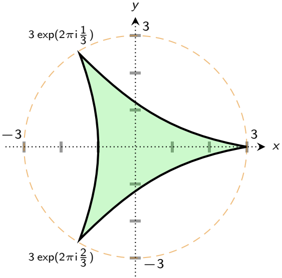
Di- and trihedral (2-)representation theory II

Marco Mackaay & Daniel Tubbenhauer



Joint work with Volodymyr Mazorchuk and Vanessa Miemietz

July 2018

Outline

• **Daniel's talk**: a close relation between Chebyshev polynomials, quantum $\mathfrak{sl}_2$ and certain quotients of $\mathrm{H}(\widehat{A}_1) = H(I_2(\infty))$ (dihedral Hecke algebras)

Outline

- **Daniel's talk**: a close relation between Chebyshev polynomials, quantum $\mathfrak{sl}_2$ and certain quotients of $\mathrm{H}(\widehat{A}_1) = H(I_2(\infty))$ (dihedral Hecke algebras)
- **This talk**: a similar relation between two-variable Chebyshev-like polynomials, quantum $\mathfrak{sl}_3$ and a certain subquotients of $\mathrm{H}(\widehat{A}_2)$ (trihedral Hecke algebras).

Outline

• **Daniel's talk**: a close relation between Chebyshev polynomials, quantum $\mathfrak{sl}_2$ and certain quotients of $\mathrm{H}(\widehat{A}_1) = H(I_2(\infty))$ (dihedral Hecke algebras)

• **This talk**: a similar relation between two-variable Chebyshev-like polynomials, quantum $\mathfrak{sl}_3$ and a certain subquotients of $\mathrm{H}(\widehat{A}_2)$ (trihedral Hecke algebras).

• **Daniel's talk**: the 2-representation theory of certain quotients of Soergel bimodules of type $\widehat{A}_1$ (involving zigzag algebras of ADE Dynkin type).

Outline

- **Daniel's talk**: a close relation between Chebyshev polynomials, quantum $\mathfrak{sl}_2$ and certain quotients of $\mathrm{H}(\widehat{A}_1) = H(I_2(\infty))$ (dihedral Hecke algebras)
- **This talk**: a similar relation between two-variable Chebyshev-like polynomials, quantum $\mathfrak{sl}_3$ and a certain subquotients of $\mathrm{H}(\widehat{A}_2)$ (trihedral Hecke algebras).
- **Daniel's talk**: the 2-representation theory of certain quotients of Soergel bimodules of type $\widehat{A}_1$ (involving zigzag algebras of ADE Dynkin type).
- **This talk**: the 2-representation theory of certain subquotients of Soergel bimodules of type $\widehat{A}_2$ (involving trihedral zigzag algebras of generalized ADE Dynkin type).

Chebyshev-like polynomials

Definition (???, Koornwinder 1974)

The polynomials $\mathrm{U}_{m,n}(x,y)$, $m,n \in \mathbb{N}^0$, are recursively defined by

$$\mathrm{U}_{0,0}(x,y)=1,\ \mathrm{U}_{1,0}(x,y)=x,\ \mathrm{U}_{m,n}(x,y)=\mathrm{U}_{n,m}(y,x),$$
$$x\mathrm{U}_{m,n}(x,y)=\mathrm{U}_{m+1,n}(x,y)+\mathrm{U}_{m-1,n+1}(x,y)+\mathrm{U}_{m,n-1}(x,y),$$
$$y\mathrm{U}_{m,n}(x,y)=\mathrm{U}_{m,n+1}(x,y)+\mathrm{U}_{m+1,n-1}(x,y)+\mathrm{U}_{m-1,n}(x,y).$$

E.g.

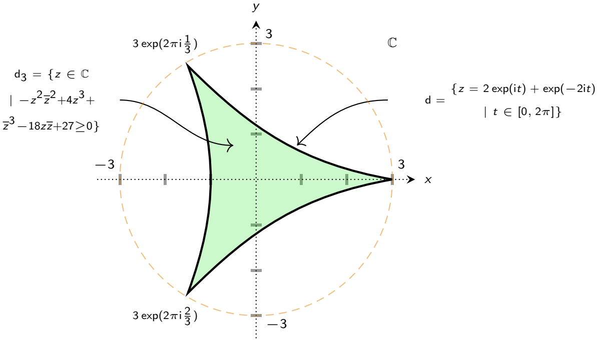
$$\mathrm{U}_{1,1}(x,y)=xy-1,\ \mathrm{U}_{2,1}(x,y)=x^2y-y^2-x,\ \mathrm{U}_{0,2}(x,y)=y^2-x,\ U_{1,0}(x,y)=x,$$

$$\rightsquigarrow$$

$$x\mathrm{U}_{1,1}(x,y)=\mathrm{U}_{2,1}(x,y)+\mathrm{U}_{0,2}(x,y)+\mathrm{U}_{1,0}(x,y)$$

The zeros of the $\mathrm{U}_{m,n}$

The zeros of the $\mathrm{U}_{m,n}$ are all of the form $(z, \overline{z})$ with $z \in \mathrm{d}_3^{\circ}$ (..., Koornwinder 1974, Evans-Pugh 2010, ...).



The disciod $\mathrm{d}_3 = \mathrm{d}_3(\mathfrak{sl}_3)$ bounded by Steiner's hypocycloid d

Note the $\mathbb{Z}/3\mathbb{Z}$-symmetry of d_3: $(z, \overline{z}) \mapsto (e^{\pm 2\pi i/3}z, e^{\mp 2\pi i/3}\overline{z})$.

Relation with quantum $\mathfrak{sl}_3$: generic case

Let $\mathrm{q} \in \mathbb{C}$ be generic.

Theorem

There exists an isomorphism of algebras:

$$\begin{aligned} \left[\mathrm{U}_{\mathrm{q}}(\mathfrak{sl}_3) - \mathrm{mod}\right]_{\mathbb{C}} &\cong \mathbb{C}[x, y] \\ [V_{m,n}] = \sum_{k,l=0}^{m,n} d_{m,n}^{k,l} \left[V_{1,0}^{\otimes k} \otimes V_{0,1}^{\otimes l}\right] &\mapsto \mathrm{U}_{m,n}(x, y) = \sum_{k,l=0}^{m,n} d_{m,n}^{k,l} x^k y^l \end{aligned}$$

for $m, n \in \mathbb{N}^0$.

The integers $d_{m,n}^{k,l}$ can be computed recursively. Note that they can be positive or negative.

Relation quantum $\mathfrak{sl}_3$: root of unity case

Theorem

Suppose $\eta^{2(e+3)} = 1$. *Then there exists an isomorphism of algebras*

$$\begin{aligned} [\mathrm{U}_\eta(\mathfrak{sl}_3) - \mathrm{mod}_{\mathrm{ss}}]_{\mathbb{C}} &\cong \mathbb{C}[x,y]/\left(\mathrm{U}_{m,n}(x,y) \mid m+n = e+1\right) \\ [V_{m,n}] &\mapsto \mathrm{U}_{m,n}(x,y) \quad (0 \leq m+n \leq e). \end{aligned}$$

The trihedral Hecke algebra of level ∞

• We are now going to define the trihedral analogue of $\mathrm{H}(I_2(\infty)) = \mathrm{H}(\widehat{A}_1)$, which is an infinite-dimensional algebra $\mathrm{T}_\infty \subset \mathrm{H}(\widehat{A}_2)$.

The trihedral Hecke algebra of level ∞

• We are now going to define the trihedral analogue of $\mathrm{H}(I_2(\infty)) = \mathrm{H}(\widehat{A}_1)$, which is an infinite-dimensional algebra $\mathrm{T}_\infty \subset \mathrm{H}(\widehat{A}_2)$.

• There is no underlying group (that we know of), so we define T_∞ directly in terms of the trihedral Kazhdan-Lusztig generators.

The trihedral Hecke algebra of level ∞

• We are now going to define the trihedral analogue of $\mathrm{H}(I_2(\infty)) = \mathrm{H}(\widehat{A}_1)$, which is an infinite-dimensional algebra $\mathrm{T}_\infty \subset \mathrm{H}(\widehat{A}_2)$.

