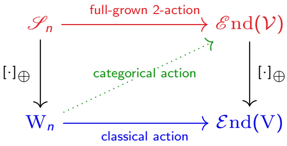
2-representation theory of Coxeter groups: some first steps

Or: The "next generation" of representation theory of Coxeter groups!?

Daniel Tubbenhauer



Joint work with Marco Mackaay, Volodymyr Mazorchuk and Vanessa Miemietz

February 2017

1. The what, how and why of categorical representation theory
   - Classical representation theory
   - Categorical representation theory

2. 2-representation theory of dihedral groups
   - Dihedral groups as Coxeter groups
   - Cooking-up candidate lists

3. Constructing 2-representations of dihedral groups
   - Some general methods
   - 2-representations via (co)algebras

Pioneers of representation theory

Let G be a finite group G.

Frobenius ∼1895++, Burnside ∼1900++. Representation theory is the study of linear group actions: (useful)

$$\mathrm{M}\colon G \longrightarrow \mathrm{End}(\mathrm{V}), \quad \mathrm{M}(g) = \text{a “matrix” in } \mathrm{End}(\mathrm{V}),$$

with V being some $\mathbb{C}$-vector space. We call V a module or a representation.

The “atoms” of such an action are called simple.

Maschke ∼1899. All modules are built out of atoms (“Jordan-Hölder”).

We want to have a categorical version of this!

Pioneers of representation theory

Let A be a finite-dimensional algebra.

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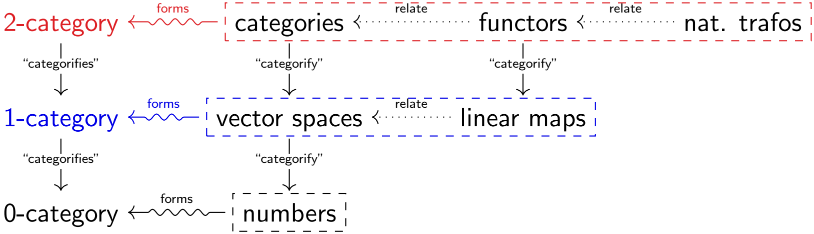
Categorification: A picture to keep in mind



An algebra A can be viewed as an one-object category $\mathcal{C}$, and a representation as a functor from $\mathcal{C}$ into the one-object category $\mathcal{E}\mathrm{nd}(\mathrm{V})$, i.e. $\mathrm{M}\colon \mathcal{C} \longrightarrow \mathcal{E}\mathrm{nd}(\mathrm{V})$.

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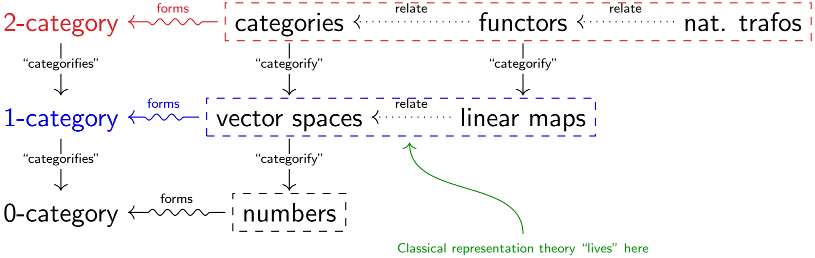

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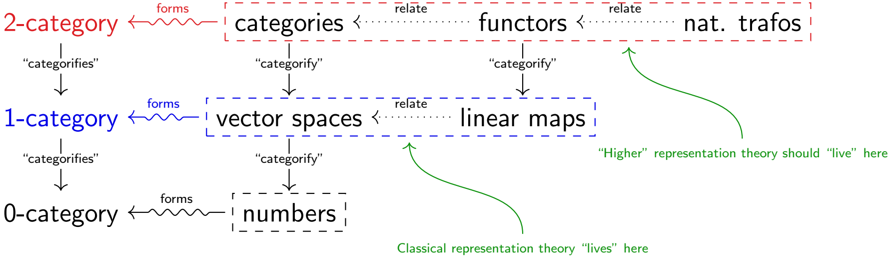

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"Lifting" representation theory

