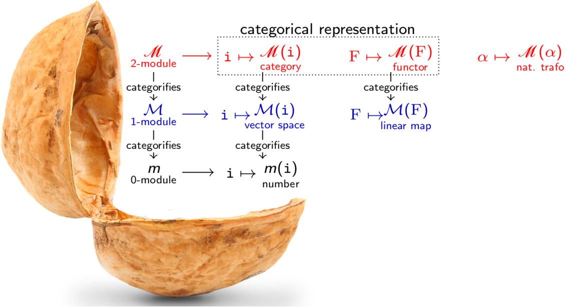
2-representation theory in a nutshell

Or: A tale of matrices and functors

Daniel Tubbenhauer



Joint with Michael Ehrig, Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz and Paul Wedrich

June 2018

1. **Classical representation theory**
   - Main ideas
   - Some classical results
   - An example

2. **Categorical representation theory**
   - The main ideas
   - Some categorical results
   - An example

Pioneers of representation theory

Let G be a finite group.

Frobenius ∼1895++, Burnside ∼1900++. Representation theory is the ▸ useful? study of linear group actions:

$$\mathcal{M}\colon \mathrm{G} \longrightarrow \mathcal{E}\mathrm{nd}(\mathtt{V}), \quad \text{“}\mathcal{M}(g) = \text{a matrix in } \mathcal{E}\mathrm{nd}(\mathtt{V})\text{”}$$

with $\mathtt{V}$ being some $\mathbb{C}$-vector space. We call $\mathtt{V}$ a module or a representation.

The “atoms” of such an action are called simple.

Maschke ∼1899. All modules are built out of simples (“Jordan–Hölder”).

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We want to have a categorical version of this!

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Let A be a finite-dimensional algebra.

Noether ∼1928++. Representation theory is the useful? study of algebra actions:

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"$\mathcal{M}(a)$ = a matrix in $\mathcal{E}\mathrm{nd}(\mathtt{V})$"

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We want to have a categorical version of this.

I am going to explain what we can do at present.

The strategy

"Groups, as men, will be known by their actions." – Guillermo Moreno

The study of group actions is of fundamental importance in mathematics and related field. Sadly, it is also very hard.

Representation theory approach. The analogous linear problem of classifying G-modules has a satisfactory answer for many groups.

Problem involving
a group action
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$$\begin{array}{ccc} \text{Problem involving} & & \text{Problem involving} \\ \text{a group action} & \cdots\cdots\cdots\rightarrow & \text{a linear group action} \\ \mathrm{G} \circlearrowright \mathtt{X} & & \mathbb{C}[\mathrm{G}] \circlearrowright \mathbb{C}\mathtt{X} \end{array}$$

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Problem involving a group action $\mathrm{G} \circlearrowright \mathtt{X}$ ⇢ Problem involving a linear group action $\mathbb{C}[\mathrm{G}] \circlearrowright \mathbb{C}\mathtt{X}$

⇣

"Decomposition of the problem" $\mathbb{C}[\mathrm{G}] \circlearrowright \bigoplus \mathtt{V}_i$

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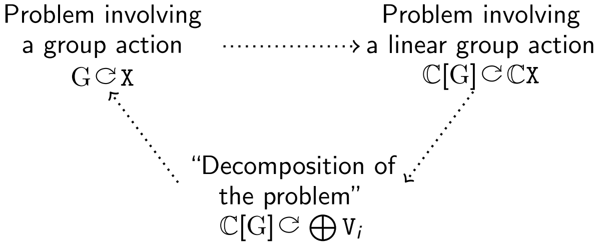

Philosophy. Turn problems into linear algebra.

Some theorems in classical representation theory

- ▷ All G-modules are built out of simples.
- ▷ The character of a simple G-module determines it.
- ▷ There is a one-to-one correspondence

$$\{\text{simple } \mathrm{G}\text{-modules}\}/\text{iso} \xleftrightarrow{1:1} \{\text{conjugacy classes in } \mathrm{G}\}.$$

- ▷ All simples can be constructed intrinsically using the regular G-module.