• There is no underlying group (that we know of), so we define T_∞ directly in terms of the trihedral Kazhdan-Lusztig generators.

Definition (MMMT 2018)

Let v be a formal parameter. Then T_∞ is the associative, unital ($\mathbb{C}(\mathrm{v})$-)algebra generated by three elements θ_g, θ_o, θ_p, subject to the following relations:

$$\theta_g^2 = [3]_\mathrm{v}!\,\theta_g, \qquad \theta_o^2 = [3]_\mathrm{v}!\,\theta_o, \qquad \theta_p^2 = [3]_\mathrm{v}!\,\theta_p,$$

$$\theta_g\theta_o\theta_g = \theta_g\theta_p\theta_g, \qquad \theta_o\theta_g\theta_o = \theta_o\theta_p\theta_o, \qquad \theta_p\theta_g\theta_p = \theta_p\theta_o\theta_p.$$

Embedding into $\mathrm{H}(\widehat{A}_2)$

- Let $W(\widehat{A}_2)$ be the affine Weyl group with simple reflections b, r, y. Then

$$byb = yby, \quad ryr = yry, \quad brb = rbr$$

are the longest elements in the (finite) type A_2 parabolic subgroups of $W(\widehat{A}_2)$.

- Let

$$\theta_{byb}, \theta_{ryr}, \theta_{brb}$$

be the corresponding Kazhdan-Lusztig basis elements in $\mathrm{H}(\widehat{A}_2)$.

Embedding into $\mathrm{H}(\widehat{A}_2)$

- Let $W(\widehat{A}_2)$ be the affine Weyl group with simple reflections b, r, y. Then

$$byb = yby, \quad ryr = yry, \quad brb = rbr$$

are the longest elements in the (finite) type A_2 parabolic subgroups of $W(\widehat{A}_2)$.

- Let

$$\theta_{byb}, \theta_{ryr}, \theta_{brb}$$

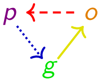
be the corresponding Kazhdan-Lusztig basis elements in $\mathrm{H}(\widehat{A}_2)$.

Lemma

There is an embedding of algebras $\mathrm{T}_\infty \hookrightarrow \mathrm{H}(\widehat{A}_2)$ such that

$$\theta_g \mapsto \theta_{byb}, \quad \theta_o \mapsto \theta_{ryr}, \quad \theta_p \mapsto \theta_{brb}.$$

Fixing a cyclic ordering on $GOP := \{g, o, p\}$, e.g.



,

we can define the *trihedral Bott-Samelson basis* of T_∞

$$\{1\} \cup \left\{\mathrm{H}_{\mathbf{u}}^{k,l} \mid \mathbf{u} \in GOP,\ m, n \in \mathbb{N}^0\right\}.$$

Main idea: T_∞ is "almost" a tricolored version of $\left[\mathrm{U_q}(\mathfrak{sl}_3) - \mathrm{mod}\right]_{\mathbb{C}} \cong \mathbb{C}[x, y]$.

Example

$$\underset{\leftrightsquigarrow x^2}{\mathrm{H}_g^{2,0} = \theta_p\theta_o\theta_g}, \quad \underset{\leftrightsquigarrow xy = yx}{\mathrm{H}_g^{1,1} = \theta_g\theta_p\theta_g = \theta_g\theta_o\theta_g}, \quad \underset{\leftrightsquigarrow y^2}{\mathrm{H}_g^{0,2} = \theta_o\theta_p\theta_g},$$

where we think of x and y as counter-clockwise and clockwise color rotation, resp.

The trihedral Kazhdan-Lusztig basis

For any $\mathbf{u} \in GOP$ and $m, n \in \mathbb{N}^0$, define

$$\mathrm{C}_{\mathbf{u}}^{m,n} := \sum_{k,l=0}^{m,n} [2]_{\mathrm{v}}^{-k-l}\, d_{m,n}^{k,l}\, \mathrm{H}_{\mathbf{u}}^{k,l}.$$

Poposition

The set

$$\{1\} \cup \left\{\mathrm{C}_{\mathbf{u}}^{m,n} \mid \mathbf{u} \in GOP,\ m, n \in \mathbb{N}^0\right\}$$

forms a positive integral basis of T_∞.

Main ingredient of the proof: the embedding $\mathrm{T}_\infty \hookrightarrow \mathrm{H}(\widehat{A}_2)$ sends trihedral KL basis elements to affine KL basis elements.

The trihedral Hecke algebra of level e

Definition

For fixed level e, let I_e be the two-sided ideal in T_∞ generated by

$$\{\mathrm{C}_{\mathbf{u}}^{m,n} \mid m+n=e+1,\ \mathbf{u}\in GOP\}.$$

We define the **trihedral Hecke algebra of level** e as

$$\mathrm{T}_e=\mathrm{T}_\infty/\mathrm{I}_e.$$

- T_e is "almost" a tricolored version of $[\mathrm{U}_\eta(\mathfrak{sl}_3)-\mathrm{mod}_{\mathrm{ss}}]_{\mathbb{C}} \cong \mathbb{C}[x,y]/\left(\mathrm{U}_{m,n}(x,y) \mid m+n=e+1\right)$

The trihedral Hecke algebra of level e

Definition

For fixed level e, let I_e be the two-sided ideal in T_∞ generated by

$$\{\mathrm{C}_{\mathbf{u}}^{m,n} \mid m+n=e+1,\ \mathbf{u}\in GOP\}\,.$$

We define the **trihedral Hecke algebra of level** e as

$$\mathrm{T}_e=\mathrm{T}_\infty/\mathrm{I}_e.$$

- T_e is "almost" a tricolored version of $[\mathrm{U}_\eta(\mathfrak{sl}_3)-\mathrm{mod}_{\mathrm{ss}}]_{\mathbb{C}}\cong\mathbb{C}[x,y]/\left(\mathrm{U}_{m,n}(x,y)\mid m+n=e+1\right)$
- T_e is actually the analogue of the **small quotient** of the dihedral Hecke algebra, obtained by killing θ_{w_0}.

Semisimplicity

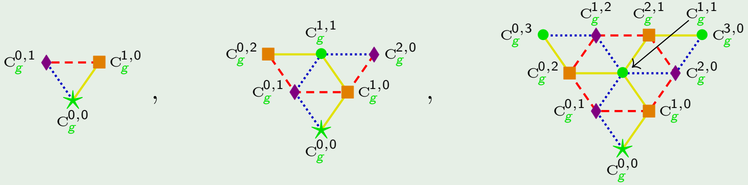
Theorem (MMMT 2018)

The algebra T_e *is semisimple and*

$$\dim \mathrm{T}_e = 3\frac{(e+1)(e+2)}{2} + 1.$$

Example

There is a 3: 1 correspondence between the non-trivial left cells of T_e and the generalized type A Dynkin diagram $\mathbf{A}_e$, which is a cut-off of the fundamental Weyl chamber of $\mathfrak{sl}_3$ (integral dominant weights), e.g.



L^g for $e = 1$ L^g for $e = 2$ L^g for $e = 3$

Complex simples of T_e

1-dimensional simples: for $\lambda_{\mathbf{u}} \in \{0, [3]_{\mathrm{v}}!\}$ s.t. relations hold.

Complex simples of T_e

1-dimensional simples: for $\lambda_{\mathbf{u}} \in \{0, [3]_{\mathrm{v}}!\}$ s.t. relations hold.

