

# On categories of tilting modules

Or: Mind your poles

Daniel Tubbenhauer

$$\boxed{v-1} = \text{Diagram 1} + (-1)^c \frac{[a,b,-c]_p}{[a,b,0]_p} \cdot \text{Diagram 2} + (-1)^{bp} \frac{[a,-b,0]_p}{[a,0,0]_p} \cdot \text{Diagram 3} + (-1)^{bp+c} \frac{[a,-b,-c]_p}{[a,0,0]_p} \cdot \text{Diagram 4}$$

The diagrams are:

- Diagram 1:** A vertical stack of three rectangular boxes connected by three vertical lines. The top and bottom boxes have dashed outlines.
- Diagram 2:** A vertical stack of three rectangular boxes connected by three vertical lines. The top and bottom boxes have dashed outlines. The top box has a curved line (arc) on its left side, labeled  $c$ . The bottom box has a curved line on its right side, labeled  $c$ .
- Diagram 3:** A vertical stack of three rectangular boxes connected by three vertical lines. The top and bottom boxes have dashed outlines. The top box has a curved line on its left side, labeled  $c$ . The bottom box has a curved line on its right side, labeled  $bp$ .
- Diagram 4:** A vertical stack of three rectangular boxes connected by three vertical lines. The top and bottom boxes have dashed outlines. The top box has a curved line on its left side, labeled  $c$ . The bottom box has a curved line on its right side, labeled  $bp$ . Additionally, there are two arcs connecting the top and bottom boxes: one on the left side and one on the right side.

Joint with Paul Wedrich

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We want to describe  $\mathcal{Tilt}(G)$  as a category, *i.e.* find an algebra  $Z$  such that there is an equivalence of additive,  $\mathbb{K}$ -linear categories

$$\mathcal{F}: \mathcal{Tilt}(G) \xrightarrow{\cong} \text{pMod-}Z_p,$$

sending indecomposable tilting modules to indecomposable projectives.

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**Goal. Describe  $\mathcal{Tilt}(G)$  by generators and relations.**

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I “do not care about objects”, but the point are morphisms and their relations.

**Folklore, Lucas ~1878.** Let  $q \in \mathbb{K}^*$ ,  $q \text{char}(\mathbb{K}) = p$ ,  $a = mp + a_0$  and  $b = np + b_0$  ( $a_0, b_0$  zeroth digit of the  $p$ -adic expansion). Then

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{pmatrix} m \\ n \end{pmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}_q.$$

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**Philosophy.** Only the vanishing order of  $\begin{bmatrix} v \\ w \end{bmatrix}_q$  matters for this lecture ;-).

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**Corollary.** We understand finite-dimensional modules for  $SL_2 = SL_2(\mathbb{K} = \overline{\mathbb{K}})$

- generically;
- for the quantum group over  $\mathbb{C}$  at  $q^{2\ell} = 1$ ;
- the quantum group over  $\mathbb{K}$ ,  $\operatorname{char}(\mathbb{K}) = p$  and  $q^{2\ell} = 1$  (mixed case);
- in prime characteristic  $\operatorname{char}(\mathbb{K}) = p$ .

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- in prime characteristic  $\text{char}(\mathbb{K}) = p$ .

**Example/Remark.**

$\mathbb{K} = \overline{\mathbb{F}}_p$ ,  $q = 1$  (known as characteristic  $p$ ),  
and  $a = [a_r, \dots, a_0]_p$ ,  $b = [b_r, \dots, b_0]_p$  (the  $p$ -adic expansions), then

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{bmatrix} a_r \\ b_r \end{bmatrix}_q \cdots \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}_q = \begin{pmatrix} a_r \\ b_r \end{pmatrix} \cdots \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}.$$

Folklore, Lucas  $\sim 1878$ . Let  $q \in \mathbb{K}^*$ ,  $q\text{char}(\mathbb{K}) = p$ ,  $a = mp + a_0$  and

**Examples for  $a = 1331 = 11^3$  and  $b = a - 1$ .**

If  $\mathbb{K} = \mathbb{C}$ ,  $q = 1$ , then  $q\text{char}(\mathbb{K}) = 0$ ,  $a = [1331]_0$  and  $b = [1330]_0$   
 $\Rightarrow [1331]_q = \binom{1331}{1330} = 1331$  does not vanish.

If  $\mathbb{K} = \mathbb{C}$ ,  $q = \exp(2\pi i/11)$ , then  $q\text{char}(\mathbb{K}) = 11$ ,  $a = [121, 0]_{11}$  and  $b = [120, 10]_{11}$   
 $\Rightarrow [1331]_q = 121 \cdot \begin{bmatrix} 0 \\ 10 \end{bmatrix}_q$  vanishes of order one.

If  $\mathbb{K} = \overline{\mathbb{F}}_{11}$ ,  $q = 3$ , then  $q\text{char}(\mathbb{K}) = 5$ ,  $a = [266, 1]_5$ ,  $b = [266, 0]_5$  and  $\frac{a-1}{5} = \frac{b}{5} = [2, 2, 2]_{11}$   
 $\Rightarrow [1331]_q = 1 \cdot 1 \cdot 1 \cdot [1]_q$  does not vanish.

If  $\mathbb{K} = \overline{\mathbb{F}}_{11}$ ,  $q = 1$ , then  $q\text{char}(\mathbb{K}) = 11$ ,  $a = [1, 0, 0, 0]_{11}$  and  $b = [0, 10, 10, 10]_{11}$   
 $\Rightarrow [1331]_q = 1 \cdot 0 \cdot 0 \cdot [0]_q$  vanishes of order three.

- the quantum group over  $\mathbb{K}$ ,  $\text{char}(\mathbb{K}) = p$  and  $q^p = 1$  (mixed case);
- in prime characteristic  $\text{char}(\mathbb{K}) = p$ .

### Example/Remark.

$\mathbb{K} = \overline{\mathbb{F}}_p$ ,  $q = 1$  (known as characteristic  $p$ ),  
 and  $a = [a_r, \dots, a_0]_p$ ,  $b = [b_r, \dots, b_0]_p$  (the  $p$ -adic expansions), then

$$\binom{a}{b} = \begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{bmatrix} a_r \\ b_r \end{bmatrix}_q \cdots \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}_q = \binom{a_r}{b_r} \cdots \binom{a_0}{b_0}.$$

**Folklore, Lucas ~1878.** Let  $q \in \mathbb{K}^*$ ,  $\text{char}(\mathbb{K}) = p$ ,  $a = mp + a_0$  and  $b = np + b_0$  ( $a_0, b_0$  zeroth digit of the  $p$ -adic expansion). Then

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{pmatrix} m \\ n \end{pmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}_q.$$

**Philosophy.** Only the vanishing order of  $\begin{bmatrix} v \\ w \end{bmatrix}_q$  matters for this lecture ;-).

**Corollary.** We have  $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{K} = \overline{\mathbb{K}}))$

- generically  $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{K} = \overline{\mathbb{K}}))$  is a quantum group
- for the quantum group  $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{K} = \overline{\mathbb{K}}))$  the quantum group is a “zeroth digit only” version of it;
- the quantum group  $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{K} = \overline{\mathbb{K}}))$  the mixed cases is a mixture of the two. (mixed case);
- in prime characteristic  $\text{char}(\mathbb{K}) = p$ .



Weyl  $\sim$ 1923. The  $SL_2$  Weyl modules  $\Delta(v-1)$ .

$\Delta(1-1)$

$x^0 y^0$

$\Delta(2-1)$

$x^1 y^0$

$x^0 y^1$

$\Delta(3-1)$

$x^2 y^0$

$x^1 y^1$

$x^0 y^2$

$\Delta(4-1)$

$x^3 y^0$

$x^2 y^1$

$x^1 y^2$

$x^0 y^3$

$\Delta(5-1)$

$x^4 y^0$

$x^3 y^1$

$x^2 y^2$

$x^1 y^3$

$x^0 y^4$

$\Delta(6-1)$

$x^5 y^0$

$x^4 y^1$

$x^3 y^2$

$x^2 y^3$

$x^1 y^4$

$x^0 y^5$

$\Delta(7-1)$

$x^6 y^0$

$x^5 y^1$

$x^4 y^2$

$x^3 y^3$

$x^2 y^4$

$x^1 y^5$

$x^0 y^6$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$  matrix whose columns are expansions of  $(aX + cY)^{v-i} (bX + dY)^{i-1}$ .

Example  $\Delta(7-1) = \mathbb{K}X^6Y^0 \oplus \dots \oplus \mathbb{K}X^0Y^6$ .