Let $\mathscr{C}$ be a (suitable) 2-category, $\mathfrak{A}^f_{\Bbbk}$ be the 2-category of (suitable) categories and $\mathbf{M}$ be a (suitable) 2-functor $\mathbf{M}\colon \mathscr{C} \longrightarrow \mathfrak{A}^f_{\Bbbk}$. Then $\mathbf{M}$ is a 2-representation, and 2-representations decategorify to representations:

2-morphisms	$\alpha \mapsto \mathbf{M}(\alpha)$ nat. trafo		
1-morphisms	$\mathrm{F} \mapsto \mathbf{M}(\mathrm{F})$ functor	$\xrightarrow[\text{decategorifies}]{[\cdot]_{\oplus}}$	$[\mathrm{F}] \mapsto [\mathbf{M}(\mathrm{F})]$ linear map
objects	$\mathtt{i} \mapsto \mathbf{M}(\mathtt{i})$ category		$[\mathtt{i}] \mapsto [\mathbf{M}(\mathtt{i})]$ vector space

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A lot of statements from classical representation theory "lift", e.g.:

Mazorchuk-Miemietz ∼2014. Notion of "2-atoms" (called simple transitive). All (suitable) 2-representations are built out of 2-atoms ("2-Jordan-Hölder"). These are "determined" on the level of the Grothendieck group $[\cdot]_{\oplus}$.

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2-atoms of the symmetric group decategorify to atoms.
Beware: This is wrong in general.

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Basic philosophy: Stay as long in the Grothendieck group as possible!

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Example(construction). We have the $\mathtt{i}$-th principal 2-representation $\mathscr{C}(\mathtt{i},\_)$.

- ▷ This “lifts” the regular representation of algebras.
- ▷ Sadly: These are usually not 2-atoms.

The state of the arts

- ▷ **Chuang-Rouquier ∼2004, Khovanov-Lauda ∼2008.** Systematic study of 2-representations of Lie algebras.
- ▷ **Chuang-Rouquier ∼2004, Khovanov-Lauda ∼2008.** All (simple) representations have categorifications.
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  - ▶ **Kildetoft-Mackaay-Mazorchuk-Zimmermann & coauthors ∼2016.** There is a classification in dihedral Coxeter type.

This is what I am going to explain today.

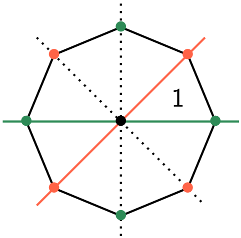
The main example today: dihedral groups

The dihedral groups are of Coxeter type $\mathrm{I}_2(n)$:

$$W_n = \langle s, t | s^2 = t^2 = 1, s_n = \underbrace{\ldots sts}_{n} = w_0 = \underbrace{\ldots tst}_{n} = t_n \rangle,$$

$$\text{e.g.: } W_4 = \langle s, t | s^2 = t^2 = 1, tsts = w_0 = stst \rangle$$

Example. These are the symmetry groups of regular n-gons, e.g. for $n = 4$; the Coxeter complex is:



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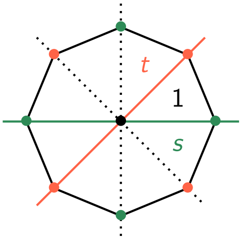

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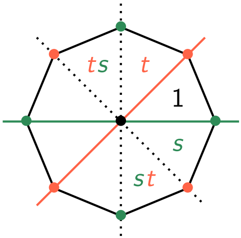

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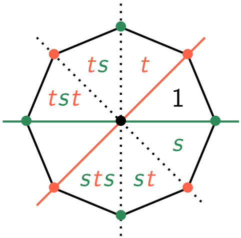

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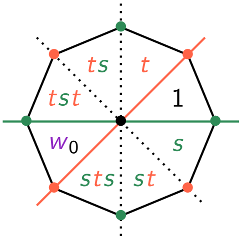

Kazhdan-Lusztig combinatorics of dihedral groups

Consider $\mathrm{W}_n = \mathbb{C}[W_n]$ for $n \in \mathbb{Z}_{>2} \cup \{\infty\}$ and define

$$\theta_s = s + 1, \qquad \theta_t = t + 1.$$

(This might remind you of the Kazhdan-Lusztig basis.)