3-dimensional simples: for $0 \neq z \in \mathrm{d}_3^{\circ}$ s.t. $\mathrm{U}_{m,n}(z, \overline{z}) = 0$ for all $m + n = e + 1$, the simple V_z is given by

$$\theta_g \mapsto [2]_{\mathrm{v}} \begin{pmatrix} [3]_{\mathrm{v}} & \overline{z} & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\theta_o \mapsto [2]_{\mathrm{v}} \begin{pmatrix} 0 & 0 & 0 \\ z & [3]_{\mathrm{v}} & \overline{z} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\theta_p \mapsto [2]_{\mathrm{v}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \overline{z} & z & [3]_{\mathrm{v}} \end{pmatrix}.$$

We have

$$V_{z_1} \cong V_{z_2} \Leftrightarrow z_1 = e^{\pm 2\pi i/3} z_2.$$

$\mathbb{N}^0$-representations

For $\mathbb{N}^0$-representations of $\mathcal{Q}_e \cong \mathbb{C}[x,y]/(U_{m,n}(x,y) \mid m+n=e+1)$:

Question 1

Are there any $X \in \mathrm{Mat}(r,\mathbb{N}^0)$, with $r \in \mathbb{N}$, such that

- $XX^T = X^TX$;
- $U_{m,n}(X,X^T) = 0$ if $m+n=e+1$;
- $U_{m,n}(X,X^T) \in \mathrm{Mat}(r,\mathbb{N}^0)$ if $0 \leq m+n \leq e$.

$\mathbb{N}^0$-representations

For $\mathbb{N}^0$-representations of $\mathcal{Q}_e \cong \mathbb{C}[x,y]/(U_{m,n}(x,y) \mid m+n=e+1)$:

Question 1

Are there any $X \in \mathrm{Mat}(r,\mathbb{N}^0)$, with $r \in \mathbb{N}$, such that

- $XX^T = X^T X$;
- $U_{m,n}(X,X^T) = 0$ if $m+n=e+1$;
- $U_{m,n}(X,X^T) \in \mathrm{Mat}(r,\mathbb{N}^0)$ if $0 \leq m+n \leq e$.

For $\mathbb{N}^0$-representations of T_e:

Question 2

How to build these from the matrices which answer Question 1?

Tricolored graphs

Let $\mathbf{\Gamma}$ be a tricolored (multi)graph without loops, and group its vertices according to color. Then the adjacency matrix $A(\mathbf{\Gamma})$ becomes of the form:

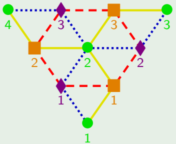
$$A(\mathbf{\Gamma}) = \begin{array}{c} \\ G \\ O \\ P \end{array} \begin{array}{c} \begin{array}{ccc} G & O & P \end{array} \\ \left(\begin{array}{c:c:c} 0 & A^{\mathrm{T}} & C \\ \hdashline A & 0 & B^{\mathrm{T}} \\ \hdashline C^{\mathrm{T}} & B & 0 \end{array}\right) \end{array}$$

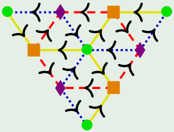
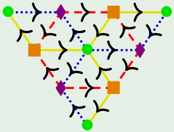
Consider also the oriented adjacency matrices $A(\mathbf{\Gamma}^{\mathrm{X}})$ and $A(\mathbf{\Gamma}^{\mathrm{Y}})$:

$$A(\mathbf{\Gamma}^{\mathrm{X}}) = A(\mathbf{\Gamma}^{\mathrm{Y}})^{\mathrm{T}} = \begin{array}{c} \\ G \\ O \\ P \end{array} \begin{array}{c} \begin{array}{ccc} G & O & P \end{array} \\ \left(\begin{array}{c:c:c} 0 & 0 & C \\ \hdashline A & 0 & 0 \\ \hdashline 0 & B & 0 \end{array}\right) \end{array}$$

Generalized Dynkin diagrams

Example (Type A, Di Francesco-Zuber 1990, Ocneanu 2002)

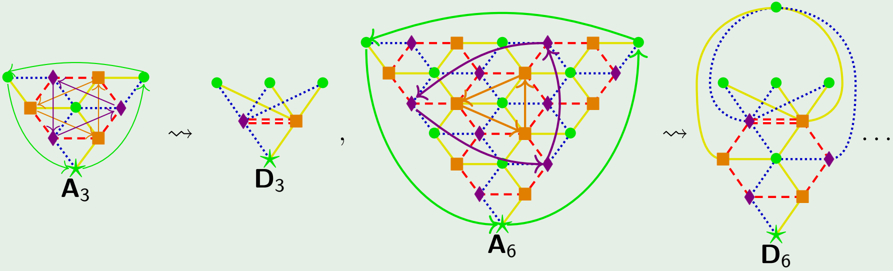
$\mathbf{A}_3 =$ 

$, \mathbf{A}_3^{\mathrm{X}} =$ 
$, \mathbf{A}_3^{\mathrm{Y}} =$ 

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

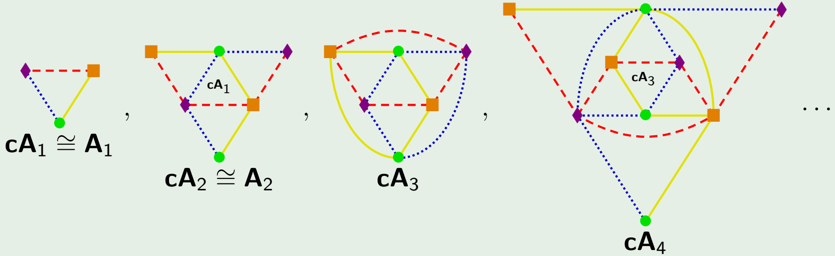
Generalized Dynkin diagrams

Example (Type D, Di Francesco-Zuber 1990, Ocneanu 2002)



Generalized Dynkin diagrams

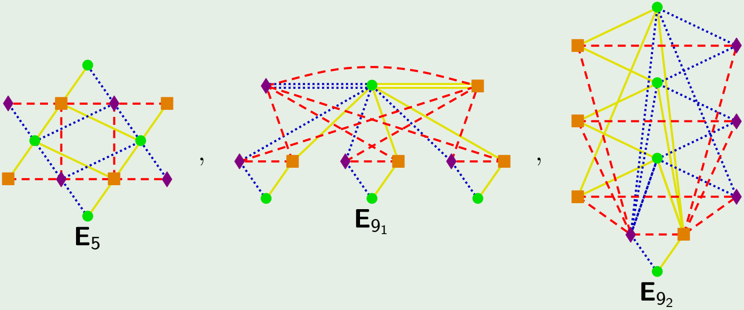
Example (Conjugate type A, Di Francesco-Zuber 1990, Ocneanu 2002)



The graph of type $\mathbf{cA}_e$ comes from an iterative procedure on the graph of type $\mathbf{A}_e$.

Generalized Dynkin diagrams

Example (Type E, Di Francesco-Zuber 1990, Ocneanu 2002)



$+$ three more

$\mathbb{N}^0$-representations of $\mathcal{Q}_e = [\mathrm{U}_\eta(\mathfrak{sl}_3) - \mathrm{mod}_{\mathrm{ss}}]_{\mathbb{C}}$

Let $\mathbf{\Gamma}$ be a tricolored generalized ADE Dynkin diagram with generalized Coxeter number $h = e + 3$.

Theorem (MMMT 2018)

The assignment

$$x \mapsto A(\mathbf{\Gamma}^{\mathrm{X}}), \quad y \mapsto A(\mathbf{\Gamma}^{\mathrm{Y}})$$

defines an integral representation of $\mathcal{Q}_e \cong \mathbb{C}[x, y] / \left(\mathrm{U}_{m,n}(x, y) \mid m + n = e + 1 \right)$. *In type A and D it is* **positive** *integral.*

Let $\mathbf{\Gamma}$ be a tricolored generalized ADE Dynkin diagram with generalized Coxeter number $h = e + 3$.

Theorem (MMMT 2018)

The assignment

$$x \mapsto A(\mathbf{\Gamma}^{\mathrm{X}}), \quad y \mapsto A(\mathbf{\Gamma}^{\mathrm{Y}})$$

defines an integral representation of $\mathcal{Q}_e \cong \mathbb{C}[x, y]/\left(\mathrm{U}_{m,n}(x, y) \mid m + n = e + 1\right)$*.*
In type A *and* D *it is* **positive** *integral.*

- In particular, we have $A(\mathbf{\Gamma}^{\mathrm{X}})A(\mathbf{\Gamma}^{\mathrm{Y}}) = A(\mathbf{\Gamma}^{\mathrm{Y}})A(\mathbf{\Gamma}^{\mathrm{X}})$.