Some theorems in classical representation theory

- ▷ All G-modules are built out of simples.
- ▷ The character of a simple G-module determines it.

The ▸ character only remembers the traces of the acting matrices.

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"Regular G-module $= \mathrm{G}$ acting on itself."

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Find categorical versions of these facts.

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The dihedral groups on one slide

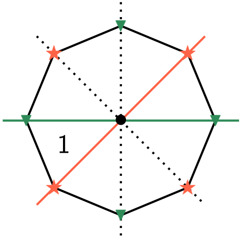
Coxeter groups have Kazhdan–Lusztig theory which makes them much easier form the categorical point of view.

The dihedral groups are of Coxeter type $\mathrm{I}_2(e+2)$:

$$\mathrm{W}_{e+2} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1, \underbrace{\ldots \mathtt{sts}}_{e+2} = w_0 = \underbrace{\ldots \mathtt{tst}}_{e+2} \rangle,$$

$$\text{e.g.: } \mathrm{W}_4 = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1, \mathtt{tsts} = w_0 = \mathtt{stst} \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, e.g. for $e=2$ the Coxeter complex is:



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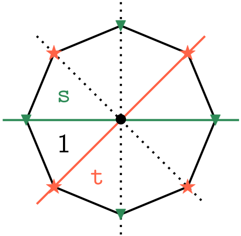

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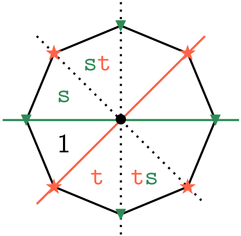

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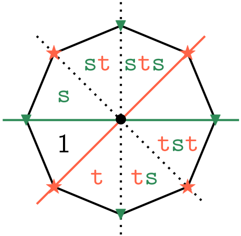

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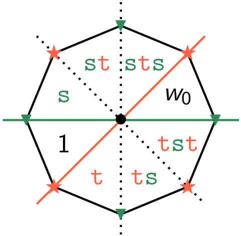

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One-dimensional representations. $\mathcal{M}_{\lambda_s,\lambda_t}$, $s \mapsto \lambda_s \in \mathbb{C}$, $t \mapsto \lambda_t \in \mathbb{C}$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
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Examp

the Co

Two-dimensional representations. $\mathcal{M}_z$, $z \in \mathbb{R}$, $s \mapsto \begin{pmatrix} 1 & z \\ 0 & -1 \end{pmatrix}$, $t \mapsto \begin{pmatrix} -1 & 0 \\ z & 1 \end{pmatrix}$.

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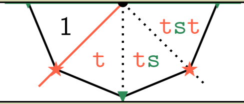
$\mathcal{M}_z$, z pos. root of U_{e+1}	$\mathcal{M}_z$, z pos. root of U_{e+1}

U_{e+1} is the Chebyshev polynomial.

$= 2$



Proposition (Lusztig?).
All of these are simple, and the list is complete and irredundant.

Pioneers of 2-representation theory

Let G be a finite group.

Plus some coherence conditions which I will not explain.

Chuang–Rouquier & many others ∼2004++. Higher representation theory is the useful? study of (certain) categorical actions, e.g.:

$$\mathscr{M}\colon \mathrm{G} \longrightarrow \mathscr{E}\mathrm{nd}(\mathcal{V}),$$ "$\mathscr{M}(g)$ = a functor in $\mathscr{E}\mathrm{nd}(\mathcal{V})$"

with $\mathcal{V}$ being some $\mathbb{C}$-linear category. We call $\mathcal{V}$ a 2-module or a 2-representation.

The "atoms" of such an action are called 2-simple.

Mazorchuk–Miemietz ∼2014. All (suitable) 2-modules are built out of 2-simples ("2-Jordan–Hölder").

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Let $\mathscr{C}$ be a finitary 2-category. Why?

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The three goals of 2-representation theory.
Improve the theory itself.
Discuss examples.
Find applications.

