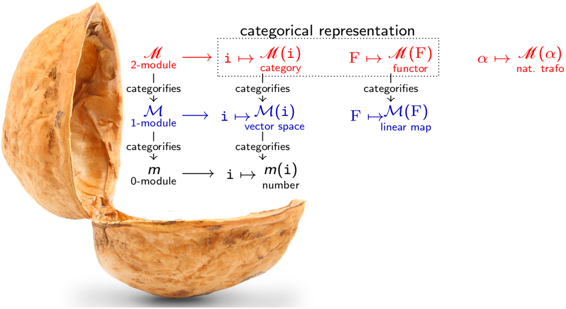
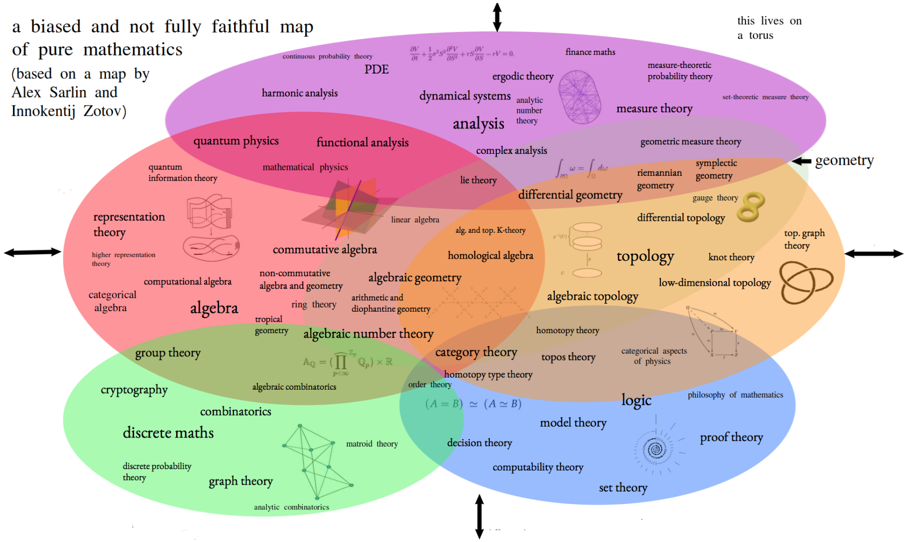
Why 2-representation theory?

Or: Representation theory of the 21th century!?