$\mathbb{N}^0$-representations of $\mathcal{Q}_e = [\mathrm{U}_\eta(\mathfrak{sl}_3) - \mathrm{mod}_{\mathrm{ss}}]_{\mathbb{C}}$

Let $\mathbf{\Gamma}$ be a tricolored generalized ADE Dynkin diagram with generalized Coxeter number $h = e + 3$.

Theorem (MMMT 2018)

The assignment

$$x \mapsto A(\mathbf{\Gamma}^{\mathrm{X}}), \quad y \mapsto A(\mathbf{\Gamma}^{\mathrm{Y}})$$

defines an integral representation of $\mathcal{Q}_e \cong \mathbb{C}[x, y]/\left(\mathrm{U}_{m,n}(x, y) \mid m + n = e + 1\right)$. *In type A and D it is* **positive** *integral.*

- In particular, we have $A(\mathbf{\Gamma}^{\mathrm{X}})A(\mathbf{\Gamma}^{\mathrm{Y}}) = A(\mathbf{\Gamma}^{\mathrm{Y}})A(\mathbf{\Gamma}^{\mathrm{X}})$.
- The first claim follows from the fact that all eigenvalues of $\mathbf{\Gamma}^{\mathrm{X}}$ (Evans-Pugh 2010) are roots of the $U_{m,n}$ with $m + n = e + 1$.

$\mathbb{N}^0$-representations of $\mathcal{Q}_e = [\mathrm{U}_\eta(\mathfrak{sl}_3) - \mathrm{mod}_{\mathrm{ss}}]_{\mathbb{C}}$

Let $\mathbf{\Gamma}$ be a tricolored generalized ADE Dynkin diagram with generalized Coxeter number $h = e + 3$.

Theorem (MMMT 2018)

The assignment

$$x \mapsto A(\mathbf{\Gamma}^{\mathrm{X}}), \quad y \mapsto A(\mathbf{\Gamma}^{\mathrm{Y}})$$

defines an integral representation of $\mathcal{Q}_e \cong \mathbb{C}[x, y]/\left(\mathrm{U}_{m,n}(x, y) \mid m + n = e + 1\right)$. *In type A and D it is* **positive** *integral.*

- In particular, we have $A(\mathbf{\Gamma}^{\mathrm{X}})A(\mathbf{\Gamma}^{\mathrm{Y}}) = A(\mathbf{\Gamma}^{\mathrm{Y}})A(\mathbf{\Gamma}^{\mathrm{X}})$.
- The first claim follows from the fact that all eigenvalues of $\mathbf{\Gamma}^{\mathrm{X}}$ (Evans-Pugh 2010) are roots of the $U_{m,n}$ with $m + n = e + 1$.
- Positivity in type A and D follows from categorification. We conjecture positivity to hold in type cA and E as well.

$\mathbb{N}^0$-representations of T_e

Let $\boldsymbol{\Gamma}$ be a tricolored generalized ADE Dynkin diagram with generalized Coxeter number $h = e + 3$.

Theorem (MMMT 2018)

There exists a unique integral T_e-representation $\mathrm{M}_{\boldsymbol{\Gamma}}$ s.t.

$$\theta_g \mapsto [2]_{\mathrm{v}} \begin{pmatrix} [3]_{\mathrm{v}}\mathrm{Id} & A^{\mathrm{T}} & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \theta_o \mapsto [2]_{\mathrm{v}} \begin{pmatrix} 0 & 0 & 0 \\ A & [3]_{\mathrm{v}}\mathrm{Id} & B^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\theta_p \mapsto [2]_{\mathrm{v}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C^{\mathrm{T}} & B & [3]_{\mathrm{v}}\mathrm{Id} \end{pmatrix}.$$

It is **positive** *integral in type A and D.*

We conjecture positivity to hold in conjugate type A and type E as well.

2-Representations of $\mathcal{Q}_e = \mathrm{U}_\eta(\mathfrak{sl}_3) - \mathrm{mod}_{\mathrm{ss}}$ using quivers

- Let $\mathbf{\Gamma}$ be the generalized type ADE Dynkin diagram with $h = e + 3$.
- Take $\mathrm{T}\nabla_e \cong \mathbb{C}^{V(\mathbf{\Gamma})}$ to be the **trivial** quiver algebra associated to $\mathbf{\Gamma}$.
- Let $P_{i,j}$ (resp. ${}_{i,j}P$) be the left (resp. right) projective $\mathrm{T}\nabla_e$-module associated to the vertex $v_{i,j}$ in $\mathbf{\Gamma}$.

Conjecture

There exists a finitary 2-representation of $\mathcal{Q}_e$ on $\mathrm{T}\nabla_e - \mathrm{fpmod}$ such that

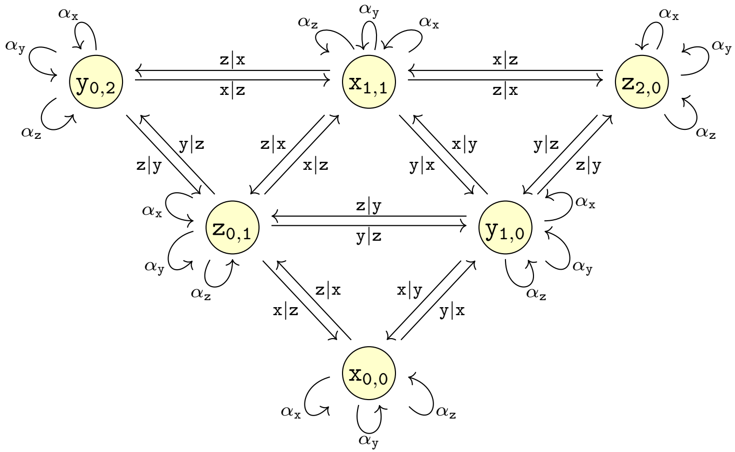
$$V_{1,0} \mapsto \bigoplus_{(i,j)\to(k,l)\in\mathbf{\Gamma}^{\mathrm{X}}} P_{k,l} \otimes {}_{i,j}P,$$

$$V_{0,1} \mapsto \bigoplus_{(i,j)\leftarrow(k,l)\in\mathbf{\Gamma}^{\mathrm{Y}}} P_{k,l} \otimes {}_{i,j}P,$$

which decategorifies to the positive integral representation of $\mathbb{C}[x,y]/\left(\mathrm{U}_{m,n}(x,y) \mid m+n=e+1\right)$ associated to $\mathbf{\Gamma}$.

Functorial representations of T_e in generalized type A

Consider the following quiver:



The trihedral zigzag algebra of generalized type A

Definition (MMMT 2018)

Let ∇_e be the complex path algebra of Γ modulo the relations:

- Any path with more than one triangle to its left (right) is equal to zero.
- $\alpha_x + \alpha_y + \alpha_z = 0$, $\alpha_x\alpha_y + \alpha_x\alpha_z + \alpha_y\alpha_z = 0$, $\alpha_x\alpha_y\alpha_z = 0$.
- Loops commute with edges.
- $\alpha_z \mathrm{y}|\mathrm{x} = 0$ etc.
- Zig-zag relation: $x|y|x = \alpha_x\alpha_y$ etc.
- Zig-zig equals zag times loop: $x|y|z = \alpha_x x|z$ etc.

The grading on ∇_e is given by twice the path length.

The trihedral zigzag algebra of generalized type A

Definition (MMMT 2018)

Let ∇_e be the complex path algebra of Γ modulo the relations:

- Any path with more than one triangle to its left (right) is equal to zero.
- $\alpha_x + \alpha_y + \alpha_z = 0$, $\alpha_x\alpha_y + \alpha_x\alpha_z + \alpha_y\alpha_z = 0$, $\alpha_x\alpha_y\alpha_z = 0$.
- Loops commute with edges.
- $\alpha_z \mathrm{y}|\mathrm{x} = 0$ etc.
- Zig-zag relation: $x|y|x = \alpha_x\alpha_y$ etc.
- Zig-zig equals zag times loop: $x|y|z = \alpha_x x|z$ etc.

