2-representations of Soergel bimodules

Or: Mind your twists

Daniel Tubbenhauer

$\mathscr{R}\mathrm{ep}(S_5)$

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/5\mathbb{Z}$	S_3	$\mathbb{Z}/6\mathbb{Z}$	D_4	D_5	A_4	D_6	$GA(1,5)$	S_4	A_5	S_5
#	1	2	1	1	2	1	2	1	1	1	1	1	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	4, 1	5	3	6	5, 2	4, 2	4, 3	6, 3	5	5, 3	5, 4	7, 5

March 2020

Slogan. Representation theory is group theory in vector spaces.

Let A be a finite-dimensional algebra, *e.g.* a group ring $\mathbb{K}[\mathrm{G}]$.

Frobenius ∼1895++ Representation theory is the useful? study of actions of algebras:

$$\mathcal{M}\colon \mathrm{A} \longrightarrow \mathcal{E}\mathrm{nd}(\mathtt{V}),$$

with $\mathtt{V}$ being some vector space. (Called modules or representations.)

The "elements" of such an action are called simple.

Maschke ∼1899. All modules are built out of simples ("Jordan–Hölder" filtration).

Main goal of representation theory. Find the periodic table of simples.

Slogan. 2-representation theory is group theory in categories.

Let $\mathscr{C}$ be a (suitable) 2-category, *e.g.* a monoidal category.

Etingof–Ostrik, Chuang–Rouquier, many others ∼2000++. Higher representation theory is the useful? study of actions of 2-categories:

$$\mathbf{M}\colon \mathscr{C} \longrightarrow \mathscr{E}\mathrm{nd}(\mathcal{V}),$$

with $\mathcal{V}$ being some (suitable) category. (Called 2-modules or 2-representations.)

The "elements" of such an action are called 2-simple.

Mazorchuk–Miemietz ∼2014. All (suitable) 2-modules are built out of 2-simples ("weak 2-Jordan–Hölder filtration").

Main goal of 2-representation theory. Find the periodic table of 2-simples.

Examples of 2-categories.

Monoidal categories, G-graded vector spaces $\mathscr{V}\mathrm{ec}(G)$, module categories $\mathscr{R}\mathrm{ep}(G)$,

$\mathscr{R}\mathrm{ep}$(Hopf algebra), tensor or fusion or modular categories,

Soergel bimodules $\mathscr{S} = \mathscr{S}(W)$ (“the Hecke category”),

categorified quantum groups, categorified Heisenberg algebras, ...

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Examples of 2-representation of these.

Categorical modules, functorial actions,

(co)algebra objects, conformal embeddings of affine Lie algebras, the LLT algorithm,

cyclotomic Hecke/KLR algebras, categorified (anti-)spherical module, ...

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Applications of 2-representations.

Representation theory (classical and modular), link homologies, combinatorics,

TQFTs, quantum physics, geometry, ...

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The "elemen...

Today: example based.

1) Fix some notation.

2) Discuss the cases $\mathscr{V}\mathrm{ec}(G)$ and $\mathscr{R}\mathrm{ep}(G)$. Semisimple

3) Discuss the case of the Hecke category. Non-semisimple

Mazorchuk–Miemietz ∼2014. All (suitable) 2-modules are built out of 2-simples ("weak 2-Jordan–Hölder filtration").

Main goal of 2-representation theory. Find the periodic table of 2-simples.

Classification problems are impossible unless you restrict yourself.

In classical representation theory one would:

Specify what should be represented,
e.g. groups, algebra, Lie groups, Lie algebras, *etc.*

Specify where one wants to represent,
e.g. on finite-dimensional vector spaces, unitary representation *etc.*

In 2-representation theory one needs do the same.

What we want to represent.

A finitary category $\mathcal{C}$ is "linear-finite":

- It is linear, additive and idempotent split.
- It has finitely many indecomposable objects (up to $\cong$).
- It has finite-dimensional hom-spaces.

A finitary 2-category $\mathscr{C}$ is also "linear-finite":

- It has finitely many objects and its hom-categories are finitary.
- The horizontal composition of 2-morphisms is bilinear.
- The identity 1-morphisms are indecomposable.

One also needs dualities, so we add "rigid":

- If additionally there is an object-preserving, linear biequivalence $^{\star}\colon \mathscr{C} \to \mathscr{C}^{\,\mathrm{coop}}$ of finite order, then $\mathscr{C}$ is called weakly fiat. (Fiat=order two.)
- Weakly fiat + semisimple is called fusion.