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  acts as

$a^6$	$6a^5c$	$15a^4c^2$	$20a^3c^3$	$15a^2c^4$	$6ac^5$	$c^6$
$a^5b$	$5a^4bc + a^5d$	$10a^3b^2c^2 + 5a^4cd$	$10a^2b^3c^3 + 10a^3c^2d$	$5ab^4c^4 + 10a^2c^3d$	$b^5c^5 + 5ac^4d$	$c^5d$
$a^4b^2$	$4a^3b^2c + 2a^4bd$	$6a^2b^2c^2 + 8a^3bcd + a^4d^2$	$4ab^2c^3 + 12a^2b^2cd + 4a^3c^2d^2$	$b^2c^4 + 8ab^3c^3d + 6a^2c^2d^2$	$2b^4c^4d + 4ac^3d^2$	$c^4d^2$
$a^3b^3$	$3a^2b^3c + 3a^3b^2d$	$3ab^3c^2 + 9a^2b^2cd + 3a^3bd^2$	$b^3c^3 + 9ab^2c^2d + 9a^2bcd^2 + a^3d^3$	$3b^2c^3d + 9ab^2c^2d^2 + 3a^2cd^3$	$3b^3c^3d^2 + 3ac^2d^3$	$c^3d^3$
$a^2b^4$	$2ab^4c + 4a^2b^3d$	$b^4c^2 + 8ab^3cd + 6a^2b^2d^2$	$4b^3c^2d + 12ab^2cd^2 + 4a^2bd^3$	$6b^2c^2d^2 + 8abc^3d^3 + a^2d^4$	$4b^2c^3d^3 + 2acd^4$	$c^2d^4$
$ab^5$	$b^5c + 5ab^4d$	$5b^4cd + 10ab^3d^2$	$10b^3cd^2 + 10ab^2d^3$	$10b^2cd^3 + 5abd^4$	$5bc^4d^4 + ad^5$	$cd^5$
$b^6$	$6b^5d$	$15b^4d^2$	$20b^3d^3$	$15b^2d^4$	$6bd^5$	$d^6$

The columns are expansions of  $(aX + cY)^{7-i}(bX + dY)^{i-1}$ . Binomials!

$\Delta(3-1)$

$\Delta(4-1)$

$\Delta(5-1)$

$\Delta(6-1)$

$\Delta(7-1)$

$X^2Y^0$

$X^1Y^1$

$X^0Y^2$

$X^3Y^0$

$X^2Y^1$

$X^1Y^2$

$X^0Y^3$

$X^4Y^0$

$X^3Y^1$

$X^2Y^2$

$X^1Y^3$

$X^0Y^4$

$X^5Y^0$

$X^4Y^1$

$X^3Y^2$

$X^2Y^3$

$X^1Y^4$

$X^0Y^5$

$X^6Y^0$

$X^5Y^1$

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$X^1Y^5$

$X^0Y^6$

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$a^5b$	$5a^4bc + a^5d$	$10a^3b^2c^2 + 5a^4cd$	$10a^2b^2c^3 + 10a^3c^2d$	$5ab^2c^4 + 10a^2c^3d$	$b^2c^5 + 5ac^4d$	$c^5d$
$a^4b^2$	$4a^3b^2c + 2a^4bd$	$6a^2b^2c^2 + 8a^3bcd + a^4d^2$	$4ab^2c^3 + 12a^2b^2cd + 4a^3c^2d^2$	$b^2c^4 + 8ab^2c^3d + 6a^2c^2d^2$	$2b^2c^4d + 4ac^3d^2$	$c^4d^2$
$a^3b^3$	$3a^2b^3c + 3a^3b^2d$	$3ab^3c^2 + 9a^2b^2cd + 3a^3bd^2$	$b^3c^3 + 9ab^2c^2d + 9a^2bcd^2 + a^3d^3$	$3b^2c^3d + 9ab^2c^2d^2 + 3a^2cd^3$	$3bc^3d^2 + 3ac^2d^3$	$c^3d^3$
$a^2b^4$	$2ab^4c + 4a^2b^3d$	$b^4c^2 + 8ab^3cd + 6a^2b^2d^2$	$4b^3c^2d + 12ab^2c^2d^2 + 4a^2bd^3$	$6b^2c^2d^2 + 8ab^2cd^3 + a^2d^4$	$4b^2c^2d^3 + 2acd^4$	$c^2d^4$
$ab^5$	$b^5c + 5ab^4d$	$5b^4cd + 10ab^3d^2$	$10b^3cd^2 + 10ab^2d^3$	$10b^2cd^3 + 5abd^4$	$5bc^2d^4 + ad^5$	$cd^5$
$b^6$	$6b^5d$	$15b^4d^2$	$20b^3d^3$	$15b^2d^4$	$6bd^5$	$d^6$

The columns are expansions of  $(aX + cY)^{7-i}(bX + dY)^{i-1}$ . Binomials!

$\Delta(3-1)$

$X^2Y^0$

$X^1Y^1$

$X^0Y^2$

Example  $\Delta(7-1)$ , characteristic 0.

No common eigensystem  $\Rightarrow \Delta(7-1)$  simple.

Example  $\Delta(7-1)$ , characteristic 2.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  acts as

$a^6$	$\emptyset$	$a^4c^2$	$\emptyset$	$a^2c^4$	$\emptyset$	$c^6$
$a^5b$	$a^4bc + a^5d$	$a^4cd$	$\emptyset$	$abc^4$	$bc^5 + ac^4d$	$c^5d$
$a^4b^2$	$\emptyset$	$a^4d^2$	$\emptyset$	$b^2c^4$	$\emptyset$	$c^4d^2$
$a^3b^3$	$a^2b^3c + a^3b^2d$	$ab^3c^2 + a^2b^2cd + a^3bd^2$	$b^3c^3 + ab^2c^2d + a^2bcd^2 + a^3d^3$	$b^2c^3d + ab^2c^2d^2 + a^2cd^3$	$bc^3d^2 + ac^2d^3$	$c^3d^3$
$a^2b^4$	$\emptyset$	$b^4c^2$	$\emptyset$	$a^2d^4$	$\emptyset$	$c^2d^4$
$ab^5$	$b^5c + ab^4d$	$b^4cd$	$\emptyset$	$abd^4$	$bc^2d^4 + ad^5$	$cd^5$
$b^6$	$\emptyset$	$b^4d^2$	$\emptyset$	$b^2d^4$	$\emptyset$	$d^6$

$(0, 0, 0, 1, 0, 0, 0)$  is a common eigenvector, so we found a submodule.

Weyl  $\sim 1923$ . The  $SL_2$  Weyl modules  $\Delta(v-1)$ .

$\Delta(1-1)$

$\Delta(2-1)$

$\Delta(3-1)$

$\Delta(4-1)$

$\Delta(5-1)$

### When is $\Delta(v-1)$ simple?

$\Delta(v-1)$  is simple

$\Leftrightarrow$

$\binom{v-1}{w-1} \neq 0$  for all  $w \leq v$

$\Leftrightarrow$  (Lucas's theorem)

$v = [a_r, 0, \dots, 0]_p$ .

$x^4 y^0$

$x^3 y^1$

$x^2 y^2$

$x^1 y^3$

$x^0 y^4$

$x^0 y^3$

#### General.

Weyl  $\Delta(\lambda)$  and dual Weyl  $\nabla(\lambda)$

are easy a.k.a. standard;

are parameterized by dominant integral weights;

are highest weight modules;

are defined over  $\mathbb{Z}$ ;

have the classical Weyl characters;

form a basis of the Grothendieck group untriangular w.r.t. simples;

satisfy (a version of) Schur's lemma  $\dim_{\mathbb{K}} \text{Ext}^i(\Delta(\lambda), \nabla(\mu)) = \delta_{i,0} \delta_{\lambda, \mu}$ ;

are simple generically;

have a root-binomial-criterion to determine whether they are simple (Jantzen's thesis  $\sim 1973$ ).

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \text{matrix}$

$y^6$

$X + dY)^{i-1}$ .

**Ringel, Donkin ~1991.** The indecomposable  $SL_2$  tilting modules  $T(v-1)$  are the indecomposable summands of  $\Delta(1)^{\otimes i} (\cong (\mathbb{K}^2)^{\otimes i})$ .

**General.**

These facts hold in general, and the first bullet point is the general definition.

Tilting modules  $T(v-1)$

- are those modules with a  $\Delta(w-1)$ - and a  $\nabla(w-1)$ -filtration;
- are parameterized by dominant integral weights;
- are highest weight modules;
- satisfy reciprocity  $(T(v-1) : \Delta(w-1)) = (T(v-1) : \nabla(w-1)) = [\Delta(w'-1) : L(v'-1)] = [\nabla(w'-1) : L(v'-1)]$ ;
- form a basis of the Grothendieck group unitriangular w.r.t. simples;
- satisfy (a version of) Schur's lemma  $\dim_{\mathbb{K}} \text{Hom}(T(v-1), T(w-1)) = \sum_{x < \min(v,w)} (T(v-1) : \Delta(x-1)) (T(w-1) : \nabla(x-1))$  [Why the name?](#);
- are simple generically;
- have a root-binomial-criterion to determine whether they are simple.

Let  $\mathcal{T}_{\text{ilt}}$  be the category of tilting modules.

**Goal. Describe  $\mathcal{T}_{\text{ilt}}$  by generators and relations.**

**Ringel, Donkin ~1991.** The indecomposable  $SL_2$  tilting modules  $T(v-1)$  are the indecomposable summands of  $\Delta(1)^{\otimes i} (\cong (\mathbb{K}^2)^{\otimes i})$ .