The grading on ∇_e is given by twice the path length.

- Let $e_{i,j}$ be the idempotent at vertex $v_{i,j}$. Paths of length > 3 are zero and

$$e_{i,j}\nabla_e e_{k,l} \cong \begin{cases} H^*(\mathcal{F}l_3, \mathbb{C}), & \text{if } v_{i,j} = v_{k,l}, \\ \mathbb{C}\{2\} \oplus \mathbb{C}\{4\}, & \text{if } v_{i,j} - v_{k,l}, \\ \{0\}, & \text{else.} \end{cases}$$

Functorial representations of T_e in generalized type A

Let $P_{i,j}$ (resp. ${}_{i,j}P$) be the left (resp. right) graded projective ∇_e-module corresponding to vertex $v_{i,j}$ in $\mathbf{\Gamma}$.

Theorem

The assignment

$$\theta_g \mapsto \bigoplus_{i-j\equiv 0 \bmod 3} P_{i,j} \otimes {}_{i,j}P$$

$$\theta_o \mapsto \bigoplus_{i-j\equiv 1 \bmod 3} P_{i,j} \otimes {}_{i,j}P$$

$$\theta_p \mapsto \bigoplus_{i-j\equiv 2 \bmod 3} P_{i,j} \otimes {}_{i,j}P$$

defines a functorial representation of T_e *on* $\nabla_e\text{−}\mathrm{fpmod}_{gr}$.

Remarks

• By using the $\mathbb{Z}/3\mathbb{Z}$-symmetry on ∇_e, for $e \equiv 0 \bmod 3$, one can easily define the corresponding type D trihedral zigzag algebra. For other generalized types it is not clear what the right definition is.

Remarks

• By using the $\mathbb{Z}/3\mathbb{Z}$-symmetry on ∇_e, for $e \equiv 0 \bmod 3$, one can easily define the corresponding type D trihedral zigzag algebra. For other generalized types it is not clear what the right definition is.

• Unfortunately, we do not know how to lift these functorial representations of T_e to full-blown 2-representations of trihedral Soergel bimodules in a straightforward way.

Remarks

• By using the $\mathbb{Z}/3\mathbb{Z}$-symmetry on ∇_e, for $e \equiv 0 \bmod 3$, one can easily define the corresponding type D trihedral zigzag algebra. For other generalized types it is not clear what the right definition is.

• Unfortunately, we do not know how to lift these functorial representations of T_e to full-blown 2-representations of trihedral Soergel bimodules in a straightforward way.

• Therefore, we use an alternative construction of simple transitive 2-representations, involving algebra objects. The two approaches are related by the quantum $SU(3)$ McKay correspondence.

But we first recall the **Quantum Satake Correspondence** and define **trihedral Soergel bimodules**.

Definition

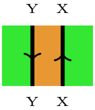
For $\mathbf{u} \in \{g, o, p\}$, let $\mathcal{Q}_{\mathrm{q}}^{\mathbf{u}}$ denote the full subcategory of $\mathcal{Q}_{\mathrm{q}}$ generated by the $V_{m,n}$ such that

$$m - n \equiv \begin{cases} 0 \bmod 3, & \text{if } \mathbf{u} = g, \\ 1 \bmod 3, & \text{if } \mathbf{u} = o, \\ 2 \bmod 3, & \text{if } \mathbf{u} = p. \end{cases}$$

Tensoring with $V_{1,0}$, resp. $V_{0,1}$, defines a functor X, resp. Y, between the $\mathcal{Q}_{\mathrm{q}}^{\mathbf{u}}$, e.g.

${}_{o}\mathrm{X}_{g} =$ 

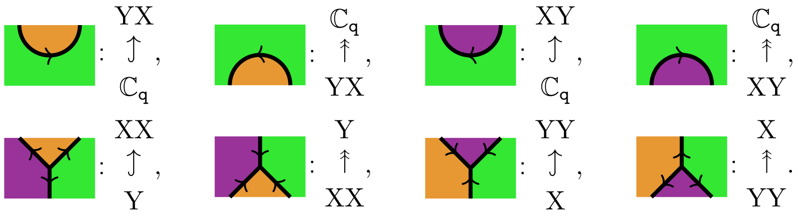
$: \mathcal{Q}_{\mathrm{q}}^{g} \to \mathcal{Q}_{\mathrm{q}}^{o}$, ${}_{g}\mathrm{Y}_{o} =$ 

$: \mathcal{Q}_{\mathrm{q}}^{o} \to \mathcal{Q}_{\mathrm{q}}^{g}$, ${}_{g}\mathrm{Y}_{o} \circ {}_{o}\mathrm{X}_{g} =$ 

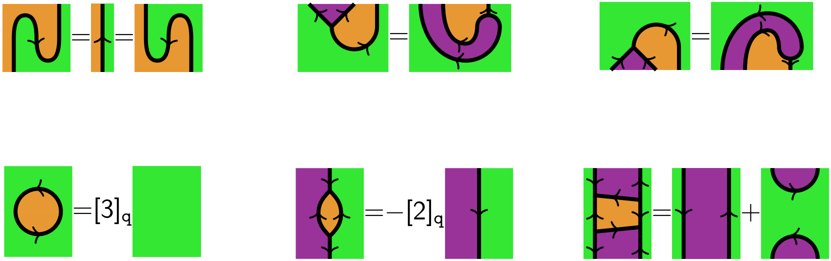
Definition (Elias 2014 motivated by Kuperberg 1996)

We define $\mathscr{Q}_{q}^{GOP}$ to be the additive, $\mathbb{C}_q$-linear closure of the 2-category whose objects are the categories $\mathcal{Q}_q^{\mathbf{u}}$, whose 1-morphisms are composites of X and Y, and whose 2-morphisms are natural transformations.

A natural transformation between composites of X and Y is the same as a $\mathrm{U}_q(\mathfrak{sl}_3)$-equivariant map, so we can use Kuperberg's diagrammatic web calculus to describe $\mathscr{Q}_{q}^{GOP}$. The generating 2-morphisms (up to color variations) are



These are subject to the relations



together with the vertical mirrors and the relations obtained by varying the orientation and the colors.

Three-colored $\mathfrak{sl}_3$-clasps

Given $m, n \in \mathbb{N}^0$, for each choice of source $\mathbf{u} \in \{g, o, p\}$, the simple $V_{m,n}$ corresponds to a direct summand of the functor $\mathrm{X}^m\mathrm{Y}^n$ in $\mathscr{Q}_{\mathrm{q}}^{GOP}$, given by a diagrammatic idempotent $\mathrm{c}_{\mathbf{u}}^{m,n}$ (Kuperberg 1996, Kim 2007).

Example (Three-colored $\mathfrak{sl}_3$-clasps)

$\mathrm{c}_{g}^{2,0} =$  $+ \frac{1}{[2]_{\mathrm{q}}}$ , $\mathrm{c}_{g}^{1,1} =$  $- \frac{1}{[3]_{\mathrm{q}}}$ ,

$\mathrm{c}_{g}^{0,2} =$  $+ \frac{1}{[2]_{\mathrm{q}}}$ 

The root of unity case

Let $\eta^{2(e+3)} = 1$.

Definition

Define $\mathscr{Q}_e^{GOP}$ as the quotient of the diagrammatic 2-category above, for $\mathrm{q} = \eta$, by the 2-ideal generated by all $\mathrm{c}_{\mathbf{u}}^{m,n}$, such that $m + n = e + 1$ and $\mathbf{u} \in GOP$.

- $\mathscr{Q}_e^{GOP}$ is nothing but a three-colored version of Kuperberg's diagrammatic calculus for $\mathcal{Q}_e = \mathrm{U}_\eta(\mathfrak{sl}_3) - \mathrm{mod}_{\mathrm{ss}}$.