These elements generate W_n and their relations are fully understood:

$$\theta_s\theta_s = 2 \cdot \theta_s, \qquad \theta_t\theta_t = 2 \cdot \theta_t, \qquad \text{a relation for } \underbrace{\ldots sts}_{n} = w_0 = \underbrace{\ldots tst}_{n}.$$

- Any categorical action will assign to these endofunctors $\boldsymbol{\theta}_s, \boldsymbol{\theta}_t$.
- The relations of $\theta_s = [\boldsymbol{\theta}_s]$ and $\theta_t = [\boldsymbol{\theta}_t]$ have to be satisfied in $[\cdot]_\oplus$.

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Working with the group is possible, but requires complexes and does not directly fit into the our setup.

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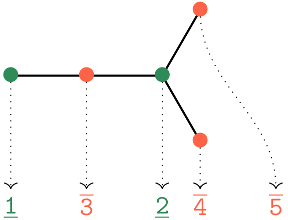
The principal graph - a preparatory example

Construct a W_∞-module V associated to a bipartite graph G:

$$\mathrm{V} = \langle \underline{1}, \underline{2}, \overline{3}, \overline{4}, \overline{5} \rangle_{\mathbb{C}}$$



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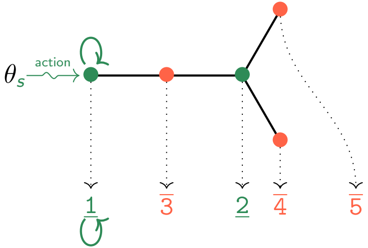

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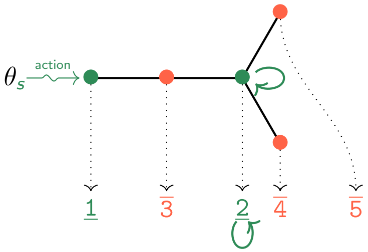

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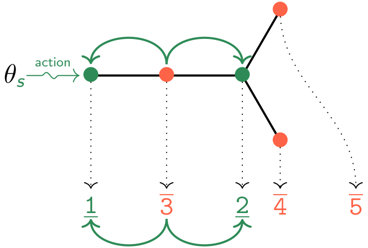

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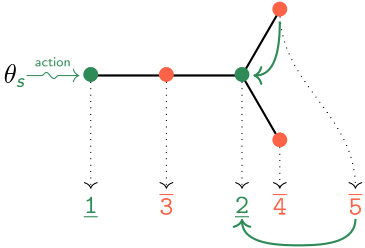

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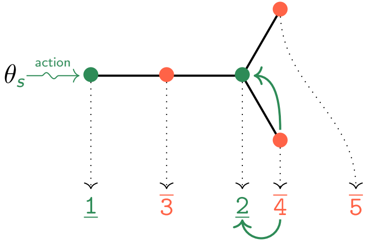

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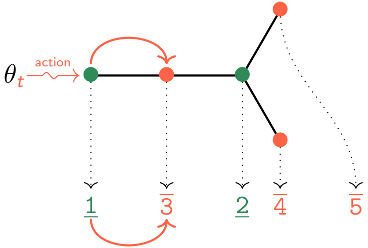

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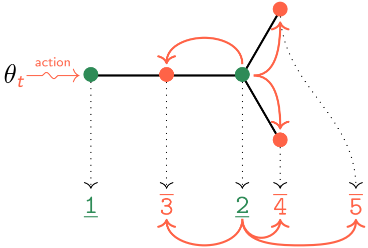

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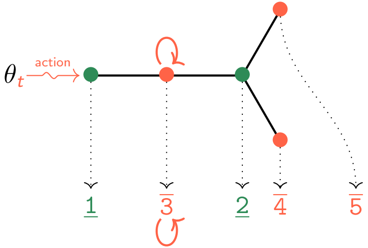

$$\theta_s = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

The principal graph - a preparatory example

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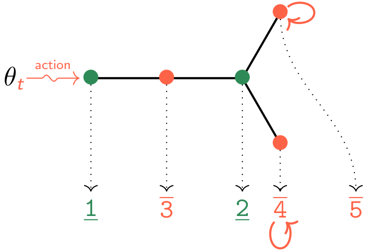

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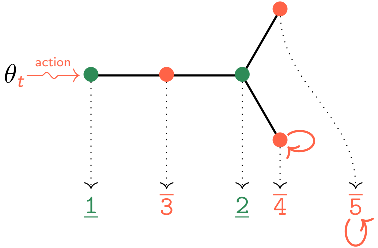

$$\theta_s = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

The first steps towards 1 and 2

Assume one has a category $\mathcal{V}$ and a "categorical action $\mathbf{m}\colon \mathrm{W}_n \to \mathscr{E}\mathrm{nd}(\mathcal{V})$".