"Lifting" classical representation theory

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- All simples can be constructed intrinsically using the regular G-module.

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- **Mazorchuk–Miemietz ∼2014.** All (suitable) 2-modules are built out of 2-simples.

  Note that we have a very particular notion what a "suitable" 2-module is.

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- ▷ **Mazorchuk–Miemietz ∼2014.** All (suitable) 2-modules are built out of 2-simples.
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- ▷ There is a one-to-one What characters were for Frobenius are these matrices for us.

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$$\{\text{2-simples of } \mathscr{C}\}/\text{equi.}$$
$$\xleftrightarrow{1:1}$$
$$\{\text{certain (co)algebra 1-morphisms}\}/\text{“2-Morita equi.”}.$$

There are some technicalities.

- ▷ All simples can be constructed intrinsically using the regular G-module.

Goal 1. Improve the theory itself.

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  Several authors including myself ∼2016. But even in well-behaved cases there are 2-simples which do not arise in this way.

These turned out to be very interesting
since their importance is only visible via categorification.

2-modules of dihedral groups

$$\text{Consider:} \quad \theta_{\mathtt{s}} = \mathtt{s} + 1, \qquad \theta_{\mathtt{t}} = \mathtt{t} + 1.$$

(Motivation. The Kazhdan–Lusztig basis has some neat integral properties.)

These elements generate $\mathbb{C}[\mathrm{W}_{e+2}]$ and their relations are fully understood:

$$\theta_{\mathtt{s}}\theta_{\mathtt{s}} = 2\theta_{\mathtt{s}}, \qquad \theta_{\mathtt{t}}\theta_{\mathtt{t}} = 2\theta_{\mathtt{t}}, \qquad \text{a relation for } \underbrace{\dots\mathtt{sts}}_{e+2} = \underbrace{\dots\mathtt{tst}}_{e+2}.$$

We want a categorical action. So we need:

- ▷ A category $\mathcal{V}$ to act on.
- ▷ Endofunctors $\Theta_{\mathtt{s}}$ and $\Theta_{\mathtt{t}}$ acting on $\mathcal{V}$.
- ▷ The relations of $\theta_{\mathtt{s}}$ and $\theta_{\mathtt{t}}$ have to be satisfied by the functors.
- ▷ A coherent choice of natural transformations. (Skipped today.)

▸ Some details.

2-modules of dihedral groups

Consider: $\theta_s = s + 1, \qquad \theta_t = t + 1.$

(Motivatior ... rties.)

Mackaay–T. ∼2016.

There is a one-to-one correspondence

$\{$(non-trivial) 2-simple W_{e+2}-modules$\}$/2-iso

$\overset{1:1}{\longleftrightarrow}$

$\{$bicolored ADE Dynkin diagrams with Coxeter number $e+1\}$.

Thus, its easy to write down a list.

These elem ... ood:

$\theta_s\theta$ ... st.

We want a

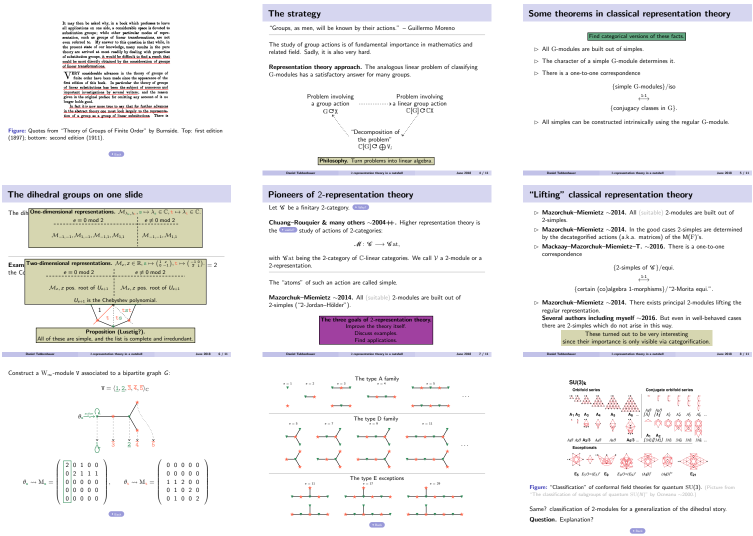
- ▷ A category $\mathcal{V}$ to act on.
- ▷ Endofunctors Θ_s and Θ **Goal 2.** Discuss examples.
- ▷ The relations of θ_s and θ_t have to be satisfied by the functors.
- ▷ A coherent choice of natural transformations. (Skipped today.)