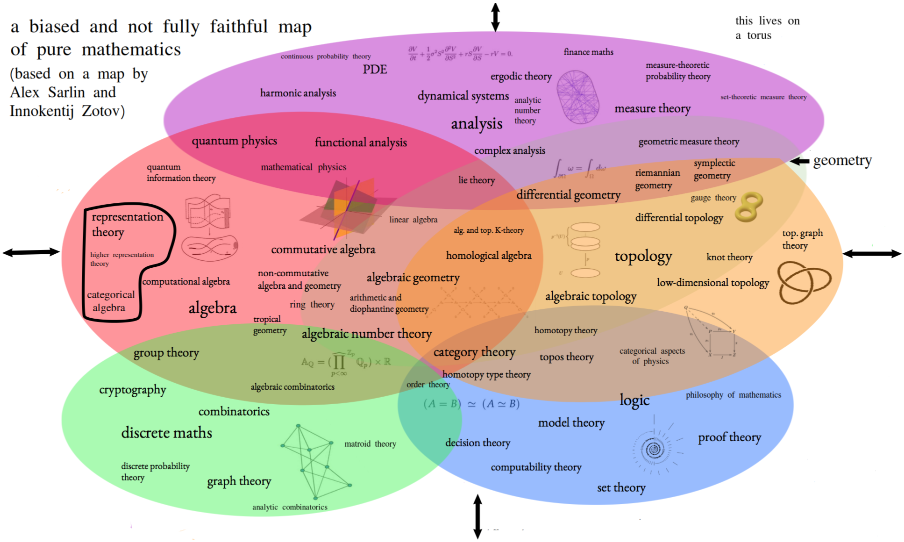
Daniel Tubbenhauer



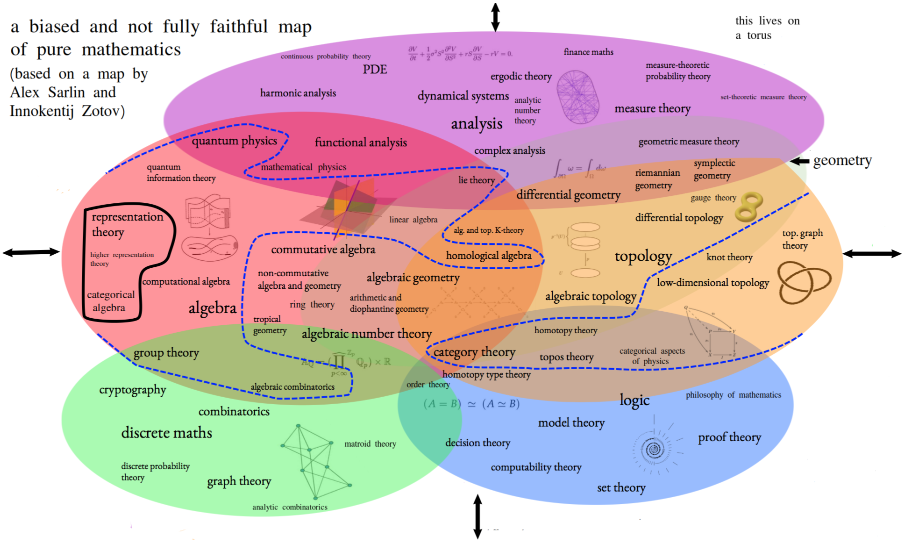
The map of pure mathematics.



The map of pure mathematics—my part of it.



The map of pure mathematics—my part of it and ramifications.



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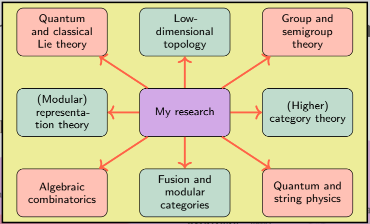
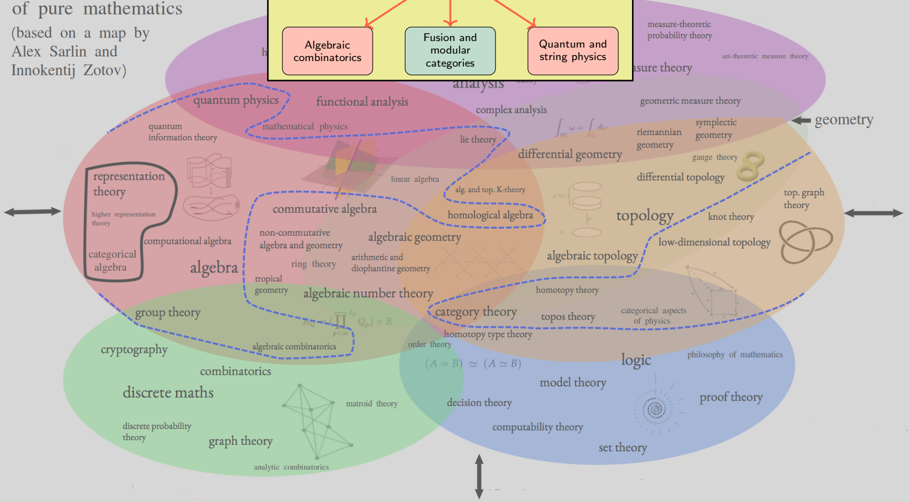


a biased and not fully faithful ... of pure mathematics

(based on a map by Alex Sarlin and Innokentij Zotov)

this lives on a torus



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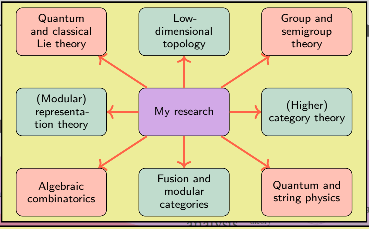


this lives on a torus

measure-theoretic probability theory

set-theoretic measure theory

This fits into to your profile, cf. Algebra and Geometry.

2-representation theory is a modern version of representation theory and has connections to quantum groups, Hopf and Lie algebras.