Mazorchuk-Miemietz ∼2014, Zimmermann ∼2015: If $\mathbf{m}$ corresponds to a 2-atom, then there are two disjoint cases:

- ▷ If $\boldsymbol{\theta}_{w_0}$ does not act as zero, then $\mathbf{m}$ is trivial.
- ▷ Otherwise, there is an ordering of indecomposable objects in $\mathcal{V}$ such that

$$[\boldsymbol{\theta}_s] = \left(\begin{array}{ccc|c} 2 & 0 & 0 & \\ 0 & \ddots & 0 & A \\ 0 & 0 & 2 & \\ \hline & 0 & & 0 \end{array}\right), \qquad [\boldsymbol{\theta}_t] = \left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline & 2 & 0 & 0 \\ A^{\mathrm{T}} & 0 & \ddots & 0 \\ & 0 & 0 & 2 \end{array}\right)$$

(A similar statement is actually true in way bigger generality.)

The graph $G_{\mathbf{m}}$ for $\left(\begin{smallmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{smallmatrix}\right) \in \mathrm{Mat}_*(\mathbb{Z}_{\geq 0})$ is called the principal graph of $\mathbf{m}$.

"Killing" 1 and 2

By doing calculations in the Grothendieck group (checking the relations for the matrices corresponding to θ_s and θ_t) one gets:

A category $\mathcal{V}$ and a simple transitive 2-representation $\mathbf{m}$ as before can only exist if $G_{\mathbf{m}}$ is of ADE Dynkin type. Hereby, the Coxeter number h of $G_{\mathbf{m}}$ is $n-2$.

Thus, it is easy to write down the list of all candidates.

It remains 3 – the construction of the 2-representations.

"Killing" 1 and 2

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A category $\mathcal{V}$ and a [...]ore can only exist if $G_{\mathbf{m}}$ is of ADE Dyn[...] $G_{\mathbf{m}}$ is $n-2$.

Surprisingly: The condition of $[\theta_s]$,$[\theta_t]$ for the "braid relation" is hereby equivalent to $G_{\mathbf{m}}$ having spectral radius < 4.

Thus, it is easy to write down the list of all candidates.

It remains 3 – the construction of the 2-representations.

Honorable mentions

- ▷ There are so-called cell 2-representations $\mathbf{C}_{\mathcal{L}}$
  - ▶ Their definition involves only combinatorics of 1-morphisms – i.e. $\mathbf{C}_{\mathcal{L}}$ is basically determined on the Grothendieck group.
  - ▶ These work for any Coxeter group and categorify the cell representations of Kazhdan-Lusztig.
- ▷ Having a 2-representation $\mathbf{M}$ and a (coherent) symmetry ϕ of it, one can construct a orbit 2-representation $\mathbf{O}_{\mathbf{M},\phi}$.
- ▷ Direct construction by guessing a quiver algebra for the principal graphs.
  - ▶ Representations of the quiver algebra provides the categories $\mathbf{M}(\mathtt{i})$.
  - ▶ In dihedral type these are "zig-zag algebras" (in the sense of **Huerfano-Khovanov ∼2000**) for the graphs in question.

Examples

Honorable mentions

▷ There are so-called cell 2-representations $\mathbf{C}_{\mathcal{L}}$

- Their definit … orphisms – i.e. $\mathbf{C}_{\mathcal{L}}$ is basically det …
- These work … the cell representations of Kazhdan- …

> In the case of the symmetric group all simple transitive 2-representations are cell 2-representations.
> This is completely wrong in general.

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▸ Examples

(Co)algebras in 2-categories

The following result is inspired by work of Ostrik and several of his coauthors on fusion categories and related notions ∼2001++.

Up to some technicalities: For any transitive 2-representation **M** of a fiat 2-category $\mathscr{C}$ one can find a (co)algebra (1-morphism) in $\mathscr{C}$ whose (co)module 2-category is equivalent to **M**.