Some details.

Concluding remarks – let me dream a bit

- ▷ The theory is still not fully developed.
  Goal 1 question. Are there finitely many 2-simples in general?
- ▷ The dihedral story is just the tip of the iceberg.
  Goal 2 question. Finite Coxeter groups in general?
- ▷ The connection to low-dimensional topology needs to be worked out.
  Goal 3 question. Impact on non-semisimple invariants of 3-manifolds?

- ▶ Connections to the study of braid groups, web calculi and geometry of Grassmanians, following **Khovanov–Seidel**, **Kuperberg**, **Cautis–Kamnitzer–Morrison**,... ▸ Click
- ▶ Connections to conformal field theory following ideas of **Zuber**,... ▸ Click
- ▶ Connections to the theory of subfactors, fusion categories (q-groups at roots of unity) etc. **à la Etingof–Gelaki–Nikshych–Ostrik, Ocneanu**,... ▸ Click



There is still much to do...

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

Back

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"Decomposition of the problem" $\mathbb{C}[G] \circlearrowright \bigoplus V_i$

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Find categorical versions of these facts.

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U_{e+1} is the Chebyshev polynomial.

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Several authors including myself ∼2016. But even in well-behaved cases there are 2-simples which do not arise in this way.

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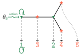
Construct a W_∞-module V associated to a bipartite graph G:

$$V = \langle 1, 2, 3, 4, 5 \rangle_{\mathbb{C}}$$



$$\theta_s \rightsquigarrow M_s = \begin{pmatrix} 2 & 0 & 1 & 0 & 0\\ 0 & 2 & 1 & 1 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \theta_t \rightsquigarrow M_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 2 & 0 & 0\\ 0 & 1 & 0 & 2 & 0\\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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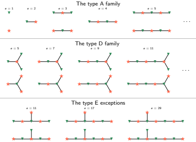

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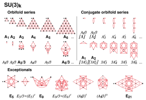

Figure: "Classification" of conformal field theories for quantum SU(3). (Picture from "The classification of subgroups of quantum SU(N)" by Ocneanu ∼2000.)

Same? classification of 2-modules for a generalization of the dihedral story.
Question. Explanation?

Back

Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

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Nowadays representation theory is pervasive across fields of mathematics, and beyond.

VERY considerable advances in the theory of groups of

But this wasn't clear at all when Frobenius started it.

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Back

Khovanov & others ∼1999++. Knot homologies are instances of 2-representation theory. Low-dim. topology & Math. Physics

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Khovanov–Seidel & others ∼2000++. Faithful 2-representations of braid groups. Low-dim. topology & Symplectic geometry

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Chuang–Rouquier ∼2004. Proof of the Broué conjecture using 2-representation theory. p-RT of finite groups & Geometry & Combinatorics

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Elias–Williamson ∼2012. Proof of the Kazhdan–Lusztig conjecture using ideas from 2-representation theory. Combinatorics & RT & Geometry

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Riche–Williamson ∼2015. Tilting characters using 2-representation theory. p-RT of reductive groups & Geometry

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Many more...

Back

Khovanov & others ∼1999++. Knot homologies are instances of 2-representation theory. Low-dim. topology & Math. Physics

Goal 3. Find application.

Khovanov–Seidel & others ... -representations of braid groups. Low-dim. t...