The Grothendieck ring $[\mathscr{C}(\mathtt{i},\mathtt{i})]$ of such $\mathscr{C}$ is a finite-dimensional algebra.

What we want to represent.

$\mathscr{A}_{\mathbb{K}}^{f}$: 2-category of finitary categories, linear functors and natural transformations.

A (left) finitary 2-representation of $\mathscr{C}$ is a linear 2-functor $\mathbf{M}\colon \mathscr{C} \to \mathscr{A}_{\mathbb{K}}^{f}$.

Concretely, it associates:

- A finitary category $\mathbf{M}(\mathtt{i})$ to each object $\mathtt{i}$.
- A linear functor $\mathbf{M}(\mathrm{F})$ to each 1-morphism F.
- A natural transformation $\mathbf{M}(\alpha)$ to each 1-morphism α.

The Grothendieck group $[\mathbf{M}(\mathtt{i})]$ is a module of $[\mathscr{C}(\mathtt{i},\mathtt{i})]$.

$[\mathbf{M}(\mathrm{F})]$ are $\mathbb{N}$-valued matrices in $\mathcal{E}\mathrm{nd}([\mathbf{M}])$.

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The $\ldots, \mathtt{i})]$.

For simplicity, let us stay with monoidal categories,

a.k.a. 2-categories with one object,

for the rest of the talk.

$[\mathbf{M}(\mathrm{F})]$ are $\mathbb{N}$-valued matrices in $\mathcal{E}\mathrm{nd}([\mathbf{M}])$.

Example ($\mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z})$: the skeleton of $\mathbb{Z}/2\mathbb{Z}$-graded ($\mathbb{C}$-)vector spaces).

- As a category $\mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z})$ is boring: two objects and no non-trivial homs.

$$\mathrm{id}_1 \circlearrowright 1 \qquad\qquad \text{-}1 \circlearrowleft \mathrm{id}_{\text{-}1}$$

- As a monoidal category this is not much more exciting:

$$\mathtt{a} \otimes \mathtt{b} = \mathtt{ab}, \quad \mathrm{id}_{\mathrm{a}} \otimes \mathrm{id}_{\mathrm{a}} = \mathrm{id}_{\mathrm{ab}}.$$

- As a fusion category this is still not complicated:

close $\mathbb{C}$-linear, take $\oplus$-sums and let $\mathtt{a}^{\star} = \mathtt{a}^{-1}$.

(I will write $\mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z})$ for the above and its linear and additive closure.)

- Clearly, $[\mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z})] \cong \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$.

Example ($\mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z})$: the skeleton of $\mathbb{Z}/2\mathbb{Z}$-graded ($\mathbb{C}$-)vector spaces).

The fusion category $\mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z})$ has two evident 2-modules:

- The trivial 2-module $\mathcal{V}(1,1)$ given by the trivial 2-representation

$$\mathbf{M}\colon \mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z}) \to \mathscr{V}\mathrm{ec}, \quad \text{“forget } \mathbb{Z}/2\mathbb{Z}\text{-grading”}.$$

  The $\mathbb{N}$-matrices are $1,\text{-}1 \rightsquigarrow \left(\begin{smallmatrix} 1 \end{smallmatrix}\right)$.

- The regular 2-module $\mathcal{V}(\mathbb{Z}/2\mathbb{Z},1)$ given by the regular 2-representation

$$\mathbf{M}\colon \mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z}) \to \mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z}), \quad \mathbf{M}(\mathrm{a}) = \mathrm{a} \otimes \_.$$

  The $\mathbb{N}$-matrices are $1 \rightsquigarrow \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\text{-}1 \rightsquigarrow \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

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Theorem (folklore?).

$\boxed{\text{Completeness.}}$ All 2-simples of $\mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z})$ are of the form $\mathcal{V}(1,1)$ or $\mathcal{V}(\mathbb{Z}/2\mathbb{Z},1)$.

$\boxed{\text{Non-redundancy.}}$ These are non-equivalent.

$$\mathbf{M}\colon \mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z}) \to \mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z}), \quad \mathbf{M}(\mathrm{a}) = \mathrm{a} \otimes \_.$$

The $\mathbb{N}$-matrices are $1 \rightsquigarrow \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and -1 $\rightsquigarrow \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

Note that $\mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z})$ has only finitely many 2-simples.

This is no coincidence.