- ▷ A (co)algebra A in $\mathscr{C}$ has some (co)multiplication 2-morphism μ satisfying suitably formulated associativity and unit axioms. Its (Co)modules are pairs (M, α) with $\mathrm{M} \in \mathscr{C}$ and $\alpha \colon \mathrm{A} \circ \mathrm{M} \Rightarrow \mathrm{M}$ being the (co)action.

Example

- ▷ Checking if some 1-morphism has a (co)algebra structure is hard.
- ▷ However, a lot of (co)algebras are determined on the level of the Grothendieck group, e.g. (pseudo) idempotents in $[\cdot]_{\oplus}$ give rise to (co)algebras.
- ▷ There is a related Morita(-Takeuchi) 2-theory for these (co)algebras.

(Co)algebras in the dihedral case

[Algebra 1-morphism]	Diagram	W_n	[Module] dimension
θ_s	A_k	$n = k$	$n-1$
$\theta_s + \theta_{s_{n-1}}$	D_k	$n = 2k-2$	$\frac{1}{2}(n+2)$
$\theta_s + \theta_{s_7}$	E_6	$n = 12$	6
$\theta_s + \theta_{s_9} + \theta_{s_{17}}$	E_7	$n = 18$	7
$\theta_s + \theta_{s_{11}} + \theta_{s_{19}} + \theta_{s_{29}}$	E_8	$n = 30$	8

Similar in "tomato".

- ▷ The type A and D algebra 1-morphisms decategorify to (pseudo) idempotents in the Grothendieck group. Hence, without further work, we see that these are indeed algebra 1-morphisms.
- ▷ This is not true for the one's of type E.

(Co)algebras in the dihedral case

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Similar in "tomato"

Up to colors:
This completes again the classification of simple transitive 2-representations.

- ▷ The type A and D al… …udo) idempotents in the Grothendieck g… …see that these are indeed algebra 1-morphisms.
- ▷ This is not true for the one's of type E.

Concluding remarks

- ▸ Not all simple modules of the dihedral group are "categorifyable".
- Everything works graded as well.
- The dihedral story is just the tip of the iceberg: We hope that the general theory has impact beyond the case of Soergel calculus for finite Coxeter groups, e.g. for "Soergel calculi associated to complex reflection groups $G(n, n, m)$" **à la Elias**. ▸ Example
- There are various connections:
  - To the theory of subfactors, fusion categories etc. **à la Etingof-Gelaki-Nikshych-Ostrik**,...
  - To quantum groups at roots of unity and their "subgroups" **à la Etingof-Khovanov, Ocneanu, Kirillov-Ostrik**,...
  - To web calculi **à la Kuperberg, Cautis-Kamnitzer-Morrison**,...
- More?

There is still much to do...

Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside – top: first edition (1897); bottom: second edition (1911).

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The philosophy: If you have a very restrictive notion of "higher" representation theory, then your theory will be boring. If you have a very flexible notion, then your theory will be uncontrollable.

The (2-)categories and 2-representations which we consider are:

finitary	finiteness conditions
fiat 2-category	"finitary + involution + adjunction"
transitive 2-representation	finitary + connectivity condition
simple 2-representation	finitary + no 2-action stable 2-ideal

Plus some less important conditions à la $\mathbb{k}$-linearity etc.

Examples. Soergel bimodules and "cut-offs" of categorified quantum groups.

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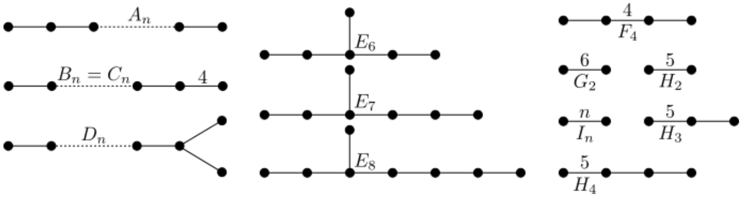

Figure: The Coxeter graphs of finite type.

Example. The type A family is given by the symmetric groups using the simple transpositions as generators.

(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Back

Define the Kazhdan-Lusztig basis elements (hereby $\leq$ denotes the Bruhat order)

$$\theta_w = \sum_{w' \leq w} w', \quad w, w' \in W_n,$$

e.g.: $\theta_s = s + 1, \qquad \theta_t = t + 1, \qquad \theta_{sts} = sts + ts + st + s + t + 1,$ etc.