### How many Weyl factors does $T(v-1)$ have?

# Weyl factors of  $T(v-1)$  is  $2^k$  where

$$k = \max\{\nu_p\left(\binom{v-1}{w-1}\right), w \leq v\}. \text{ (Order of vanishing of } \binom{v-1}{w-1}\text{.)}$$

determined by (Lucas's theorem)

non-zero non-leading digits of  $v = [a_r, a_{r-1}, \dots, a_0]_p$ .

### Example $T(220540-1)$ for $p = 11$ ?

$$v = 220540 = [1, 4, 0, 7, 7, 1]_{11};$$

Maximal vanishing for  $w = 75594 = [0, 5, 1, 8, 8, 2]_{11}$ ;

$$\binom{v-1}{w-1} = (\text{HUGE}) = [\dots, \neq 0, 0, 0, 0, 0]_{11}.$$

$\Rightarrow T(220540-1)$  has  $2^4$  Weyl factors.

**Ringel, Donkin ~1991.** The indecomposable  $SL_2$  tilting modules  $T(v-1)$  are the indecomposable summands of  $\Delta(1)^{\otimes i} (\cong (\mathbb{K}^2)^{\otimes i})$ .

Tilting modules  $T(v-1)$

- are those modules with a  $\Delta(w-1)$ - and a  $\nabla(w-1)$ -filtration;

**Which Weyl factors does  $T(v-1)$  have a.k.a. the negative digits game?**

Weyl factors of  $T(v-1)$  are

$$\Delta([a_r, \pm a_{r-1}, \dots, \pm a_0]_{p-1}) \text{ where } v = [a_r, \dots, a_0]_p.$$

- satisfy (a version of) Schur's lemma  $\dim_{\mathbb{K}} \text{Hom}(T(v-1), T(w-1)) =$

$$\sum_{x < \min(v, w)}$$

- are simple g
- have a root

**Example  $T(220540-1)$  for  $p = 11$ ?**

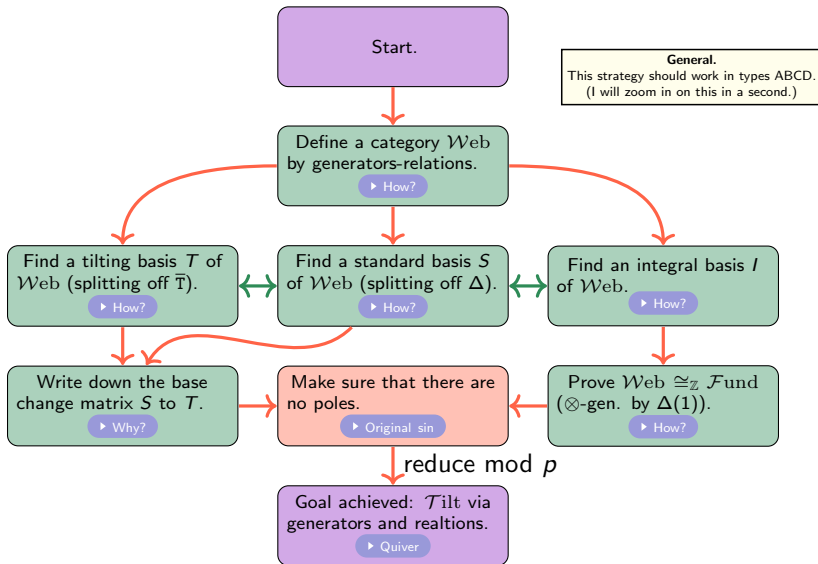
$$v = 220540 = [1, 4, 0, 7, 7, 1]_{11};$$

has Weyl factors  $[1, \pm 4, 0, \pm 7, \pm 7, \pm 1]_{11};$

e.g.  $\Delta(218690 = [1, 4, 0, -7, -7, -1]_{11}-1)$  appears.

Let  $T$ ilt be the

# Strategical interlude.





## Strategical interlude.

Start.

**What remains to be done?**

No more sins!

What is the diagrammatic incarnation of the Frobenius  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$ ?

The mixed case will be easier but might be a pain to write down.

Up next: the first steps towards higher ranks,  
*i.e.* let us try  $U_q(\mathfrak{sl}_3)$  for  $q$  a primitive complex  $2\ell$ th root of unity.

Write down the base change matrix  $S$  to  $T$ .

► Why?

Make sure that there are no poles.

► Original sin

Prove  $\text{Web} \cong_{\mathbb{Z}} \mathcal{F}\text{und}$   
( $\otimes$ -gen. by  $\Delta(1)$ ).

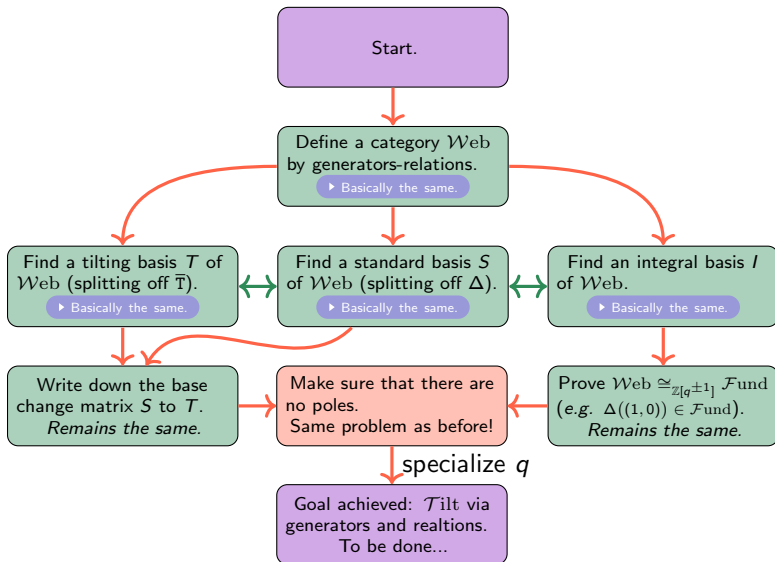
► How?

reduce mod  $p$

Goal achieved:  $T$  tilt via generators and relations.

► Quiver

## Strategical interlude.



Folklore, Lucas – 1878. Let  $q \in \mathbb{K}^*$ ,  $qchar(\mathbb{K}) = p$ ,  $a = mp + a_0$  and  $b = np + b_0$  ( $a_0, b_0$  smooth digit of the  $p$ -adic expansion). Then

$$\begin{bmatrix} a \\ b \end{bmatrix}_p = \binom{m}{n} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}_p.$$

Philosophy. Only the vanishing order of  $\begin{bmatrix} a \\ b \end{bmatrix}_p$  matters for the lecture  $\rightarrow$ .

Corollary. We understand finite-dimensional modules for  $SL_2 = SL_2(\mathbb{K} = \overline{\mathbb{F}}_p)$

- generically,
- for the quantum group over  $\mathbb{C}$  at  $q^2 = 1$ ,
- the quantum group over  $\mathbb{K}$ ,  $char(\mathbb{K}) = p$  and  $q^2 = 1$  (mixed case);
- in prime characteristic  $char(\mathbb{K}) = p$ .

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The  $SL_2$  fusion rules for  $\Delta(1) = \mathbb{C}\{r_1, r_{-1}\}$ :

$$\Delta(\lambda) \otimes \Delta(1) = \Delta(\lambda+1) \oplus \Delta(\lambda-1),$$

Rumer–Teller–Weyl – 1933, Eliaş – 2015 & la Littelmann – 1995. For any path  $\pi$  in the dominant Weyl chamber define  $\varepsilon(\pi)$  inductively by

$$\varepsilon_1(\pi) = \begin{array}{|c|} \hline \square \\ \hline \end{array} \quad \varepsilon_{-1}(\pi) = \begin{array}{|c|} \hline \square \\ \hline \end{array}$$

Flip to obtain  $\varepsilon(\pi)$  and stick them together. This gives an integral basis  $I$  of  $\mathbb{W}(\mathbb{K})$ .

Remark  
 Module of an indecomposable projective

The result. There exists a  $\mathbb{K}$ -algebra  $Z_p$  defined as a (very explicit) quotient of the path algebra of an infinite, fractal-like quiver. Let  $p\text{-Mod-}Z_p$  denote the category of finitely-generated, projective (right-)modules for  $Z_p$ . There is an equivalence of additive,  $\mathbb{K}$ -linear categories

$$\mathcal{F}: \mathbb{W}(\mathbb{K}) \xrightarrow{\sim} p\text{-Mod-}Z_p,$$

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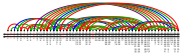


Figure: The full subquiver containing the first  $\Omega$  vertices of the quiver underlying  $Z_p$ .

Weyl – 1923. The  $SL_2$  Weyl module  $\Delta(v-1)$ .



$(\pm 2) \leftrightarrow$  matrix whose columns are expansions of  $(x\mathbb{K} + zY)^{-1}(x\mathbb{K} + yY)^{-1}$ .