Example ($\mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z})$: the skeleton of $\mathbb{Z}/2\mathbb{Z}$-graded ($\mathbb{C}$-)vector spaces).

- One can twist the $\otimes$ by a sign:

$$\mathrm{id}_{\text{-}1} \otimes \mathrm{id}_{\text{-}1} = -\mathrm{id}_1,$$

  and get another fusion category $\mathscr{V}\mathrm{ec}^{\omega}(\mathbb{Z}/2\mathbb{Z})$.
- $\mathscr{V}\mathrm{ec}^{\omega}(\mathbb{Z}/2\mathbb{Z})$ is skeletal with non-trivial associator.
- There is no trivial 2-module $\mathcal{V}(1,1)$ since $\mathscr{V}\mathrm{ec}$ has a trivial associator. However, $\mathcal{V}(\mathbb{Z}/2\mathbb{Z},1)$ still makes sense.
- Moreover, $[\mathscr{V}\mathrm{ec}^{\omega}(\mathbb{Z}/2\mathbb{Z})] \cong \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$.

Example ($\mathscr{V}\mathrm{ec}(\mathbb{Z}/2\mathbb{Z})$: the skeleton of $\mathbb{Z}/2\mathbb{Z}$-graded ($\mathbb{C}$-)vector spaces).

- C

  a

Theorem (folklore?).

Completeness. All 2-simples of $\mathscr{V}\mathrm{ec}^{\omega}(\mathbb{Z}/2\mathbb{Z})$ are of the form $\mathcal{V}(\mathbb{Z}/2\mathbb{Z},1)$.

Non-redundancy. (I have nothing to say in this case...)

- $\mathscr{V}\mathrm{ec}^{\omega}(\mathbb{Z}/2\mathbb{Z})$ is skeletal with non-trivial associator.
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Note that $\mathscr{V}\mathrm{ec}^{\omega}(\mathbb{Z}/2\mathbb{Z})$ has only finitely many 2-simples.

This is no coincidence.

Note: twisting, even in this toy example,

is non-trivial and affects the 2-representation theory.

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$ for G a finite group.
- For any $\mathtt{M}, \mathtt{N} \in \mathscr{C}$, we have $\mathtt{M} \otimes \mathtt{N} \in \mathscr{C}$:

$$g(m \otimes n) = gm \otimes gn$$

  for all $g \in G$, $m \in \mathtt{M}$, $n \in \mathtt{N}$. There is a trivial representation $\mathbb{1} = \mathbb{C}$.
- Thus, $\mathscr{C}$ is fusion.
- Example: the regular 2-representation $\mathbf{M}\colon \mathscr{C} \to \mathscr{E}\mathrm{nd}(\mathscr{C})$ is

$$\begin{tikzcd} \mathtt{M} \arrow[r] \arrow[d, "f"'] & \mathtt{M} \otimes \_ \arrow[d, "f \otimes \_"] \\ \mathtt{N} \arrow[r] & \mathtt{N} \otimes \_ \end{tikzcd} .$$

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $K \subset G$ be a subgroup.
- $\mathcal{R}\mathrm{ep}(K)$ is a 2-module of $\mathscr{R}\mathrm{ep}(G)$, with 2-action

$$\mathbf{Res}_K^G \otimes \_ \colon \mathscr{R}\mathrm{ep}(G) \to \mathscr{E}\mathrm{nd}\big(\mathcal{R}\mathrm{ep}(K)\big),$$

  which is indeed a 2-action because $\mathbf{Res}_K^G$ is a $\otimes$-functor.
- In words, $\mathbf{Res}_K^G \otimes \_$ assigns to simple G-modules endofunctors on $\mathcal{R}\mathrm{ep}(K)$.
- The decategorifications of these endofunctors are $\mathbb{N}$-valued matrices. ▸ Example

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $\psi \in H^2(K, \mathbb{C}^*)$. Let $\mathcal{V}(K, \psi)$ be the category of projective K-modules with Schur multiplier ψ, *i.e.* vector spaces $\mathtt{V}$ with $\rho\colon K \to \mathcal{E}\mathrm{nd}(\mathtt{V})$ such that

$$\rho(g)\rho(h) = \psi(g, h)\rho(gh), \text{ for all } g, h \in K.$$

- Note that $\mathcal{V}(K, 1) = \mathcal{R}\mathrm{ep}(K)$ and

$$\otimes\colon \mathcal{V}(K, \phi) \boxtimes \mathcal{V}(K, \psi) \to \mathcal{V}(K, \phi\psi).$$

- $\mathcal{V}(K, \psi)$ is also a 2-representation of $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$:

$$\mathscr{R}\mathrm{ep}(G) \boxtimes \mathcal{V}(K, \psi) \xrightarrow{\mathbf{Res}_K^G \boxtimes \mathrm{Id}} \mathcal{R}\mathrm{ep}(K) \boxtimes \mathcal{V}(K, \psi) \xrightarrow{\otimes} \mathcal{V}(K, \psi).$$

- In short, one can twist the 2-representations $\mathbf{Res}_K^G$.