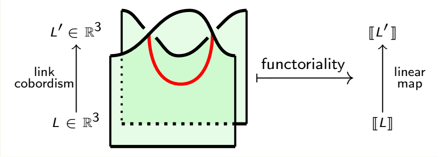
Chuang–Rouquier ... sing 2-representation theory. p-RT of fin...

Elias–Williamson ... njecture using ideas from 2-representatio... ...try

Riche–Williamson ... entation theory. p-RT of reductive g...

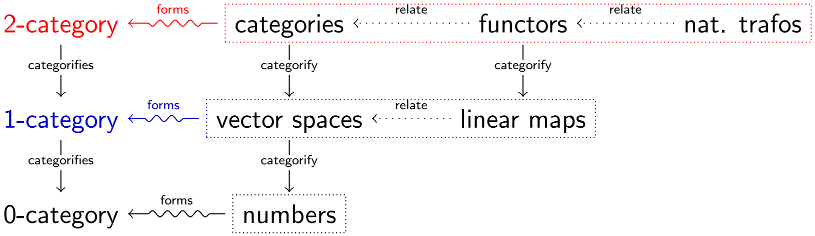
Many more...

In joint work with Ehrig–Wedrich ∼2017 we proved the functoriality of Khovanov–Rozansky's invariants.



(This was conjectured from about 10 years, but seemed infeasible to prove, and has some impact on 4-dim. topology.)
One of our main ingredient? 2-representation theory.

Back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forms
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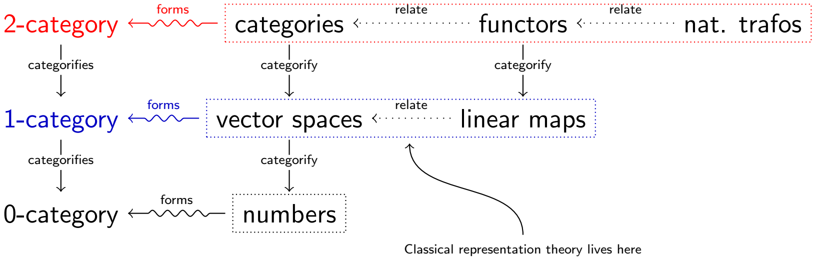
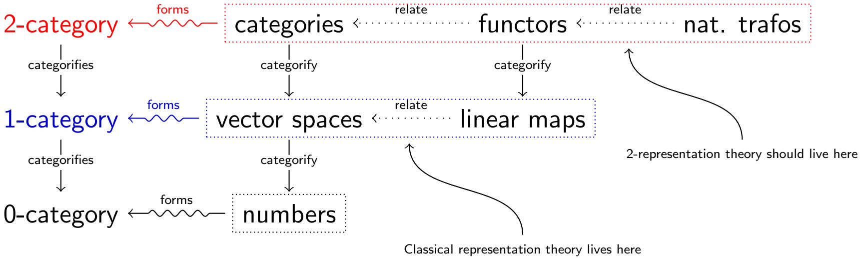


A group G can be viewed as an one-object category $\mathcal{G}$, and a representation as a functor from $\mathcal{G}$ into the one-object category $\mathrm{End}(\mathtt{V})$, i.e. $\mathcal{M}\colon \mathcal{G} \longrightarrow \mathrm{End}(\mathtt{V})$.

Back



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vector spaces
linear maps
2-representation theory should live here
categorifies
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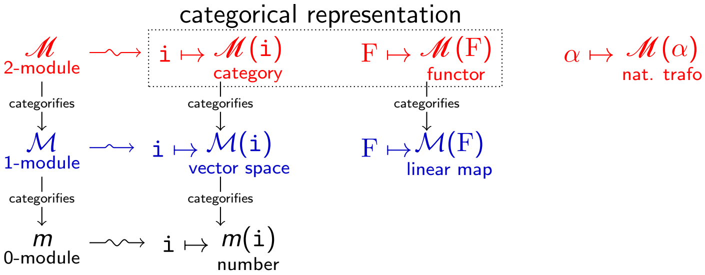
forms
0-category
numbers
Classical representation theory lives here

◂ Back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2-module
$\mathtt{i} \mapsto \mathscr{M}(\mathtt{i})$
category
$\mathrm{F} \mapsto \mathscr{M}(\mathrm{F})$
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$\mathscr{M}$
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What one can hope for.