Diagrammatic Soergel calculus in type $\widehat{A}_2$

Using a q-deformation of the usual $\widehat{A}_2$ Cartan matrix, Elias (2014) constructed a linear representation of $\mathrm{W} = \mathrm{W}(\widehat{A}_2)$ on the root space $\mathrm{Span}_{\mathbb{C}(\mathsf{q})}\{\alpha_{b}, \alpha_{r}, \alpha_{y}\}$.

Diagrammatic Soergel calculus in type $\widehat{A}_2$

Using a q-deformation of the usual $\widehat{A}_2$ Cartan matrix, Elias (2014) constructed a linear representation of $\mathrm{W} = \mathrm{W}(\widehat{A}_2)$ on the root space $\mathrm{Span}_{\mathbb{C}(\mathsf{q})}\{\alpha_b, \alpha_r, \alpha_y\}$.

We can specialize q to a complex number to get a complex representation:

- for generic q, it is reflection faithful.
- for q a root of unity, the representation is not faithful and descends to a finite complex reflection group.

Let $\mathrm{R}_{\mathsf{q}} = \mathbb{C}(q)[\alpha_b, \alpha_r, \alpha_y]$, where $\alpha_b, \alpha_r, \alpha_y$ are given degree 2. The above representation extends to a degree-preserving action of W on R_{q} by automorphisms.

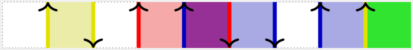
Soergel calculus in type $\widehat{A}_2$

Definition (The 2-cat $\mathbf{s}\mathscr{B}\mathscr{S}_{\mathrm{q}}^{*}$, Elias 2014, Elias-Williamson 2013)

- **Objects:** proper subsets of $\{b, y, r\}$:

$$\emptyset,\ b,\ y,\ r,\ g := \{b, y\},\ o := \{r, y\},\ p := \{b, r\}.$$

- 1-**morphisms:** finite strings of compatible colors, e.g.:



- 2-**morphisms:** generated by

degree 1, degree -1, degree 2, degree -2, degree 0

and decorations of the regions by partially invariant polynomials in R_{q}, and subject to a whole list of relations (which depend on q).

Remarks

Let $\boldsymbol{s}\mathscr{B}\mathscr{S}_{\mathrm{q}}$ be the 2-category obtained from $\boldsymbol{s}\mathscr{B}\mathscr{S}_{\mathrm{q}}^{*}$ by allowing formal grading shifts on 1-morphisms and considering only degree-zero 2-morphisms, i.e. for any $t \in \mathbb{Z}$ we define

$$2\boldsymbol{s}\mathscr{B}\mathscr{S}_{\mathrm{q}}(x\{t\}, y) := 2\boldsymbol{s}\mathscr{B}\mathscr{S}_{\mathrm{q}}^{*}(x, y)_{t}.$$

Theorem (Elias 2014, Elias-Williamson 2013)

Let $\mathrm{q} \in \mathbb{C}$ be generic.

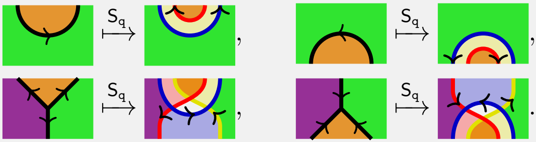
- $\mathscr{K}\mathrm{ar}(\boldsymbol{s}\mathscr{B}\mathscr{S}_{\mathrm{q}})$ *is equivalent to the* 2*-category of* **all** *Soergel bimodules of type* $\widehat{A}_2$ *and decategorifies to the Hecke algebroid of that type, such that the indecomposable* 1*-morphisms correspond to the KL-basis elements.*
- *Let* $\mathscr{B}\mathscr{S}_{\mathrm{q}} := \boldsymbol{s}\mathscr{B}\mathscr{S}_{\mathrm{q}}(\emptyset, \emptyset)$*. Then* $\mathscr{K}\mathrm{ar}(\mathscr{B}\mathscr{S}_{\mathrm{q}})$ *is equivalent to the monoidal category of* **regular** *Soergel bimodules of type* $\widehat{A}_2$ *and decategorifies to* $\mathrm{H}_{\mathrm{v}}(\widehat{A}_2)$*.*

The Quantum Satake Correspondence (QSC)

- The 2-category of **maximally singular** Soergel bimodules $\mathscr{K}\mathrm{ar}(\boldsymbol{m}\mathscr{B}\mathscr{S}_{\mathrm{q}})$ is defined as the Karoubi envelope of the 2-full 2-subcategory of $\boldsymbol{s}\mathscr{B}\mathscr{S}_{\mathrm{q}}$ generated by diagrams whose left- and rightmost colors are secondary.

Definition (Elias 2014)

The **Satake** 2-**functor** $\mathrm{S}_{\mathrm{q}}\colon \mathscr{Q}_{\mathrm{q}}^{GOP} \to \boldsymbol{m}\mathscr{B}\mathscr{S}_{\mathrm{q}}$ is defined as indicated below:



Theorem (Elias 2014)

The Satake 2-functor is a well-defined degree-zero 2-equivalence.

Trihedral Soergel bimodules of level ∞

Assume that $q \in \mathbb{C}$ is generic.

Definition (MMMT 2018)

Let $\mathscr{T}_\infty$ be the additive closure of the 2-full 2-subcategory of $\mathscr{BS}_q$, whose 1-morphisms are generated by all grading shifts of

$$\emptyset, \quad \emptyset bgb\emptyset, \quad \emptyset yoy\emptyset, \quad \emptyset bpb\emptyset,$$

and the 1-morphisms obtained from these by changing the intermediate primary colors.

Example

By the relations on 2-morphisms in $\mathscr{BS}_q$, we have

$$\emptyset bgb\emptyset \cong \emptyset bgy\emptyset \cong \emptyset ygb\emptyset \cong \emptyset ygy\emptyset.$$

Similar isomorphisms hold for the strings with o and p.

The categorification theorem at level ∞

Theorem

The decategorification of $\mathscr{T}_\infty$ is isomorphic to T_∞, such that the indecomposable objects correspond to the tricolored KL basis elements.

- We can always remove intermediate $\emptyset$, e.g.

$$\emptyset bg\, b\emptyset bpb\emptyset \cong \emptyset bg\, bpb\emptyset \oplus \emptyset bg\, bpb\emptyset\{2\}$$

This shows that all 1-morphisms in $\mathscr{T}_\infty$ can be obtained from $\boldsymbol{s}\mathscr{B}\mathscr{S}_{\mathtt{q}}$ by **biinduction**.

The categorification theorem at level ∞

Theorem

The decategorification of $\mathscr{T}_\infty$ is isomorphic to T_∞, such that the indecomposable objects correspond to the tricolored KL basis elements.

- We can always remove intermediate $\emptyset$, e.g.

$$\emptyset bg\, b\emptyset bpb\emptyset \cong \emptyset bg\, bpb\emptyset \oplus \emptyset bg\, bpb\emptyset\{2\}$$

This shows that all 1-morphisms in $\mathscr{T}_\infty$ can be obtained from $\boldsymbol{s}\mathscr{B}\mathscr{S}_{\mathtt{q}}$ by **biinduction**.

- For every pair of 1-morphisms x and y in $\boldsymbol{s}\mathscr{B}\mathscr{S}_{\mathtt{q}}$, biinduction gives a functor

$$\mathrm{BI}(x,y)\colon \boldsymbol{s}\mathscr{B}\mathscr{S}_{\mathtt{q}}(x,y) \to \mathscr{T}_\infty(\mathrm{BI}(x),\mathrm{BI}(y)).$$

However, it is **not** a 2-functor, because it does not behave well under horizontal composition.