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Example.  $\Delta(1) \otimes \Delta(1) = \Delta(2) \oplus \Delta(0)$

Non-example.  $\Delta(1) \otimes \Delta(1) \neq \Delta(2) \oplus \Delta(0)$

Example (four boundary points).

Flip to obtain  $\varepsilon(\pi)$  and stick them together. This gives an integral basis  $I$  of  $\mathbb{W}(\mathbb{K})$ .

Example, generation 2, i.e. only three non-zero digit.

In this case every connected component of the quiver is a bunch of type  $A$  graphs glued together in a multi-grid. Each row and column is a zigzag algebra, with arrows acting on the 0th digit or 1st, and there are "square commutative" relations.

Continuing this periodically gives a quiver for projective  $G_2$   $T$ -modules (See to Andersen – 2009).

Strategic iterates.



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Bases of  $\text{hom}(\Delta(1)^{\otimes p}, \Delta(1)^{\otimes q})$ .

- The integral basis  $I$ .
- Defined over  $\mathbb{Z}$ .
  - Needed for the transition from characteristic 0 to  $p$ .
  - Algebraically:  $\Delta(1)^{\otimes p} = \Delta(1) \otimes \Delta(1)^{\otimes (p-1)}$ .
  - Bottleneck principle:  $\varepsilon_1^{\otimes p} = \sum_{\lambda \vdash p} w(\lambda) \cdot \varepsilon_{\lambda}$ .
- The standard basis  $S$ .
- Defined generically, but without poles.
  - Actin–Weidemann basis  $\Rightarrow$  trivial relations.
  - Algebraically:  $\Delta(1)^{\otimes p} = \Delta(1) \otimes \Delta(1)^{\otimes (p-1)}$ .
  - Bottleneck principle:  $\varepsilon_1^{\otimes p} = \sum_{\lambda \vdash p} w(\lambda) \cdot \varepsilon_{\lambda}$ .
- The tilting basis  $T$ .
- Defined generically, but without poles.
  - The one we want for  $\mathbb{F}_p$ .
  - Algebraically:  $\Delta(1)^{\otimes p} = \Delta(1) \otimes \Delta(1)^{\otimes (p-1)}$ .
  - Bottleneck principle:  $\varepsilon_1^{\otimes p} = \sum_{\lambda \vdash p} w(\lambda) \cdot \varepsilon_{\lambda}$ .

$$\varepsilon_1^{\otimes p} = \sum_{\lambda \vdash p} w(\lambda) \cdot \varepsilon_{\lambda} \quad \varepsilon_1^{\otimes 3} = \sum_{\lambda \vdash 3} w(\lambda) \cdot \varepsilon_{\lambda} \quad \varepsilon_1^{\otimes 2} = \sum_{\lambda \vdash 2} w(\lambda) \cdot \varepsilon_{\lambda}$$

This is a reformulation of the Bottleneck principle. See also Andersen – 2009, Andersen – 2010, Andersen – 2011, Andersen – 2012, Andersen – 2013, Andersen – 2014, Andersen – 2015, Andersen – 2016, Andersen – 2017, Andersen – 2018, Andersen – 2019, Andersen – 2020, Andersen – 2021, Andersen – 2022, Andersen – 2023, Andersen – 2024, Andersen – 2025, Andersen – 2026, Andersen – 2027, Andersen – 2028, Andersen – 2029, Andersen – 2030, Andersen – 2031, Andersen – 2032, Andersen – 2033, Andersen – 2034, Andersen – 2035, Andersen – 2036, Andersen – 2037, Andersen – 2038, Andersen – 2039, Andersen – 2040, Andersen – 2041, Andersen – 2042, Andersen – 2043, Andersen – 2044, Andersen – 2045, Andersen – 2046, Andersen – 2047, Andersen – 2048, Andersen – 2049, Andersen – 2050, Andersen – 2051, Andersen – 2052, Andersen – 2053, Andersen – 2054, Andersen – 2055, Andersen – 2056, Andersen – 2057, Andersen – 2058, Andersen – 2059, Andersen – 2060, Andersen – 2061, Andersen – 2062, Andersen – 2063, Andersen – 2064, Andersen – 2065, Andersen – 2066, Andersen – 2067, Andersen – 2068, Andersen – 2069, Andersen – 2070, Andersen – 2071, Andersen – 2072, Andersen – 2073, Andersen – 2074, Andersen – 2075, Andersen – 2076, Andersen – 2077, Andersen – 2078, Andersen – 2079, Andersen – 2080, Andersen – 2081, Andersen – 2082, Andersen – 2083, Andersen – 2084, Andersen – 2085, Andersen – 2086, Andersen – 2087, Andersen – 2088, Andersen – 2089, Andersen – 2090, Andersen – 2091, Andersen – 2092, Andersen – 2093, Andersen – 2094, Andersen – 2095, Andersen – 2096, Andersen – 2097, Andersen – 2098, Andersen – 2099, Andersen – 2100.

Examples (blue: "all positive", red: "non-example").

The tilting basis  $T$  of  $\mathbb{W}(\mathbb{K})$ .

Example for  $\lambda = 2 + 1 = (2, 1)$ .

$\Delta(1) \otimes \Delta(1) = \Delta(2) \oplus \Delta(0)$  and  $\Delta(1) \otimes \Delta(1) \neq \Delta(2) \oplus \Delta(0)$ .

There is still much to do...

Folklore, Lucas –1878. Let  $q \in \mathbb{K}^*$ ,  $qchar(\mathbb{K}) = p$ ,  $a = mp + a_0$  and  $b = np + b_0$ ,  $a_0, b_0$  smooth digit of the  $p$ -adic expansion. Then

$$\begin{bmatrix} a \\ b \end{bmatrix}_n = \begin{bmatrix} m \\ n \end{bmatrix}_a \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}_n.$$

Philosophy. Only the vanishing order of  $\begin{bmatrix} a \\ b \end{bmatrix}_n$  matters for the lecture  $\rightarrow$ .

Corollary. We understand finite-dimensional modules for  $SL_2 = SL_2(\mathbb{K} = \overline{\mathbb{F}}_p)$

- generically,
- for the quantum group over  $\mathbb{C}$  at  $q^2 = 1$ ,
- the quantum group over  $\mathbb{K}$ ,  $char(\mathbb{K}) = p$  and  $q^2 = 1$  (mixed case);
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The  $SL_2$  fusion rules for  $\Delta(1) = \mathbb{C}\{r_1, r_{-1}\}$ :

$$\Delta(\lambda) \otimes \Delta(1) \cong \Delta(\lambda+1) \oplus \Delta(\lambda-1),$$

Rumer–Teller–Weyl –1933, Ekus –2015 & la Littelmann –1995. For any path  $\pi$  in the dominant Weyl chamber define  $l(\pi)$  inductively by

$$l(\uparrow) = 1, \quad l(\downarrow) = 1, \quad l(\uparrow \downarrow) = l(\downarrow \uparrow) = 0.$$

Flip to obtain  $u(\pi)$  and stick them together. This gives an integral basis  $I$  of  $\mathbb{W}(\mathfrak{h})$ .

Remark  
This is an integral basis of  $\mathbb{W}(\mathfrak{h})$ .

The result. There exists a  $\mathbb{K}$ -algebra  $Z_p$  defined as a (very explicit) quotient of the path algebra of an infinite, fractal-like quiver. Let  $p\text{-Mod-}Z_p$  denote the category of finitely-generated, projective (right-)modules for  $Z_p$ . There is an equivalence of additive,  $\mathbb{K}$ -linear categories

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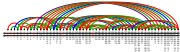


Figure: The full subquiver containing the first 11 vertices of the quiver underlying  $Z_p$ .

Weyl –1923. The  $SL_2$  Weyl module  $\Delta(v-1)$ .



$(\pm 2) \leftrightarrow$  matrix whose columns are expansions of  $(xK + yY)^{v-1} (xK + yY)^{-1}$ .

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**Example.**  
 $0 = (y/x)^{v-1} \dots$

**Non-example.**  
 $1 = (y/x)^{v-1} \dots$

**Example (four boundary points).**

Flip  $\pi$  to obtain  $u(\pi)$  and stick them together. This gives an integral basis  $I$  of  $\mathbb{W}(\mathfrak{h})$ .

**Example, generation 2, i.e. only three non-zero digit.**

In this case every connected component of the quiver is a bunch of type  $A$  graphs glued together in a multi-grid. Each row and column is a zigzag algebra, with arrows acting on the 0th digit or 1st, and there are "square commutative" relations.

Continuing this periodically gives a quiver for projective  $G_2$   $T$ -modules (See to Andersen –2009).

Strategic integrals.



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Bases of  $\text{hom}(\Delta(1)^{\otimes p}, \Delta(1)^{\otimes q})$ .

- The integral basis  $I$ .
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  - Algebraically:
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- The citing basis  $T$ .
- Defined generically, but without poles.
  - The one we want for  $\mathbb{F}_p$ .
  - Algebraically:
  - Bottleneck principle:

$$e_1^{\otimes p} = \sum_{\lambda \in I} w(\lambda) \cdot \lambda, \quad e_1^{\otimes q} = \sum_{\lambda \in S} \lambda, \quad e_1^{\otimes p} = \sum_{\lambda \in T} \lambda.$$

This is a reference to the work of Andersen and Grothendieck, and other important authors. Notice: The table below is not a complete list of references.