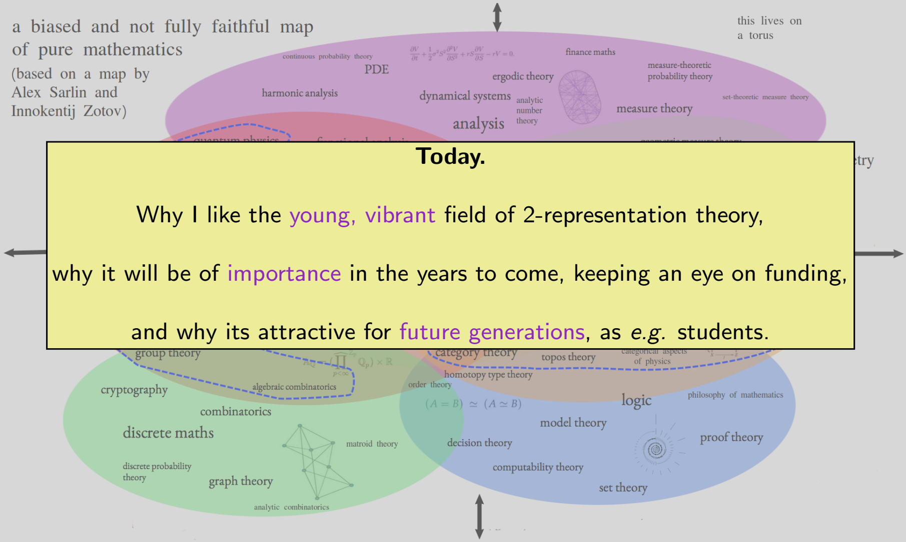
It grow out of and runs in parallel to the study of knot invariants, tensor categories and operator algebras.

My interest in semigroup theory, via cells, has overlap with group and ring theory, and permutation groups.

2-representation theory, via modular representation theory, connects to algebraic and fractal geometry.

Its ramifications, *e.g.* in topology, category theory, mathematical physics, computational mathematics, are currently explored.

The map of pure mathematics—my part of it and ramifications.



Today.

Why I like the young, vibrant field of 2-representation theory,

why it will be of importance in the years to come, keeping an eye on funding,

and why its attractive for future generations, as *e.g.* students.

Slogan. Representation theory is group theory in vector spaces

Let A be a finite-dimensional algebra, *e.g.* a group ring $\mathbb{K}[\mathrm{G}]$.

Frobenius ∼1895++ Representation theory is the useful? study of algebra actions

$$\mathcal{M}\colon \mathrm{A} \longrightarrow \mathcal{E}\mathrm{nd}(\mathtt{V}),$$

with $\mathtt{V}$ being some vector space. (Called modules or representations.)

The "elements" of such an action are called simple.

Maschke ∼1899. All modules are built out of simples ("Jordan–Hölder" filtration).

Main goal of representation theory. Find the periodic table of simples.

Slogan. Representation theory is group theory in vector spaces

The representation theory approach.

Reduce a non-linear problem to questions in linear algebra.

Problem involving a group action $\mathrm{G} \circlearrowright \mathtt{X}$

“linearize”

Problem involving a linear group action $\mathbb{K}[\mathrm{G}] \circlearrowright \mathbb{K}\mathtt{X}$

Decomposition of the problem into simples

new insights?

Let A be a fi

Frobenius $\sim$ … lgebra actions

with $\mathtt{V}$ being …)

The “element

Maschke $\sim$**1899.** All modules are built out of simples (“Jordan–Hölder” filtration).

Main goal of representation theory. Find the periodic table of simples.

Slogan. 2-representation theory is group theory in linear categories.

Let $\mathscr{C}$ be a (suitable) 2-category.

Etingof–Ostrik, Chuang–Rouquier, many others ∼2000++. Higher representation theory is the useful? study of actions of 2-categories:

$$\mathscr{M} : \mathscr{C} \longrightarrow \mathscr{E}\mathrm{nd}(\mathcal{V}),$$

with $\mathcal{V}$ being some (suitable) category. (Called 2-modules or 2-representations.)

The "elements" of such an action are called 2-simple.

Mazorchuk–Miemietz ∼2014. All (suitable) 2-modules are built out of 2-simples ("weak 2-Jordan–Hölder filtration").

Main goal of 2-representation theory. Find the periodic table of 2-simples.

Slogan. 2-representation theory is group theory in linear categories.

Let $\mathscr{C}$ be a (su

Etingof–Ostri ... igher

representation ... :

with $\mathcal{V}$ being so ... resentations.)

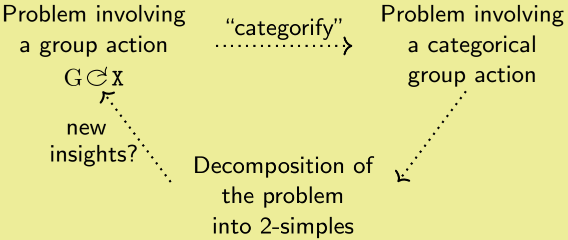
The "elements"

Mazorchuk–M ... out of 2-simples

("weak 2-Jordan–Hölder filtration").

The 2-representation theory approach.

Get new insights by studying richer structures.



Main goal of 2-representation theory. Find the periodic table of 2-simples.

Slogan. 2-representation theory is group theory in linear categories.