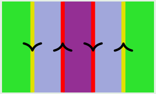
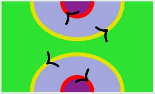
Biinduction

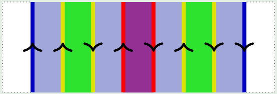
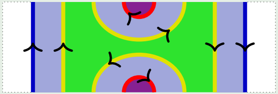
For any $\mathbf{u} \in GOP$:

- the Satake 2-functor S_{q} maps the tricolored clasps $\mathrm{c}_{\mathbf{u}}^{m,n}$ in $\mathcal{Q}_{\mathrm{q}}^{GOP}$ to the primitive idempotent 2-endomorphisms $\mathrm{S}_{\mathrm{q}}(\mathrm{c}_{\mathbf{u}}^{m,n})$ in $s\mathscr{B}\mathscr{S}_{\mathrm{q}}$;
- biinduction maps the $\mathrm{S}_{\mathrm{q}}(\mathrm{c}_{\mathbf{u}}^{m,n})$ in $s\mathscr{B}\mathscr{S}_{\mathrm{q}}$ to the primitive idempotent (2-)endomorphisms $\mathrm{C}_{\mathbf{u}}^{m,n}$ in $\mathscr{T}_{\infty}$.

Example

$\mathrm{c}_{g}^{1,1} =$  $- \frac{1}{[3]_{\mathrm{q}}}$  $\xmapsto{\mathrm{S}_{\mathrm{q}}}$

$\mathrm{S}_{\mathrm{q}}(\mathrm{c}_{g}^{1,1}) =$  $- \frac{1}{[3]_{\mathrm{q}}}$  $\xmapsto{\mathrm{BI}}$

$\mathrm{C}_{g}^{1,1} = \;=$  $- \frac{1}{[3]_{\mathrm{q}}}$ 

Maximally singular Soergel bimodules at level e

Let $\eta^{2(e+3)} = 1$.

Definition (MMMT 2018)

Define $\boldsymbol{m}\mathscr{B}\mathscr{S}_e$ as the quotient of $\boldsymbol{m}\mathscr{B}\mathscr{S}_{\mathrm{q}}$, at $\mathrm{q} = \eta$, by the two-sided 2-ideal generated by

$$\left\{\mathsf{S}_{\mathrm{q}}(\mathrm{c}_{\mathbf{u}}^{m,n}) \mid m+n=e+1,\ \mathbf{u}\in GOP\right\} = \left\{\mathsf{S}_{\mathrm{q}}({}^{m,n}_{\ \mathbf{u}}\mathrm{c}) \mid m+n=e+1,\ \mathbf{u}\in GOP\right\}.$$

The Karoubi envelope $\mathscr{K}\mathrm{ar}(\boldsymbol{m}\mathscr{B}\mathscr{S}_e)$ is by definition the 2-category of **maximally singular type $\widehat{A}_2$ Soergel bimodules at level e**.

Corollary

The Satake 2-functor S_{q}, at $\mathrm{q} = \eta$, descends to a degree-zero 2-equivalence

$$\mathsf{S}_e\colon \mathscr{Q}_e^{GOP} \to \mathscr{K}\mathrm{ar}(\boldsymbol{m}\mathscr{B}\mathscr{S}_e).$$

Trihedral Soergel bimodules at level e

Let $\eta^{2(e+3)} = 1$.

Definition (MMMT 2018)

Define $\mathscr{T}_e$ as the quotient of $\mathscr{T}_\infty$, at $q = \eta$, by the two-sided 2-ideal generated by

$$\{C_{\mathbf{u}}^{m,n} \mid m+n = e+1,\ \mathbf{u} \in GOP\} = \{{}^{m,n}_{\mathbf{u}}C \mid m+n = e+1,\ \mathbf{u} \in GOP\}.$$

Theorem

The decategorification of $\mathscr{T}_e$ is isomorphic to T_e, such that the indecomposable objects correspond to the tricolored KL basis elements.

Algebra and module objects

Let $\mathscr{C}$ be a finitary monoidal category.

- An **algebra object** (X, μ, ι) in $\mathscr{C}$ is an object X together with a multiplication morphism $\mu\colon X \otimes X \to X$ and a unit morphism $\iota\colon I \to X$ satisfying the usual axioms.

Algebra and module objects

Let $\mathscr{C}$ be a finitary monoidal category.

- An **algebra object** (X, μ, ι) in $\mathscr{C}$ is an object X together with a multiplication morphism $\mu\colon X \otimes X \to X$ and a unit morphism $\iota\colon I \to X$ satisfying the usual axioms.
- A (right) X-**module object** (M, μ_M) in $\mathscr{C}$ is an object M together with a morphism $\mu_M\colon M \otimes X \to M$ satisfying the usual axioms. Similarly, one can define **intertwiners** between (right) X-modules in $\mathscr{C}$.

Algebra and module objects

Let $\mathscr{C}$ be a finitary monoidal category.

- An **algebra object** (X, μ, ι) in $\mathscr{C}$ is an object X together with a multiplication morphism $\mu\colon X \otimes X \to X$ and a unit morphism $\iota\colon I \to X$ satisfying the usual axioms.

- A (right) X-**module object** (M, μ_M) in $\mathscr{C}$ is an object M together with a morphism $\mu_M\colon M \otimes X \to M$ satisfying the usual axioms. Similarly, one can define **intertwiners** between (right) X-modules in $\mathscr{C}$.

- In this way, we get the **finitary category of right** X**-module objects** in $\mathscr{C}$, denoted $\mathrm{mod}_{\mathscr{C}} - X$.

Algebra and module objects

Let $\mathscr{C}$ be a finitary monoidal category.

- An **algebra object** (X, μ, ι) in $\mathscr{C}$ is an object X together with a multiplication morphism $\mu\colon X \otimes X \to X$ and a unit morphism $\iota\colon I \to X$ satisfying the usual axioms.
- A (right) X-**module object** (M, μ_M) in $\mathscr{C}$ is an object M together with a morphism $\mu_M\colon M \otimes X \to M$ satisfying the usual axioms. Similarly, one can define **intertwiners** between (right) X-modules in $\mathscr{C}$.
- In this way, we get the **finitary category of right X-module objects** in $\mathscr{C}$, denoted $\mathrm{mod}_{\mathscr{C}} - X$.
- Let (X, μ, ι) be an algebra object in $\mathscr{C}$. For any object Y in $\mathscr{C}$, the monoidal product $Y \otimes X$ is naturally a right X-module object in $\mathscr{C}$, with $\mu_{Y \otimes X} = \mathrm{id}_Y \otimes \mu$.

Algebra and module objects

Let $\mathscr{C}$ be a finitary monoidal category.

- An **algebra object** (X, μ, ι) in $\mathscr{C}$ is an object X together with a multiplication morphism $\mu\colon X \otimes X \to X$ and a unit morphism $\iota\colon I \to X$ satisfying the usual axioms.

- A (right) X-**module object** (M, μ_M) in $\mathscr{C}$ is an object M together with a morphism $\mu_M\colon M \otimes X \to M$ satisfying the usual axioms. Similarly, one can define **intertwiners** between (right) X-modules in $\mathscr{C}$.

- In this way, we get the **finitary category of right** X**-module objects** in $\mathscr{C}$, denoted $\mathrm{mod}_{\mathscr{C}} - X$.

- Let (X, μ, ι) be an algebra object in $\mathscr{C}$. For any object Y in $\mathscr{C}$, the monoidal product $Y \otimes X$ is naturally a right X-module object in $\mathscr{C}$, with $\mu_{Y \otimes X} = \mathrm{id}_Y \otimes \mu$.

- In this way, the category $\mathrm{mod}_{\mathscr{C}} - X$ becomes naturally a (left) finitary 2-representation of $\mathscr{C}$.

Algebra and module objects

Let $\mathscr{C}$ be a finitary monoidal category.