Problem involving a group action $G \circlearrowright X$

“lift”

Problem involving a categorical group action

Decomposition of the problem into 2-simples

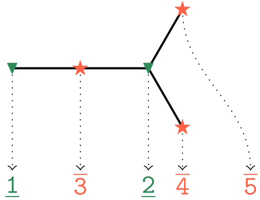
new insights?

$\mathscr{M}(\alpha)$
at. trafo

Back

Construct a W_∞-module $\mathtt{V}$ associated to a bipartite graph G:

$$\mathtt{V} = \langle \underline{1}, \underline{2}, \overline{3}, \overline{4}, \overline{5} \rangle_{\mathbb{C}}$$



$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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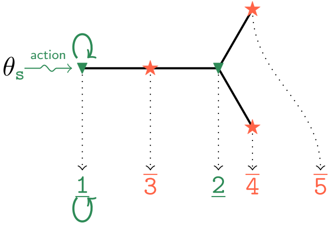

$$\theta_s \rightsquigarrow \mathrm{M}_s = \begin{pmatrix} \boxed{2} & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_t \rightsquigarrow \mathrm{M}_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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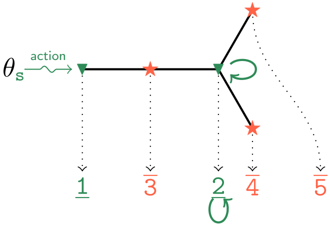

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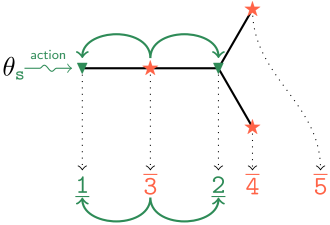

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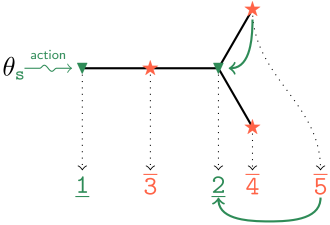

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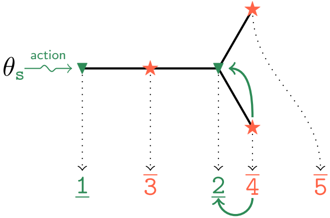

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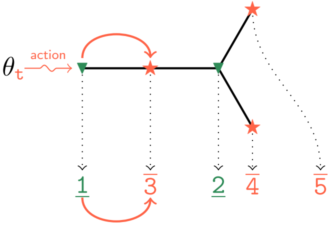

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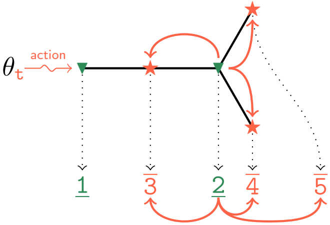

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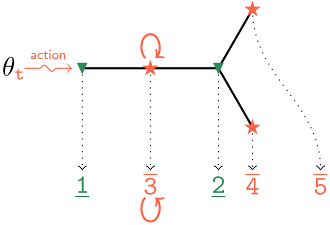

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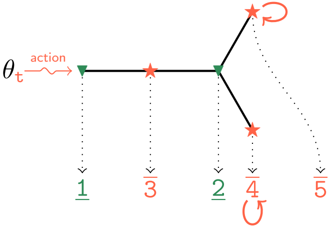

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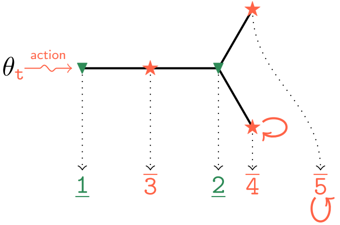

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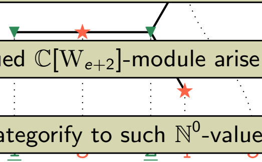
Lemma. For certain values of e these are $\mathbb{N}^0$-valued $\mathbb{C}[\mathrm{W}_{e+2}]$-modules.