**Examples (blue: "all positive", red: "non-example").**

**Example for  $\lambda = v_1 + v_2 + \dots + v_{n-1}$ .**

$\Delta(\lambda) = \dots$  and  $\mathbb{W}(\mathfrak{h}) = \dots$

Thanks for your attention!

**Weyl**  $\sim 1923$ . The  $SL_2$  simples  $L(v-1)$  in  $\nabla(v-1)$  for  $p = 5$ .

$$\nabla(1-1) \qquad x^0 y^0 \qquad L(1-1)$$

$$\nabla(2-1) \qquad x^1 y^0 \quad x^0 y^1 \qquad L(2-1)$$

$$\nabla(3-1) \qquad x^2 y^0 \quad x^1 y^1 \quad x^0 y^2 \qquad L(3-1)$$

$$\nabla(4-1) \qquad x^3 y^0 \quad x^2 y^1 \quad x^1 y^2 \quad x^0 y^3 \qquad L(4-1)$$

$$\nabla(5-1) \qquad x^4 y^0 \quad x^3 y^1 \quad x^2 y^2 \quad x^1 y^3 \quad x^0 y^4 \qquad L(5-1)$$

$$\nabla(6-1) \qquad x^5 y^0 \quad x^4 y^1 \quad x^3 y^2 \quad x^2 y^3 \quad x^1 y^4 \quad x^0 y^5 \qquad L(6-1)$$

$$\nabla(7-1) \qquad x^6 y^0 \quad x^5 y^1 \quad x^4 y^2 \quad x^3 y^3 \quad x^2 y^4 \quad x^1 y^5 \quad x^0 y^6 \qquad L(7-1)$$

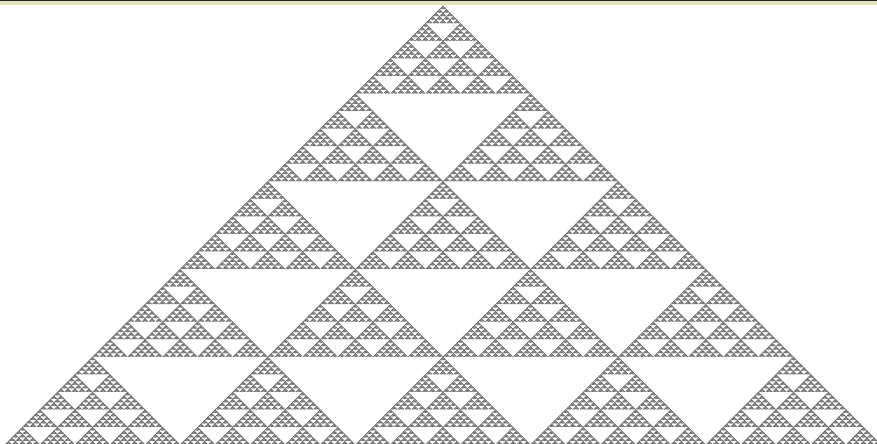
$\nabla(7-1)$  has  $L(7-1)$  and  $L(3-1)$  as factors.

Weyl  $\sim 1923$ . The  $SL_2$  simples  $L(v-1)$  in  $\nabla(v-1)$  for  $p = 5$ .

$\nabla(1-1)$

$x^0 y^0$

$L(1-1)$



Pascal's triangle modulo  $p = 5$  picks out the simples, e.g. an unbroken east-west line is a Weyl module which is simple.

Picture from [https://commons.wikimedia.org/wiki/File:Pascal\\_triangle\\_modulo\\_5.png](https://commons.wikimedia.org/wiki/File:Pascal_triangle_modulo_5.png)

## “Schur’s tilting lemma a.k.a. Weyl clustering”.

In the Grothendieck group:  $[\mathbb{T}(\lambda)] = [\Delta(\lambda)] + \sum_{\mu < \lambda} (\mathbb{T}(\lambda) : \Delta(\mu))[\Delta(\mu)]$ .

Let  $\bar{\mathbb{T}}(\lambda) = \Delta(\lambda) \oplus \bigoplus_{\mu < \lambda} (\mathbb{T}(\lambda) : \Delta(\mu))\Delta(\mu)$ , seen generically.

---

**Philosophy.** Never ever go to characteristic  $p$  – its too complicated. Work with  $\bar{\mathbb{T}}(\lambda)$  instead, “the characteristic 0 cousin of  $\mathbb{T}(\lambda)$ ”.

---

Then

$$\dim_{\mathbb{K}} \text{End}(\mathbb{T}(\lambda)) = \dim_{\text{gen}} \text{End}(\bar{\mathbb{T}}(\lambda)) = 1 + \sum_{\mu < \lambda} (\mathbb{T}(\lambda) : \Delta(\mu))^2,$$

by Schur’s lemma. (Similarly for hom-spaces, of course.)

## “Schur’s tilting lemma a.k.a. Weyl clustering”.

In the Grothendieck ring of varieties

Let  $\bar{T}(\lambda) = \Delta(\lambda) \oplus \dots$

**Philosophy.** Never use  $\bar{T}(\lambda)$  instead, “the ch

### Weyl clustering algorithm.

$\Delta(1)^k$  has the following tilting summands.

Take the highest appearing weight  $\nu - 1$ ;  
set  $\bar{T}(\nu - 1) = \bigoplus_{w \in \text{NDG}} \Delta(w - 1)$ ;  
repeat.

$(\lambda) : \Delta(\mu) [\Delta(\mu)]$ .

lly.

licated. Work with

Then

$$\dim_{\mathbb{K}} \text{End}(T(\lambda)) = \dim_{\text{gen}} \text{End}(\bar{T}(\lambda)) = 1 + \sum_{\mu < \lambda} (T(\lambda) : \Delta(\mu))^2,$$

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◀ Back



## “Schur’s tilting lemma a.k.a. Weyl clustering”.

In the Grothendieck ring of varieties

Let  $\bar{T}(\lambda) = \Delta(\lambda) \oplus \dots$

**Philosophy.** Never use  $\bar{T}(\lambda)$  instead, “the class of  $\lambda$ ”

Then

$\dim_{\mathbb{K}} \text{End}(\bar{T}(\lambda))$

by Schur’s lemma.

### Weyl clustering algorithm.

$\Delta(1)^k$  has the following tilting summands.

Take the highest appearing weight  $\nu - 1$ ;  
set  $\bar{T}(\nu - 1) = \bigoplus_{w \in \text{NDG}} \Delta(w - 1)$ ;  
repeat.

### $T(\nu - 1)$ vs. $\bar{T}(\nu - 1)$ .

The idempotents in  $\text{End}(\bar{T}(\nu - 1))$  inducing the splitting into summands have poles, and  $T(\nu - 1)$  does not split into Weyl factors.

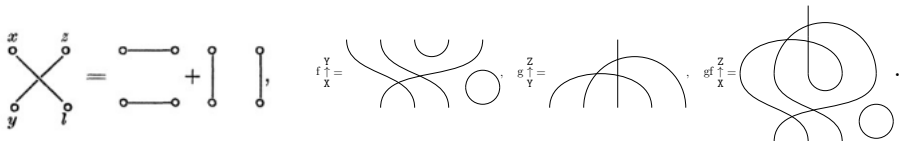
◀ Back

**Rumer–Teller–Weyl  $\sim 1932$ , Temperley–Lieb  $\sim 1971$ , Kauffman  $\sim 1987$ .**

The category  $\mathcal{Web}$  is the monoidal  $\mathbb{Z}$ -linear category monoidally generated by

object generators :  $\bullet$ , morphism generators :  $\cap : \mathbb{1} \rightarrow \bullet^{\otimes 2}$ ,  $\cup : \bullet^{\otimes 2} \rightarrow \mathbb{1}$ ,

relations :  $\bigcirc = -2$ ,  $\text{cup} = \text{cap}$ .



**Figure: Conventions and examples.** The crossing is from "G. Rumer, E. Teller, H. Weyl. Eine für die Valenztheorie geeignete

Basis der binären Vektorinvarianten. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (1932),

Volume: 1932, pages 499–504."

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**General.**

For type A we have webs  
à la Kuperberg  $\sim 1997$ , Cautis–Kamnitzer–Morrison  $\sim 2012$ .  
For types BCD there are some partial results,  
e.g. Brauer  $\sim 1937$ , Kuperberg  $\sim 1997$ ,  
Sartori  $\sim 2017$ , Rose–Tatham  $\sim 2020$ .  
Outside of these types I do not even  
expect our approach to work anyway.