Example ($\mathscr{R}\mathrm{ep}(G)$).

Theorem (folklore?).

$\boxed{\text{Completeness.}}$ All 2-simples of $\mathscr{R}\mathrm{ep}(G)$ are of the form $\mathcal{V}(K,\psi)$.

$\boxed{\text{Non-redundancy.}}$ We have $\mathcal{V}(K,\psi) \cong \mathcal{V}(K',\psi')$

$\Leftrightarrow$

the subgroups are conjugate and $\psi' = \psi^g$, where $\psi^g(k,l) = \psi(gkg^{-1}, glg^{-1})$.

- ... es ... hat

- Note that $\mathcal{V}(K,1) = \mathcal{R}\mathrm{ep}(K)$ and

Note that $\mathscr{R}\mathrm{ep}(G)$ has only finitely many 2-simples.

This is no coincidence.

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Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $\psi \in H$ ... K-modules with Schu ... $\mathrm{d}(\mathtt{V})$ such that

Ocneanu rigidity (Etingof–Nikshych–Ostrik ∼2004).

If $\mathscr{C}$ is fusion, then it has only finitely many 2-simples.

Proof? Find a computable obstruction for twists.

- Note that $\mathcal{V}(K, 1) = \mathcal{R}\mathrm{ep}(K)$ and

Problems in general.

If $\mathscr{C}$ is non-semisimple, then things get complicated:

There can be uncountably (twists giving uncountably) many 2-simples. ▸ Example

Schur's lemma does not hold.

Twist are not given by any reasonable obstruction *etc.*

Is the case of Soergel bimodules hopeless?

Theorem (Soergel–Elias–Williamson $\sim$1990,2012).
There exists a non-semisimple, graded, fiat category $\mathscr{S}^v = \mathscr{S}^v(W)$ such that:

(1) For every $w \in W$, there exists an indecomposable object $\mathtt{C}_w$.

(2) The $\mathtt{C}_w$, for $w \in W$, form a complete set of pairwise non-isomorphic indecomposable objects up to shifts.

(3) The identity object is $\mathtt{C}_1$, where 1 is the unit in W.

(4) $\mathscr{S}^v$ categorifies the Hecke algebra with $[\mathtt{C}_w] = c_w$ being the KL basis; forgetting the grading $[\mathscr{S}^v] \cong \mathbb{Z}[W]$

(5) $\mathrm{grdim}\big(\mathrm{hom}_{\mathscr{S}^v}(\mathtt{C}_v, v^k\mathtt{C}_w)\big) = \delta_{v,w}\delta_{0,k}$. (Soergel's hom formula *a.k.a.* positively graded.)

v degree, $W = (W, S)$ a (finite) Coxeter group, ground field $\mathbb{C}$, using the coinvariant algebra attached to the geometric representation.

Is the cas

Theorem

There exist ch that:

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(2) The C nic indecomposable objects up to shifts.

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Examples ($W = S_n$).

In this case $\mathscr{S}^v$ has $n!$ indecomposable objects up to shifts.

Beyond some very small cases, they may be difficult to describe.

The classification problem appears to be very hard.

Examples (W of type E_8).

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(5) grdim ... graded.)

v degree, ... the coinvariant

By the way: Why should one care, a.k.a. motivation for $\mathscr{S}^v$.

1) $\mathscr{S}^v$ categorifies the Hecke algebra.
Its 2-representation theory categorifies the representation theory of Hecke algebras.

2) $\mathscr{S}^v$ originates from projective functors acting on category $\mathcal{O}$, and $\mathrm{proj}(\mathcal{O}_0)$ is a 2-module of $\mathscr{S}^v$.
This was already used to solve questions in Lie theory.

3) $\mathscr{S}^v$ and its 2-representations appear in low-dimensional topology and we are working on applications therein.

4) $\mathscr{S}^v$ and its 2-representations appear in quantum and modular representation, which albeit needs affine Weyl groups.