- An **algebra object** (X, μ, ι) in $\mathscr{C}$ is an object X together with a multiplication morphism $\mu\colon X \otimes X \to X$ and a unit morphism $\iota\colon I \to X$ satisfying the usual axioms.
- A (right) X-**module object** (M, μ_M) in $\mathscr{C}$ is an object M together with a morphism $\mu_M\colon M \otimes X \to M$ satisfying the usual axioms. Similarly, one can define **intertwiners** between (right) X-modules in $\mathscr{C}$.
- In this way, we get the **finitary category of right** X**-module objects** in $\mathscr{C}$, denoted $\mathrm{mod}_{\mathscr{C}} - X$.
- Let (X, μ, ι) be an algebra object in $\mathscr{C}$. For any object Y in $\mathscr{C}$, the monoidal product $Y \otimes X$ is naturally a right X-module object in $\mathscr{C}$, with $\mu_{Y\otimes X} = \mathrm{id}_Y \otimes \mu$.
- In this way, the category $\mathrm{mod}_{\mathscr{C}} - X$ becomes naturally a (left) finitary 2-representation of $\mathscr{C}$.
- Under certain conditions, there is a bijection between the equivalence classes of simple transitive 2-representations of $\mathscr{C}$ and the Morita equivalence classes of simple algebra objects in $\overline{\mathscr{C}}$, its projective abelianization. [MMMT 2016]

Algebra objects in $\mathcal{Q}_e = \mathrm{U}_\eta(\mathfrak{sl}_3) - \mathrm{mod}_{\mathrm{ss}}$

Example (Generalized type A)

- The identity object $\mathrm{I} = V_{0,0}$ is an algebra object, because $\mathrm{I} \otimes \mathrm{I} \cong \mathrm{I}$.
- Since $Y \otimes \mathrm{I} \cong Y$ for all objects Y in $\mathcal{Q}_e$, we see that

$$\mathrm{mod}_{\mathcal{Q}_e} - \mathrm{I} \simeq \mathcal{Q}_e,$$

which is the regular 2-representation of $\mathcal{Q}_e$.

- It is also the unique cell 2-representation of $\mathcal{Q}_e$. In particular, it is simple transitive.
- Conjecture: it is equivalent to the generalized type A quiver 2-representation of $\mathcal{Q}_e$ from a couple of slides ago.

Algebras in $\mathcal{Q}_e = \mathrm{U}_\eta(\mathfrak{sl}_3) - \mathrm{mod}_{\mathrm{ss}}$

Let $e \equiv 0 \bmod 3$.

Example (Generalized type D, Schopieray 2017, MMMT 2018)

As an object in $\mathcal{Q}_e$ the algebra object X decomposes as

$$X \cong V_{0,0} \oplus V_{e,0} \oplus V_{0,e}.$$

The unit morphism $\iota\colon \mathrm{I} = V_{0,0} \to X$ is given by $\big(\mathrm{id}_{V_{0,0}}, 0, 0\big)$.
Furthermore, there are morphisms

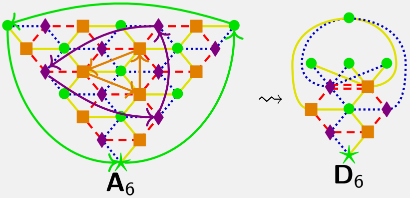
$$\begin{aligned} V_{e,0} \otimes V_{e,0} &\to V_{0,e}, \\ V_{0,e} \otimes V_{0,e} &\to V_{e,0}, \\ V_{e,0} \otimes V_{0,e} &\to V_{0,0}, \\ V_{0,e} \otimes V_{e,0} &\to V_{0,0}, \end{aligned}$$

which, together with the canonical isomorphisms $V_{0,0} \otimes V_{i,j} \cong V_{i,j} \cong V_{i,j} \otimes V_{0,0}$, assemble into a unital and associative multiplication morphism $\mu\colon X \otimes X \to X$.

Conjectures

Conjecture

The 2-representation of $\mathcal{Q}_e$ on $\mathrm{mod}_{\mathcal{Q}_e} - X$ is equivalent to the generalized type D quiver 2-representation of $\mathcal{Q}_e$.



• If simple transitive quiver 2-representations of $\mathcal{Q}_e$ exist for all simply laced generalized Dynkin diagrams (as we conjectured a couple of slides back), then so do simple algebra objects, but we do not know of any explicit construction of X in conjugate type A and type E.

Algebra objects in $\mathscr{T}_e$

- Every simple algebra object X in $\mathcal{Q}_e = \mathrm{U}_\eta(\mathfrak{sl}_3) - \mathrm{mod}_{\mathrm{ss}}$ gives rise to three algebra 1-morphism $X_{\mathbf{u}} \in \mathscr{Q}_e^{GOP}(\mathbf{u}, \mathbf{u})$, for $\mathbf{u} \in GOP$.

Proposition

For every simple algebra object (X, μ, ι) in $\mathcal{Q}_e$ and every $\mathbf{u} \in GOP$, there exist degree zero multiplication and unit morphisms such that

$$\mathrm{BI} \circ \mathsf{S}_e(X_{\mathbf{u}})\{-3\}$$

becomes a graded algebra object in $\mathscr{T}_e$.

Multiplication and unit morphisms of in $\mathscr{T}_e$

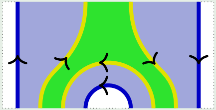
- Because biinduction is not a 2-functor, one has to be slightly careful with the definition of the multiplication morphism of $\mathrm{BI} \circ \mathsf{S}_e(X_{\mathbf{u}})\{-3\}$.

Example (Generalized type A)

For $(X, \mu, \iota) = (\mathrm{I}, \mathrm{id}_{\mathrm{I}}, \mathrm{id}_{\mathrm{I}})$ in $\mathscr{Q}_e$ and $\mathbf{u} = g$, the algebra object in $\mathscr{T}_e$ is

$\Big(\emptyset bgb\emptyset\{-3\}$,

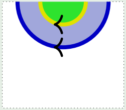
multiplication



degree -3

,

unit



degree 3 $\Big)$

Conjecture: the quiver algebra underlying the simple transitive 2-representation of $\mathscr{T}_e$ is the trihedral zigzag algebra of generalized type A.

Final remarks

- **Open problem** (for $e > 3$): classify all admissible graphs $\mathbf{\Gamma}$ such that

$$\mathrm{U}_{m,n}(A(\mathbf{\Gamma}^{\mathrm{X}}), A(\mathbf{\Gamma}^{\mathrm{Y}})) = 0, \quad \text{for all } m + n = e + 1.$$

Final remarks

- **Open problem** (for $e > 3$): classify all admissible graphs $\mathbf{\Gamma}$ such that

$$\mathsf{U}_{m,n}(A(\mathbf{\Gamma}^{\mathrm{X}}), A(\mathbf{\Gamma}^{\mathrm{Y}})) = 0, \quad \text{for all } m + n = e + 1.$$

- **Open problem**: classify all simple algebra objects in $\mathrm{U}_{\eta} - \mathrm{mod}_{\mathrm{ss}}$.

Final remarks

- **Open problem** (for $e > 3$): classify all admissible graphs $\mathbf{\Gamma}$ such that

$$\mathsf{U}_{m,n}(A(\mathbf{\Gamma}^{\mathrm{X}}), A(\mathbf{\Gamma}^{\mathrm{Y}})) = 0, \quad \text{for all } m + n = e + 1.$$

- **Open problem**: classify all simple algebra objects in $\mathrm{U}_{\eta} - \mathrm{mod}_{\mathrm{ss}}$.
- **Question**: The ordinary zigzag algebras have nice properties and interesting relations to other mathematics. Do some of those generalize to the trihedral zigzag algebras?

Final remarks

- **Open problem** (for $e > 3$): classify all admissible graphs $\mathbf{\Gamma}$ such that

$$\mathsf{U}_{m,n}(A(\mathbf{\Gamma}^{\mathrm{X}}), A(\mathbf{\Gamma}^{\mathrm{Y}})) = 0, \quad \text{for all } m + n = e + 1.$$

- **Open problem**: classify all simple algebra objects in $\mathrm{U}_\eta - \mathrm{mod}_{\mathrm{ss}}$.
- **Question**: The ordinary zigzag algebras have nice properties and interesting relations to other mathematics. Do some of those generalize to the trihedral zigzag algebras?
- **Possible generalizations**: Does our story generalize to type A_n for $n \geq 3$?

THANKS!!!