The  $SL_2$  fusion rules for  $\Delta(1) = \mathbb{C}\{\varepsilon_1, \varepsilon_{-1}\}$ :

$$\Delta(\lambda) \otimes \Delta(1) \cong \Delta(\lambda+1) \oplus \Delta(\lambda-1),$$

$$\left| \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \right. \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \varepsilon_1, \quad \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \varepsilon_{-1} \leftarrow \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array}.$$

**Rumer–Teller–Weyl**  $\sim$ 1933, **Elias**  $\sim$ 2015 à la **Littelmann**  $\sim$ 1995. For any path  $\pi$  in the dominant Weyl chamber define  $d(\pi)$  inductively by

$$\varepsilon_1(f): \boxed{f} \mapsto \boxed{f} \mid, \quad \varepsilon_{-1}(f): \boxed{f} \mapsto \boxed{f} \begin{array}{c} \uparrow \\ \downarrow \end{array}.$$

Flip to obtain  $u(\pi)$  and stick them together. This gives an integral basis  $I$  of  $\mathcal{Web}$ .

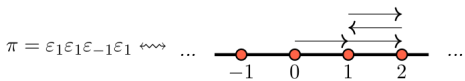
[← Back](#)

**General.**

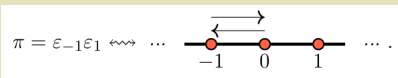
As long as you have a web calculus, this works in general.

The  $SL_2$  fusion rule

### Example.



### Non-example.

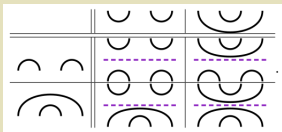
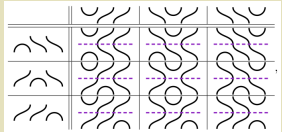
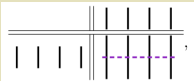
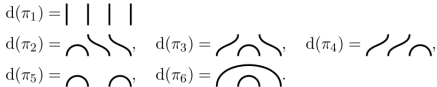


1995. For any

Rumer-Teller-W  
path  $\pi$  in the dom

### Example (four boundary points).

- $\pi_1 = \varepsilon_1 \varepsilon_1 \varepsilon_1 \varepsilon_1,$
- $\pi_2 = \varepsilon_1 \varepsilon_{-1} \varepsilon_1 \varepsilon_1, \quad \pi_3 = \varepsilon_1 \varepsilon_1 \varepsilon_{-1} \varepsilon_1, \quad \pi_4 = \varepsilon_1 \varepsilon_1 \varepsilon_1 \varepsilon_{-1},$
- $\pi_5 = \varepsilon_1 \varepsilon_{-1} \varepsilon_1 \varepsilon_{-1}, \quad \pi_6 = \varepsilon_1 \varepsilon_1 \varepsilon_{-1} \varepsilon_{-1},$



Flip to

f Web.

The  $SL_2$  fusion rules for  $\Delta(1) = \mathbb{C}\{\varepsilon_1, \varepsilon_{-1}\}$ :

$$\Delta(\lambda) \otimes \Delta(1) \cong \Delta(\lambda+1) \oplus \Delta(\lambda-1),$$

$$\left| \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \right. \rightsquigarrow \longrightarrow \varepsilon_1, \quad \left( \begin{array}{c} \frown \\ \smile \end{array} \right) \rightsquigarrow \varepsilon_{-1} \longleftarrow \left. \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \right.$$

**Jones ~1985, Wenzl ~1989, Cooper–Hogancamp ~2012.** For any path  $\pi$  in the dominant Weyl chamber define  $\tilde{d}(\pi)$  inductively by

$$\tilde{e}_1(f): \boxed{f} \mapsto \begin{array}{|c|} \hline \tilde{e}_i \\ \hline f \\ \hline \end{array}, \quad \tilde{e}_{-1}(f): \boxed{f} \mapsto \begin{array}{|c|} \hline \tilde{e}_{i-2} \\ \hline f \\ \hline \end{array} \begin{array}{l} \text{---} \\ \text{---} \end{array} \begin{array}{l} \text{---} \\ \text{---} \end{array}$$

Flip to obtain  $\tilde{u}(\pi)$  and stick them together. This gives a standard basis  $S$  of  $\mathcal{Web}$ .

[◀ Back](#)

**General.**

As long as you have a web calculus, this works in general, e.g. Elias has explained how to define the highest weight projectors “ $\tilde{e}$ ”.

The  $SL_2$  ...

### Example.

$$\tilde{d}(\pi_1) = \boxed{\tilde{e}_4},$$

$$\tilde{d}(\pi_2) = \begin{array}{c} \boxed{\tilde{e}_2} \\ \downarrow \downarrow \downarrow \\ \boxed{\tilde{e}_1} \end{array},$$

$$\tilde{d}(\pi_3) = \begin{array}{c} \boxed{\tilde{e}_2} \\ \downarrow \downarrow \downarrow \\ \boxed{\tilde{e}_2} \end{array},$$

$$\tilde{d}(\pi_4) = \begin{array}{c} \boxed{\tilde{e}_2} \\ \downarrow \downarrow \downarrow \\ \boxed{\tilde{e}_3} \end{array},$$

$$\tilde{d}(\pi_5) = \begin{array}{c} \boxed{\tilde{e}_0} \\ \downarrow \downarrow \\ \boxed{\tilde{e}_1} \quad \boxed{\tilde{e}_1} \end{array},$$

$$\tilde{d}(\pi_6) = \begin{array}{c} \boxed{\tilde{e}_0} \\ \downarrow \downarrow \downarrow \\ \boxed{\tilde{e}_2} \end{array},$$

$$\tilde{d}(\pi_6) = \begin{array}{c} \boxed{\tilde{e}_2} \\ \downarrow \downarrow \downarrow \\ \boxed{\tilde{e}_2} \end{array},$$

Jones  
the dom

ath  $\pi$  in

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◀ Back

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$$\left| \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \right. \rightsquigarrow \longrightarrow \varepsilon_1, \quad \left( \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \right) \rightsquigarrow \varepsilon_{-1} \leftarrow \leftarrow \leftarrow.$$

**Burrull–Libedinsky–Sentinelli** ~2019. For any path  $\pi$  in the dominant Weyl chamber define  $\bar{d}(\pi)$  inductively by

$$\bar{e}_1(f): \begin{array}{|c|} \hline f \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline \bar{e}_i \\ \hline f \\ \hline \end{array}, \quad \bar{e}_{-1}(f): \begin{array}{|c|} \hline f \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline \bar{e}_{i-2} \\ \hline f \\ \hline \end{array}$$

Flip to obtain  $\bar{u}(\pi)$  and stick them together. This gives a tilting basis  $T$  of  $Web$ .

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**General.**

As long as you know the tilting characters, this works in general  
*e.g.* one can define the highest weight tilting projectors “ $\bar{e}$ ”.

The  $SL_2$  fusion rules for  $\Delta(1) = \mathbb{C}\{\varepsilon_1, \varepsilon_{-1}\}$ :

$$\Delta(\lambda) \otimes \Delta(1) \cong \Delta(\lambda+1) \oplus \Delta(\lambda-1).$$

**Example.**

$$\bar{d}(\pi_1) = \boxed{\bar{e}_4},$$

$$\bar{d}(\pi_2) = \begin{array}{c} \boxed{\bar{e}_2} \\ \swarrow \quad \downarrow \quad \searrow \\ \boxed{\bar{e}_1} \quad \quad \quad \end{array}$$

$$\bar{d}(\pi_5) = \boxed{\bar{e}_0}$$

$$\bar{d}(\pi_5) = \begin{array}{c} \swarrow \quad \searrow \\ \boxed{\bar{e}_1} \quad \boxed{\bar{e}_1} \end{array}$$

$$\bar{d}(\pi_3) = \begin{array}{c} \boxed{\bar{e}_2} \\ \swarrow \quad \downarrow \quad \searrow \\ \boxed{\bar{e}_2} \quad \quad \quad \end{array}$$

$$\bar{d}(\pi_6) = \boxed{\bar{e}_0}$$

$$\bar{d}(\pi_6) = \begin{array}{c} \swarrow \quad \searrow \\ \boxed{\bar{e}_2} \end{array}$$

$$\bar{d}(\pi_4) = \begin{array}{c} \boxed{\bar{e}_2} \\ \swarrow \quad \downarrow \quad \searrow \\ \boxed{\bar{e}_3} \quad \quad \quad \end{array}$$

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In order to prove  $\mathcal{Web} \cong \mathcal{Fund}$  we need

- a functor  $\Gamma: \mathcal{Web} \rightarrow \mathcal{Fund}$  defined integrally;
- an integral basis  $I$  of  $\mathcal{Web}$ ;
- that  $\Delta(1)$  is tilting regardless of  $\mathbb{K}$  (by a very general argument, which I learned from Andersen–Stroppel  $\sim 2015$ , this implies that hom-spaces in  $\mathcal{Fund}$  are flat);
- to prove fully faithfulness  $\Gamma$  generically.

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**General.**

The first, second and last bullet points are known in type A and should work more generally.

The third bullet point works *verbatim* for tensor products of any minuscule modules.

Example. Exterior powers of  $\Delta(\omega_1)$  in type A

$\Rightarrow$  the Cautis–Kamnitzer–Morrisson exterior web calculus works *verbatim* in characteristic  $p$  (as observed by Elias  $\sim 2015$ ).