Let $\mathscr{C}$ be a

Example. $\mathscr{C} = \mathcal{R}\mathrm{ep}(G)$.
Status. Semisimple, classification of 2-simples well-understood.
Comments. $\mathcal{R}\mathrm{ep}(G)$ can be see as the categorical analog of G.

Etingof–O r
representation theory is the useful? study of actions of 2-categories:

Example. $\mathscr{C} = \mathcal{R}\mathrm{ep}$(Hopf algebra).
Status. Highly studied from various directions,
but pretty much open in general.
Comments. These arose from quantum physics and knot theory.

with $\mathcal{V}$ bei tations.)

The

Example. $\mathscr{C}$ = Hecke category.
Status. Non-semisimple, we, after 10 years, have now a complete classification
Comments. The Hecke category (a categorification of the Hecke algebra)
and its 2-representation play a crucial role in modern mathematics.
Our main result is a categorification of the theory
of representations of Hecke algebras.

Ma nples
("w

Main goal of 2-representation theory. Find the periodic table of 2-simples.

Research outlook.

(1) Most classification problems are still widely open $\Longrightarrow$ huge source of future research problems.

(2) The potential applications of 2-representation theory are still to be developed $\Longrightarrow$ strengthen and find new connections to its ramifications, *e.g.* in collaboration with members from Algebra/Geometry or Quantum/string physics.

Funding opportunities.

(1) Most grant applications are based on its deep connection to its ramifications $\Longrightarrow$ applications for grants in *e.g.* topology or mathematical physics are often successful.

(2) Connection to work done at Trinity College $\Longrightarrow$ potentially joint grant applications with members of Trinity College, *e.g.* via the aforementioned connection to Algebra/geometry or Quantum/string physics.

The future generation.

(1) Very attractive for students due to its accessibility $\Longrightarrow$ PhD students tend to have papers before they finish.

(2) Plenty of open problems $\Longrightarrow$ big source of master and PhD projects.

Research outlook.

One particular future project.

We stay in a finite-dimensional setup so far, so:

Problem. Extend the general theory to certain infinite-dimensional cases.

This fits very well to research done at Trinity college—usually uses complex analysis.

One particular grant proposal.

The above gives a possibility for a joint application, *e.g.* via *EPSRC-SFI* funding:

Proposal. "Algebraic structures of monoidal categories and their representations.".

Selling point: being beneficial to a wide cross-section of pure mathematics and beyond.

Why students care.

Topic 1. Add a grading to (parts of) the general theory. (Abstract.)

Topic 2. Study braid group actions on 2-categories. (Diagrammatic.)

Topic 3. Computer calculations for numerical data of 2-modules. (Computational.)

Research outlook.

(1) Most classification problems are still widely open $\Longrightarrow$ huge source of future research problems.

(2) The potential applications of 2-representation theory are still to be ... fications,

Fun...

(1) ... s ... thematical

(2) ... nt grant ... mentioned

The...