5) $\mathscr{S}^v$ and its 2-representations are helpful to study braid groups as they tend to give faithful representations.

6) More...

The "crystal limit" (ignoring some details, sorry).

Theorem (Lusztig, Elias–Williamson ∼2012).
There exists a (multi)fusion bicategory $\mathscr{A}^0 = \mathscr{A}^0(W)$ such that:

(1) For every $w \in W$, there exists a simple object $\mathtt{A}_w$.

(2) The $\mathtt{A}_w$, for $w \in W$, form a complete set of pairwise non-isomorphic simple objects.

(3) The local identity objects are $\mathtt{A}_d$, where d are Duflo involutions.

(4) $\mathscr{A}^0$ categorifies the asymptotic Hecke algebra with $[\mathtt{A}_w] = a_w$ being the degree zero of the KL basis.

(5) $\mathscr{A}^0$ is the degree zero part of $\mathscr{S}^v$; roughly:

$$\mathscr{A}^0 = \mathrm{add}\big(\{v^k \mathtt{C}_w \mid w \in \mathcal{H}, k \geq 0\}\big)/\mathrm{add}\big(\{v^k \mathtt{C}_w \mid w \in \mathcal{H}, k > 0\}\big).$$

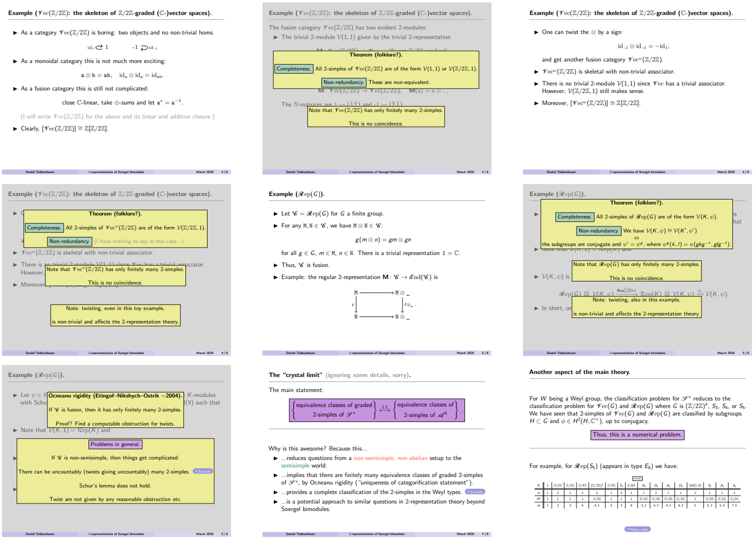
The "crystal limit" (ignoring some details, sorry).

The main statement:

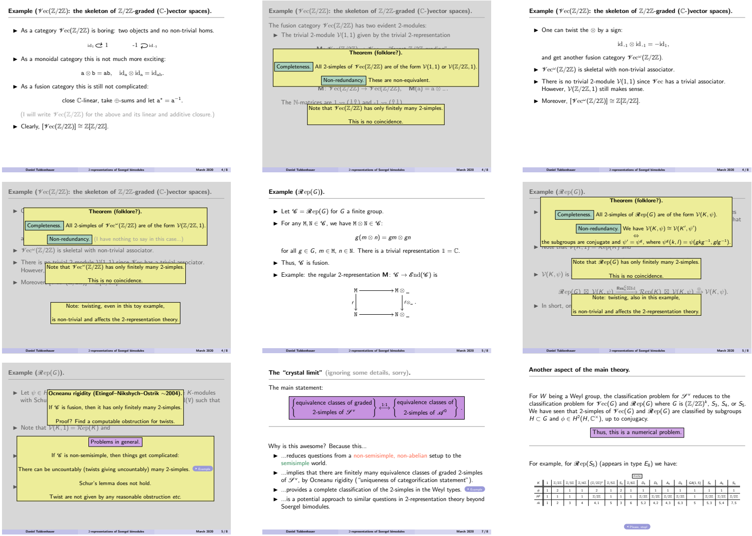
$$\left\{\begin{matrix}\text{equivalence classes of graded}\\ \text{2-simples of } \mathscr{S}^{v}\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{equivalence classes of}\\ \text{2-simples of } \mathscr{A}^{0}\end{matrix}\right\}.$$

Why is this awesome? Because this...