Non-example. Symmetric powers of  $\Delta(\omega_1)$  in type A

$\Rightarrow$  the Rose (Vaz–Wedrich) symmetric web calculus in characteristic  $p$  is still to be found.

## Bases of $\text{hom}(\Delta(1)^{\otimes i}, \Delta(1)^{\otimes j})$ .

The integral basis  $I$ .

- Defined over  $\mathbb{Z}$ .
- Needed for the transition from characteristic 0 to  $p$ .
- Algebraically:

$$\Delta(1)^{\otimes i} \rightarrow \text{wt}(\lambda) \hookrightarrow \Delta(1)^{\otimes j}.$$

- Bottleneck principle:

$$c_{\lambda}^{u,d} = \begin{array}{c} \text{u} \\ \text{d} \end{array} \text{wt}(\lambda).$$

The standard basis  $S$ .

- Defined generically, having poles.
- Artin–Wedderburn basis  $\Rightarrow$  trivial relations.
- Algebraically:

$$\Delta(1)^{\otimes i} \rightarrow \Delta(\lambda) \hookrightarrow \Delta(1)^{\otimes j}.$$

- Bottleneck principle:

$$\tilde{c}_{\lambda}^{\tilde{u},\tilde{d}} = \begin{array}{c} \tilde{u} \\ \tilde{d} \end{array} \Delta(\lambda).$$

The tilting basis  $T$ .

- Defined generically, but without poles.
- The one we want for  $\mathcal{T}\text{ilt}$ .
- Algebraically:

$$\Delta(1)^{\otimes i} \rightarrow \bar{\mathbb{T}}(\lambda) \hookrightarrow \Delta(1)^{\otimes j}.$$

- Bottleneck principle:

$$\bar{c}_{\lambda}^{\bar{u},\bar{d}} = \begin{array}{c} \bar{u} \\ \bar{d} \end{array} \bar{\mathbb{T}}(\lambda).$$

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### General.

This is a well-known strategy which works in quite some generality, e.g. for cellular categories à la Graham–Lehrer, Westbury, Elias–Lauda.

Modern examples. Light leaves à la Libedinsky, light ladders à la Elias, bases of End(tilting) à la Andersen–Stroppel, KLR-type-bases à la Hu–Mathas, more...

Bases of  $\text{hom}(\Delta(1)^{\otimes i}, \Delta(1)^{\otimes j})$ .

**Base change for  $\bar{T}([1, 1]_{11}) = \Delta([1, 1]_{11}) \oplus \Delta([1, -1]_{11})$ .**

$S = \{\tilde{c}_{[1,1]_{11}}, \tilde{c}_{[1,-1]_{11}}\}$ ,  $\tilde{c}_{[1,1]_{11}}$  and  $\tilde{c}_{[1,-1]_{11}}$  are orthogonal idempotents.

$T = \{\bar{c}_{[1,1]_{11}}, \bar{c}_{[1,-1]_{11}}\}$ , and relations to be found.

Base change matrix  $T \rightarrow S$  is  $\begin{pmatrix} 1 & 0 \\ 1 & \kappa^{-1/2} \end{pmatrix}$ , where  $\kappa = [1, -1]_{11}/[1, 0]_{11} = 10/11$ , gives

$$\bar{c}_{[1,1]_{11}}^2 = (\tilde{c}_{[1,1]_{11}} + \tilde{c}_{[1,-1]_{11}})^2 = \tilde{c}_{[1,1]_{11}} + \tilde{c}_{[1,-1]_{11}} = \bar{c}_{[1,1]_{11}},$$

$$\bar{c}_{[1,1]_{11}} \bar{c}_{[1,-1]_{11}} = \bar{c}_{[1,-1]_{11}} \bar{c}_{[1,1]_{11}},$$

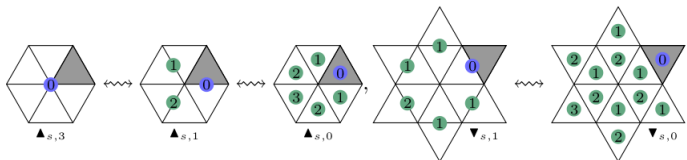
$$\bar{c}_{[1,-1]_{11}}^2 = 11/10 \cdot \tilde{c}_{[1,-1]_{11}} = 0 \pmod{11}.$$

Thus, the endomorphism space is  $\mathbb{K}[X]/(X^2)$ .

$$c_{\lambda}^{u,d} = \begin{array}{c} u \\ \text{---} \\ \text{d} \end{array} \text{wt}(\lambda) \quad \tilde{c}_{\lambda}^{\tilde{u},\tilde{d}} = \begin{array}{c} \tilde{u} \\ \text{---} \\ \tilde{d} \end{array} \Delta(\lambda) \quad \bar{c}_{\lambda}^{\bar{u},\bar{d}} = \begin{array}{c} \bar{u} \\ \text{---} \\ \bar{d} \end{array} \bar{T}(\lambda).$$

**Original sin.** In order to get  $\bar{T}(\lambda)$  I need to know the tilting characters.

So I cannot use the presentation of  $\mathcal{T}_{\text{ilt}}$  to say anything new about the objects, a.k.a. tilting modules.

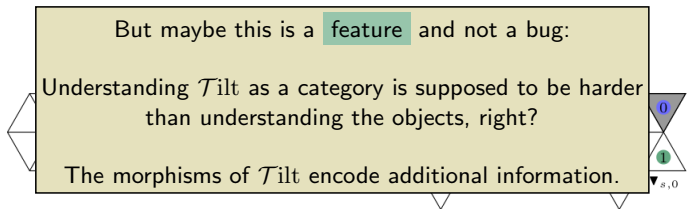


**Figure:** The quantum tilting characters for  $SL_3$ , due to Soergel and Stroppel  $\sim 1997$ .

Not much more is known in general, but there are some notable exceptions e.g. Jensen  $\sim 2000$ , Parker  $\sim 2008$ , Lusztig–Williamson  $\sim 2017$ .

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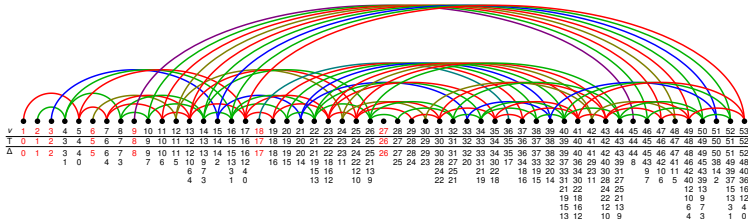
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Not much more is known in general, but there are some notable exceptions e.g. Jensen  $\sim 2000$ , Parker  $\sim 2008$ , Lusztig–Williamson  $\sim 2017$ .

**The result.** There exists a  $\mathbb{K}$ -algebra  $Z_p$  defined as a (very explicit) quotient of the path algebra of an infinite, fractal-like quiver. Let  $p\text{Mod-}Z_p$  denote the category of finitely-generated, projective (right-)modules for  $Z_p$ . There is an equivalence of additive,  $\mathbb{K}$ -linear categories

$$\mathcal{F}: \text{Tilt} \xrightarrow{\cong} p\text{Mod-}Z_p,$$

sending indecomposable tilting modules to indecomposable projectives.



**Figure:** The full subquiver containing the first 53 vertices of the quiver underlying  $Z_3$ .

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[← Time is over, you fool](#)

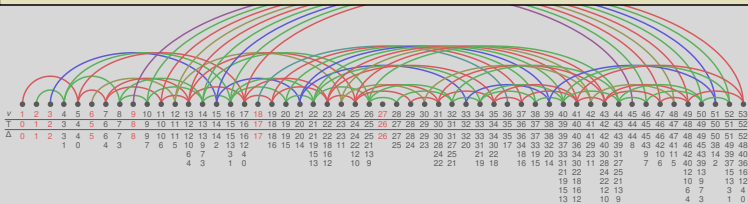
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**Example, generation 0, i.e. only one non-zero digit.**

In this case the quiver has no edges.

Continuing this periodically gives a quiver for  $\mathcal{Tilt}$  in characteristic zero.

(This is the semisimple case: the quiver has to be boring.)



**Figure:** The full subquiver containing the first 53 vertices of the quiver underlying  $Z_3$ .

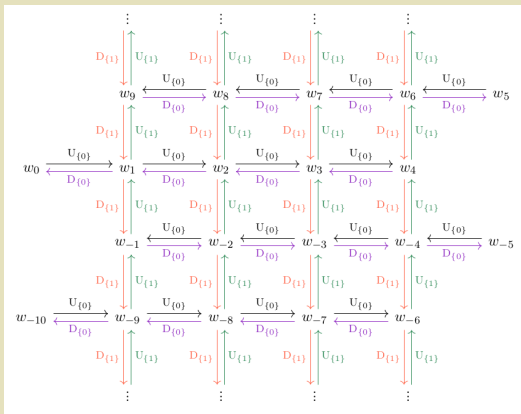




## Example, generation 2, i.e. only three non-zero digit.

In this case every connected component of the quiver is a bunch of type A graphs glued together in a matrix-grid. Each row and column is a zigzag algebra, with arrows acting on the 0th digit or 1digit, and there are “squares commute” relations.