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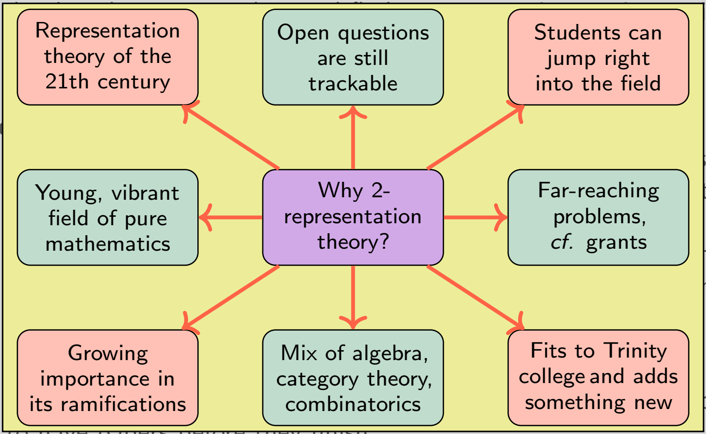
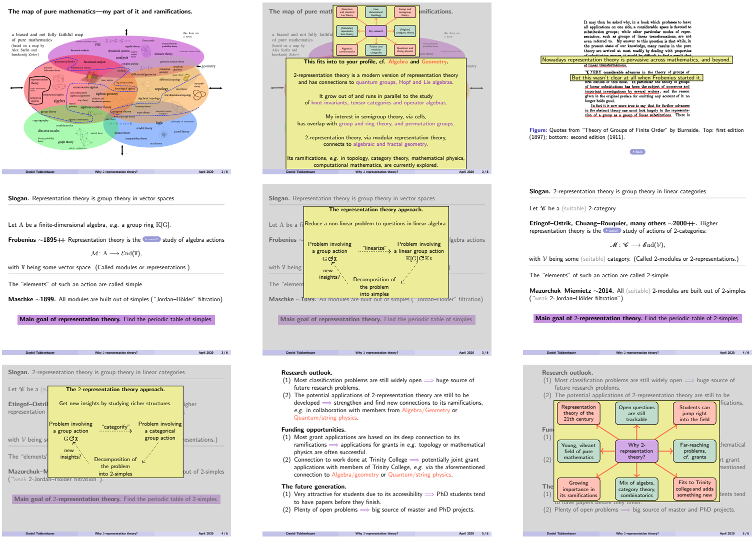
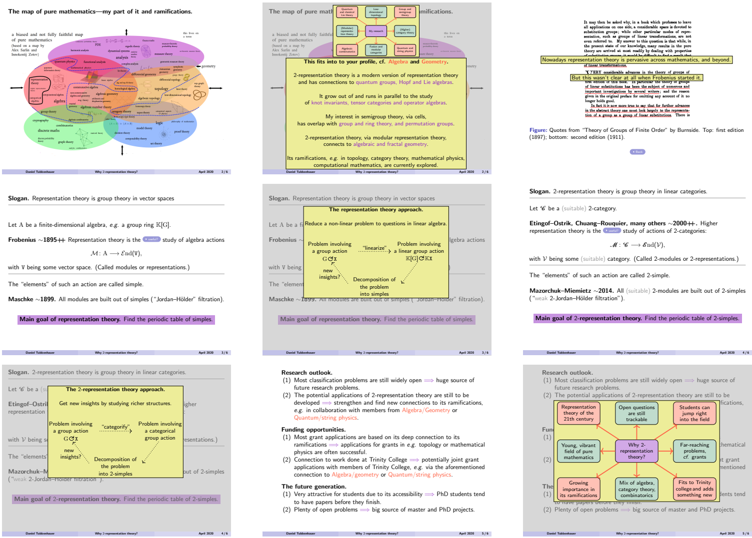


(2) Plenty of open problems $\Longrightarrow$ big source of master and PhD projects.



There is still much to do...



Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

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Nowadays representation theory is pervasive across mathematics, and beyond.

of linear transformations.

VERY considerable advances in the theory of groups of

But this wasn't clear at all when Frobenius started it.

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Applications of the theory

Khovanov & others ∼1999++. Knot homologies are instances of 2-representation theory. $\boxed{\text{Low-dim. topology \& Math. Physics}}$

Khovanov–Seidel & others ∼2000++. Faithful 2-modules of braid groups. $\boxed{\text{Low-dim. topology \& Symplectic geometry}}$

Chuang–Rouquier ∼2004. Proof of the Broué conjecture using 2-representation theory. $\boxed{p\text{-RT of finite groups \& Geometry \& Combinatorics}}$

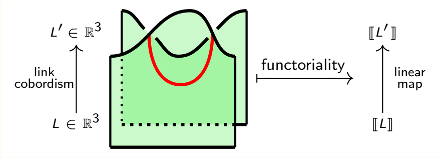
Elias–Williamson ∼2012. Proof of the Kazhdan–Lusztig conjecture using ideas from 2-representation theory. $\boxed{\text{Combinatorics \& RT \& Geometry}}$

Riche–Williamson ∼2015. Tilting characters using 2-representation theory. $\boxed{p\text{-RT of reductive groups \& Geometry}}$

Many more...

Some of my contributions ot the game. My main ingredient? 2-representation theory.

In topology. Functoriality of Khovanov–Rozansky's invariants ∼2017.



(This was conjectured for about 10 years, but seemed infeasible to prove, and has some impact on 4-dim. topology.)

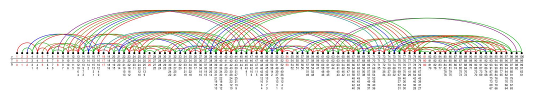
In quantum algebra. Positivity results for certain canonical/crystal bases. ∼2013.

Our tool—categorical Howe duality—turned out to be very useful.

In Lie theory. Disprove of a conjecture on properties of algebras in category $\mathcal{O}$. ∼2019.

The properties of these algebras are shadows of categorical structures.

In modular RT theory. Fractal structures in $\mathcal{R}\mathrm{ep}(\mathrm{SL}_2, \overline{\mathbb{F}}_p)$. ∼2019.



We confirmed that modular representation theory behaves like a discrete dynamical system.

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