- ...reduces questions from a non-semisimple, non-abelian setup to the semisimple world.
- ...implies that there are finitely many equivalence classes of graded 2-simples of $\mathscr{S}^{v}$, by Ocneanu rigidity ("uniqueness of categorification statement").
- ...provides a complete classification of the 2-simples in the Weyl types. ▸ Example
- ...is a potential approach to similar questions in 2-representation theory beyond Soergel bimodules.



There is still much to do...



Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

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Nowadays representation theory is pervasive across mathematics, and beyond.

of linear transformations.

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But this wasn't clear at all when Frobenius started it.

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Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

Example ($G = S_3$).

$\#$ of subgroups (up to conjugacy), Schur multipliers H^2 and ranks rk of the 2-simples.

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	S_3
$\#$	1	1	1	1
H^2	1	1	1	1
rk	1	2	3	3

Example ($K = S_3$); the $\mathbb{N}$-matrices.

$\otimes$	$\ydiagram{3}$	$\ydiagram{2,1}$	$\ydiagram{1,1,1}$
$\ydiagram{3}$	$\ydiagram{3}$	$\ydiagram{2,1}$	$\ydiagram{1,1,1}$
$\ydiagram{2,1}$	$\ydiagram{2,1}$	$\ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1}$	$\ydiagram{2,1}$
$\ydiagram{1,1,1}$	$\ydiagram{1,1,1}$	$\ydiagram{2,1}$	$\ydiagram{3}$

$$\ydiagram{3} \otimes \_ \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \ydiagram{2,1} \otimes \_ \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \ydiagram{1,1,1} \otimes \_ \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

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Example (Taft Hopf algebra).

Let $\mathrm{T}_2 = \mathbb{C}\langle g, x\rangle/(g^2 = 1,\ x^2 = 0,\ gx = -xg) \overset{\mathsf{vs}}{\cong} \mathbb{C}[\mathbb{Z}/2\mathbb{Z}] \otimes \mathbb{C}[x]/(x^2)$.

- $\mathrm{T}_2\text{-proj}$ is a non-semisimple, weakly fiat category with $[\mathrm{T}_2\text{-proj}] \cong \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$.
- It has only two indecomposable objects:

$$\begin{array}{c}\text{1-dim.}\\ \text{simples}\end{array} : \; S_+, S_- \begin{cases} g.m = \pm m, \\ x.m = 0, \end{cases} \qquad \begin{array}{c}\text{2-dim.}\\ \text{pr.in.}\end{array} : \; P_+ = \begin{smallmatrix} S_+ \\ S_- \end{smallmatrix}, P_- = \begin{smallmatrix} S_- \\ S_+ \end{smallmatrix}.$$

- Two evident 2-simples $\mathcal{V}_\pm$ obtained via:

$$P_\pm \otimes \_ : \mathrm{T}_2\text{-proj} \to \mathrm{T}_2\text{-proj}.$$

Looks harmless, but:

- Twisted by $\lambda \in \mathbb{C}$ gives other 2-simples $\mathcal{V}_\pm^\lambda$.
- One gets two one-parameter families of 2-simples.
- $[\mathcal{V}_\pm^\lambda] \cong [\mathcal{V}_\pm^\mu]$, *i.e.* this is not detectable on the Grothendieck level.

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Another aspect of the main theory.

For W being a Weyl group, the classification problem for $\mathscr{S}^{v}$ reduces to the classification problem for $\mathscr{V}\mathrm{ec}(G)$ and $\mathscr{R}\mathrm{ep}(G)$ where G is $(\mathbb{Z}/2\mathbb{Z})^k$, S_3, S_4, or S_5. We have seen that 2-simples of $\mathscr{V}\mathrm{ec}(G)$ and $\mathscr{R}\mathrm{ep}(G)$ are classified by subgroups $H \subset G$ and $\phi \in H^2(H, \mathbb{C}^\times)$, up to conjugacy.

$\boxed{\text{Thus, this is a numerical problem.}}$

For example, for $\mathscr{R}\mathrm{ep}(S_5)$ (appears in type E_8) we have:

$\mathscr{R}\mathrm{ep}(S_5)$

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/5\mathbb{Z}$	S_3	$\mathbb{Z}/6\mathbb{Z}$	D_4	D_5	A_4	D_6	$GA(1,5)$	S_4	A_5	S_5
#	1	2	1	1	2	1	2	1	1	1	1	1	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	4, 1	5	3	6	5, 2	4, 2	4, 3	6, 3	5	5, 3	5, 4	7, 5

Please, stop!