym}^e(\mathrm{L}_1) \rightsquigarrow \emptyset \oplus e.$$

The multiplication $m\colon \mathtt{D}^e \otimes \mathtt{D}^e \to \mathtt{D}^e$ is

	$\emptyset\emptyset$	$\emptyset e$	$e\emptyset$	ee
$\emptyset$	$\emptyset$	0	0	
e	0			0

Check associativity, e.g.:



The algebra object in type D:

$$\mathtt{A}^{\mathrm{D}} \cong \mathrm{L}_0 \oplus \mathrm{L}_e \cong \mathbb{C}_{\mathrm{v}} \oplus \mathrm{Sym}^e(\mathrm{L}_1) \leftrightsquigarrow \emptyset \oplus e.$$

The multiplication $m\colon \mathtt{D}^e \otimes \mathtt{D}^e \to \mathtt{D}^e$ is

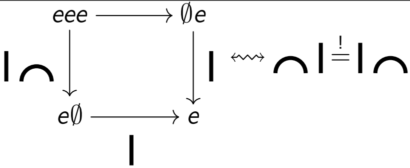
	$\emptyset\emptyset$	$\emptyset e$	$e\emptyset$	ee
$\emptyset$	$\emptyset$	0	0	
e	0			0

And this holds because

$$= \begin{bmatrix} e+1 \\ e \end{bmatrix}_{\mathrm{v}} = [e+1]_{\mathrm{v}} = 1,$$

using that we are working with a $2(e+2)^{\mathrm{th}}$ root of unity.

Check associa



Done!