Continuing this periodically gives a quiver for projective  $G_2 T$ -modules (due to Andersen  $\sim$ 2019).

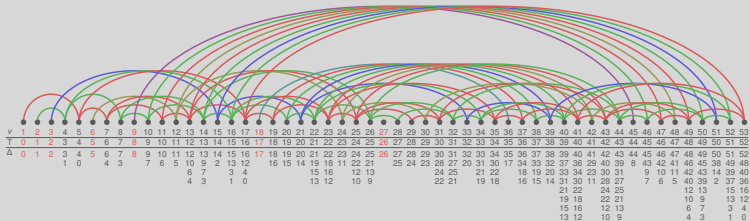


**The result.** There exists a  $\mathbb{K}$ -algebra  $Z_p$  defined as a (very explicit) quotient of the path algebra of an infinite, fractal-like quiver. Let  $p\text{Mod-}Z_p$  denote the category of finitely-generated, projective (right-)modules for  $Z_p$ . There is an equivalence of additive,  $\mathbb{K}$ -linear categories

$$\mathcal{F}: \text{Tilt} \xrightarrow{\cong} p\text{Mod-}Z_p,$$

sending

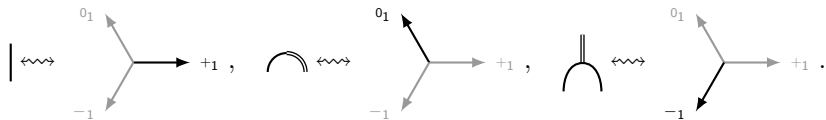
In general,  $Z_p$  is basically a bunch of zigzag algebras  
 (there are scalars and a lower-order-error term, but never mind)  
 glued together in a fractal-way, according to the digits of  $v = [a_r, \dots, a_0]_p$ .



**Figure:** The full subquiver containing the first 53 vertices of the quiver underlying  $Z_3$ .

The  $SL_3$  fusion rules for  $\Delta((1, 0)) = \mathbb{C}\{\varepsilon_1, \varepsilon_0, \varepsilon_{-1}\}$ :

$$\Delta(\lambda) \otimes \Delta((1, 0)) \cong \Delta(\lambda+(1, 0)) \oplus \Delta(\lambda+(-1, 1)) \oplus \Delta(\lambda+(0, -1)),$$



**Elias ~2015 à la Littelmann ~1995.** For any path  $\pi$  in the dominant Weyl chamber define  $d(\pi)$  inductively by

$$\varepsilon_{+1}(f): \boxed{f} \mapsto \boxed{f} \mid, \quad \varepsilon_{0_1}(f): \boxed{f} \mapsto \boxed{f} \text{ with a loop on top}, \quad \varepsilon_{-1}(f): \boxed{f} \mapsto \boxed{f} \text{ with a loop on the right}.$$

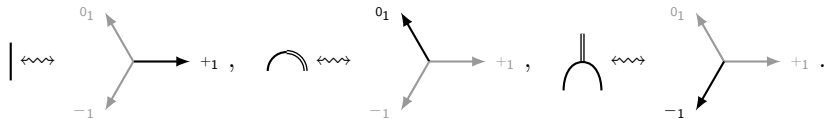
Flip to obtain  $u(\pi)$  and stick them together. This gives an integral basis  $I$  of  $\text{Web}$ .

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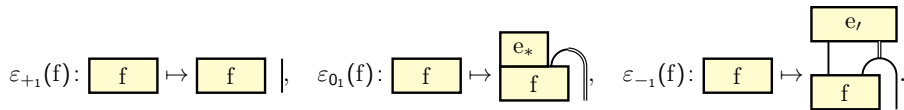
There is of course the dual picture for the second fundamental module – it is omitted to make this slide less cumbersome.

The  $SL_3$  fusion rules for  $\Delta((1, 0)) = \mathbb{C}\{\varepsilon_1, \varepsilon_0, \varepsilon_{-1}\}$ :

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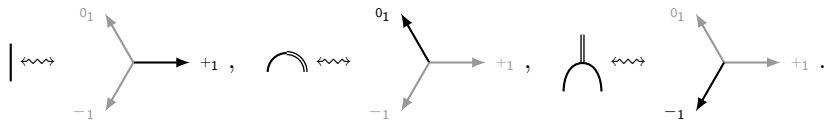
**Kuperberg ~1995, Kim ~2006, Elias ~2015.** For any path  $\pi$  in the dominant Weyl chamber define  $\tilde{d}(\pi)$  inductively by



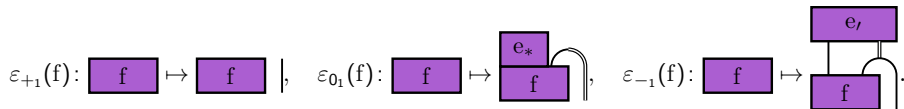
Flip to obtain  $\tilde{u}(\pi)$  and stick them together. This gives a standard basis  $S$  of  $\mathcal{Web}$ .

The  $SL_3$  fusion rules for  $\Delta((1, 0)) = \mathbb{C}\{\varepsilon_1, \varepsilon_0, \varepsilon_{-1}\}$ :

$$\Delta(\lambda) \otimes \Delta((1, 0)) \cong \Delta(\lambda+(1, 0)) \oplus \Delta(\lambda+(-1, 1)) \oplus \Delta(\lambda+(0, -1)),$$



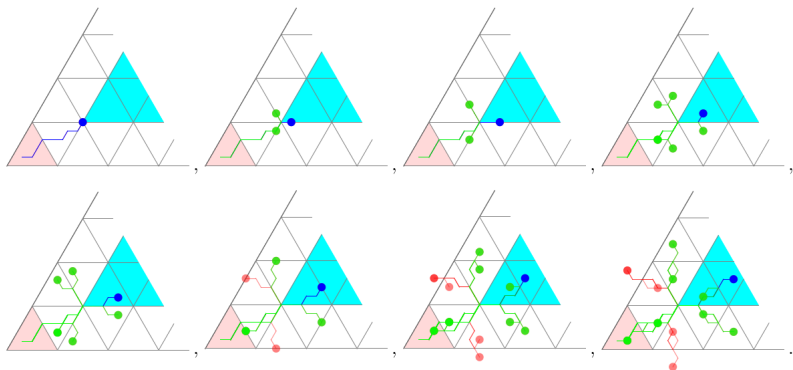
**Libedinsky–Patimo ~2020.** For any path  $\pi$  in the dominant Weyl chamber define  $\bar{d}(\pi)$  inductively by



Flip to obtain  $\bar{u}(\pi)$  and stick them together. This gives a tilting basis  $T$  of  $Web$ .

# Examples (blue="all positive", red="non-examples").

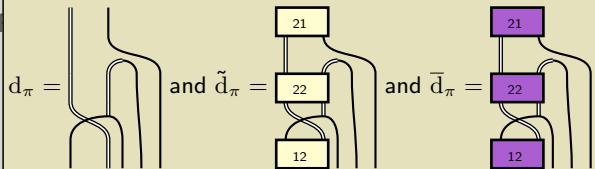
The



Libe  
defi

$\varepsilon_{+1}$

Example for  $\pi = +1 + 2 - 1 0_2 + 1$ .

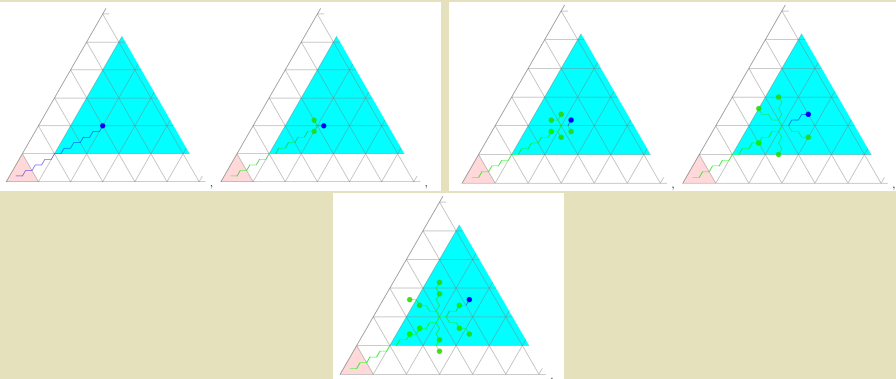


es a tilting basis  $T$  of  $\text{Web}$ .

From rank 2 onward you have crossings since e.g.  
 $\Delta((1, 0)) \otimes \Delta((0, 1)) \cong \Delta((0, 1)) \otimes \Delta((1, 0))$  but  $\neq$ .  
 They are mostly harmless – ignore them for today.

The tilting characters, and thus the tilting projectors, are given by path folding.

**Examples (blue=“leading summand”, green=“other summands”).**



Flip to obtain  $\bar{u}(\pi)$  and stick them together. This gives a tilting basis  $T$  of  $\mathcal{W}eb$ .

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