These are our main players!

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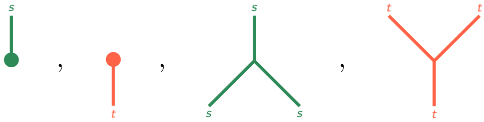
Elias-Khovanov ∼2009, Elias-Williamson ∼2013. For any Coxeter group W the Hecke 2-category $\mathscr{S}_W$ is given by diagrammtic generators and relations, e.g.:



Yes: Everything works graded as well.

Soergel ∼1992, Elias-Khovanov ∼2009, Elias-Williamson ∼2013. $\mathscr{S}_W$ categorifies W and indecomposable 1-morphisms decategorify to the Kazhdan-Lusztig basis.

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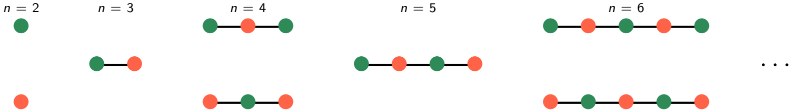
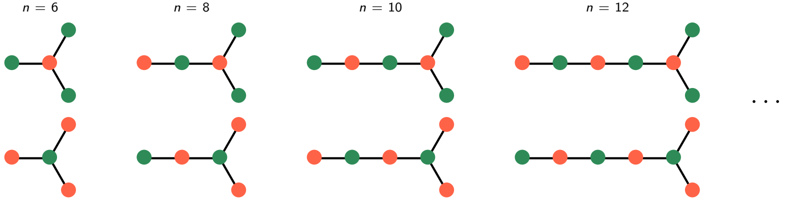
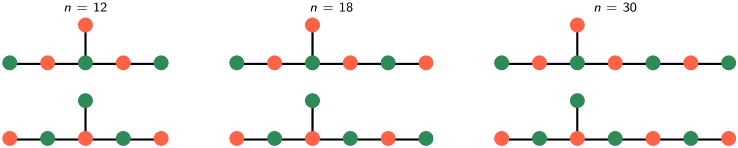
The type A family



The type D family



The type E exceptions



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Dihedral case	Type A	Type D	Type E
Cell	✓	✗	✗
Orbit	✓	✓	?
Quiver	✓	✓	✓

$n = 6$

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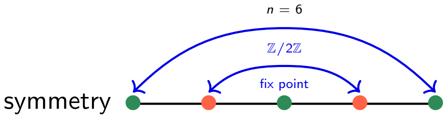
Dihedral case	Type A	Type D	Type E
Cell	✓	✗	✗
Orbit	✓	✓	?
Quiver	✓	✓	✓



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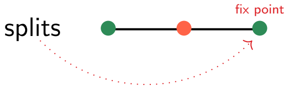
Dihedral case	Type A	Type D	Type E
Cell	✓	✗	✗
Orbit	✓	✓	?
Quiver	✓	✓	✓

$n = 6$



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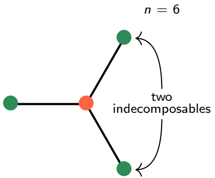
Dihedral case	Type A	Type D	Type E
Cell	✓	✗	✗
Orbit	✓	✓	?
Quiver	✓	✓	✓

orbit



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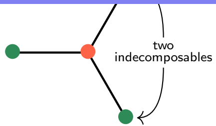
Dihedral case	Type A	Type D	Type E
Cell	✓	✗	✗
Orbit	✓	✓	?
Quiver			✓

In particular:
there are 2-representations
associated to the type
ADE principal graphs

orbit



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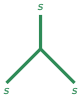

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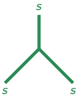

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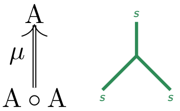

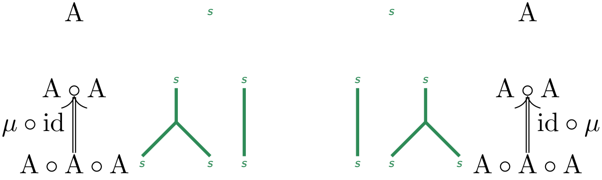

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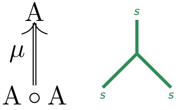