Lemma. All $\mathbb{N}^0$-valued $\mathbb{C}[\mathrm{W}_{e+2}]$-module arise in this way.

Lemma. All 2-modules decategorify to such $\mathbb{N}^0$-valued $\mathbb{C}[\mathrm{W}_{e+2}]$-module.

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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Categorification.

Category $\rightsquigarrow \mathcal{V} = \mathrm{Z}\text{-}\mathcal{M}\mathrm{od}$,
Z quiver algebra with underlying graph G.

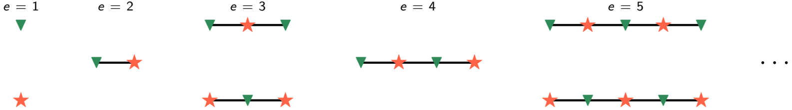
Endofunctors $\rightsquigarrow$ tensoring with Z-bimodules.

Lemma. These satisfy the relations of $\mathbb{C}[\mathrm{W}_e]$.

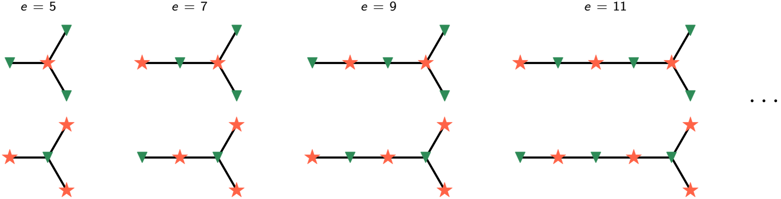
$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Back

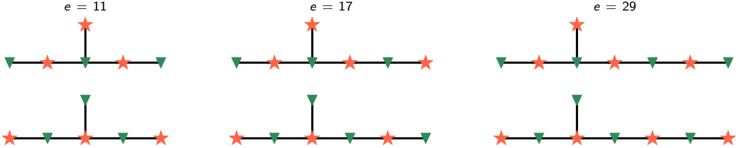
The type A family



The type D family



The type E exceptions



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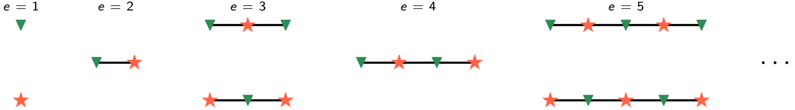

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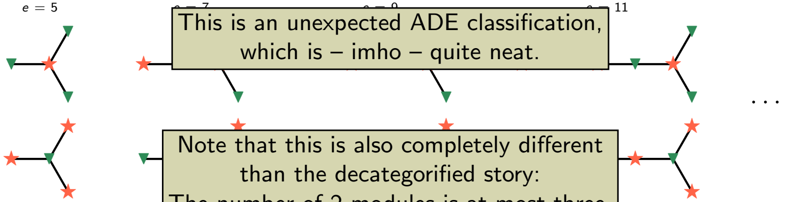
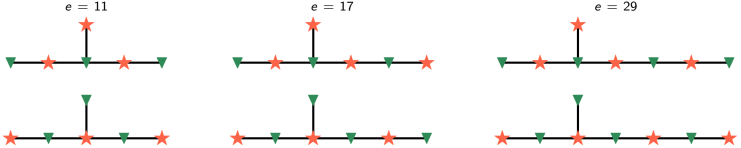


This is an unexpected ADE classification, which is – imho – quite neat.

Note that this is also completely different than the decategorified story:
The number of 2-modules is at most three, but they grow in dimension when e grows.



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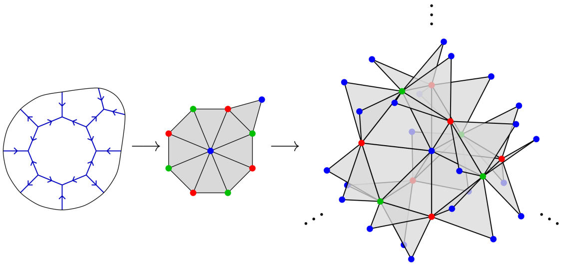

Figure: From spiders to $\mathrm{Cat}(0)$-diskoid to affine buildings. (Picture from "Buildings, spiders, and geometric Satake" by Fontaine–Kamnitzer–Kuperberg ∼2012.)

Spiders are special cases of our story, and also use them in some proofs. Spiders are known to be related to e.g. $\mathrm{Cat}(0)$-geometry.

Question. Anything one can say about this geometry using 2-modules?

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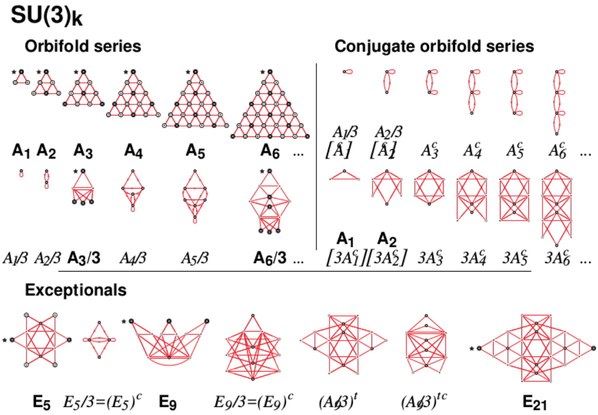

Figure: "Classification" of conformal field theories for quantum $\mathrm{SU}(3)$. (Picture from "The classification of subgroups of quantum $\mathrm{SU}(N)$" by Ocneanu ~2000.)

Same? classification of 2-modules for a generalization of the dihedral story.

Question. Explanation?

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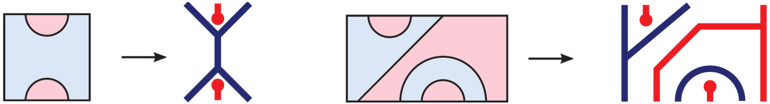

Figure: The quantum Satake; from Temperley–Lieb to Soergel bimodules. (Picture from "The two-color Soergel calculus" by Elias ∼2013.)

Elias' quantum Satake correspondence shows that the Soergel bimodules of dihedral type "are a non-semisimple generalization of semisimplyfied $\mathrm{U}_q(\mathfrak{sl}_2)$-$\mathcal{M}\mathrm{od}$ at roots of unity". (This works in more generality.)

Question. Is there impact for both sides?

Back

samen Factor f abgesehen) einen relativen Charakter von $\mathfrak{H}$, und umgekehrt lässt sich jeder relative Charakter von $\mathfrak{H}$, $\chi_0, \cdots \chi_{k-1}$, auf eine oder mehrere Arten durch Hinzufügung passender Werthe $\chi_k, \cdots \chi_{k'-1}$ zu einem Charakter von $\mathfrak{H}'$ ergänzen.

§ 8.

Ich will nun die Theorie der Gruppencharaktere an einigen Beispielen erläutern. Die geraden Permutationen von 4 Symbolen bilden eine Gruppe $\mathfrak{H}$ der Ordnung $h=12$. Ihre Elemente zerfallen in 4 Classen, die Elemente der Ordnung 2 bilden eine zweiseitige Classe (1), die der Ordnung 3 zwei inverse Classen (2) und (3) = (2′). Sei ρ eine primitive cubische Wurzel der Einheit.

Tetraeder. $h=12$.

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	h_α
χ_0	1	3	1	1	1
χ_1	1	−1	1	1	3
χ_2	1	0	ρ	ρ^2	4
χ_3	1	0	ρ^2	ρ	4

Figure: "Über Gruppencharaktere (i.e. characters of groups)" by Frobenius (1896). Bottom: first ever published character table.

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