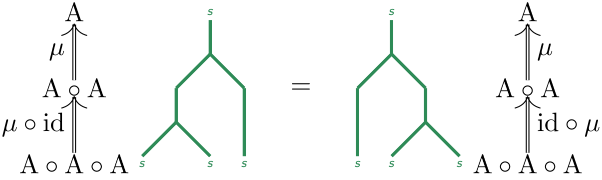

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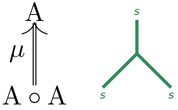

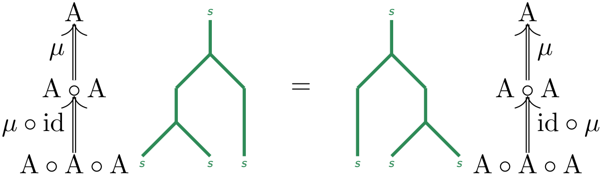

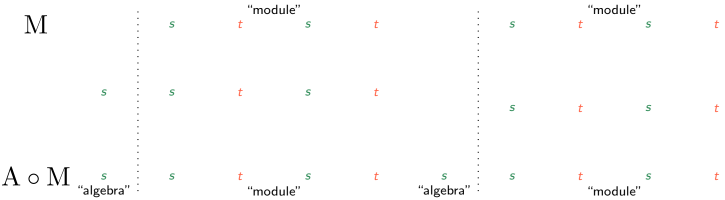

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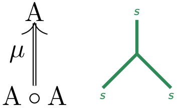
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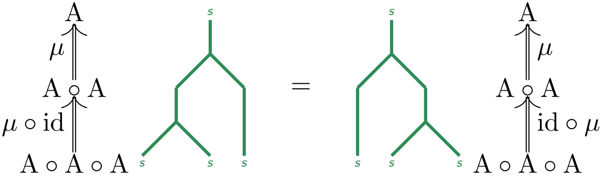
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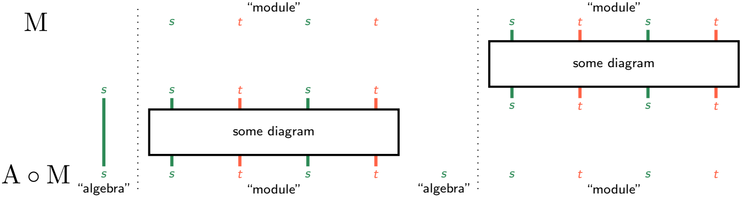
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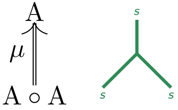
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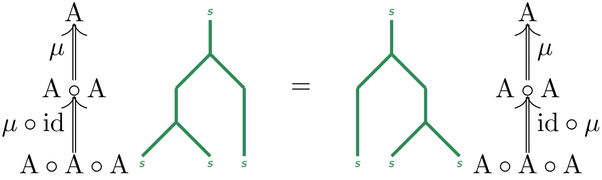
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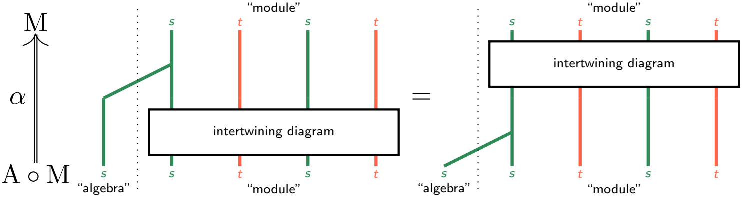
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Back

Let n be even. (The odd case is similar.) Then the simple W_n-modules are either one-dimensional or two-dimensional (for $k = 1, \dots, \frac{n-2}{2}$):

$$\mathrm{V}_{\pm\pm} = \mathbb{C}; \begin{cases} s \rightsquigarrow +1, -1; t \rightsquigarrow +1, -1, \\ \theta_s \rightsquigarrow 2, 0; \theta_t \rightsquigarrow 2, 0, \end{cases}$$

$$\mathrm{V}_k = \mathbb{C}^2; \begin{cases} s \rightsquigarrow \begin{pmatrix} \cos(\frac{2\pi k}{n}) & \sin(\frac{2\pi k}{n}) \\ \sin(\frac{2\pi k}{n}) & -\cos(\frac{2\pi k}{n}) \end{pmatrix}; t \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \theta_s \rightsquigarrow \begin{pmatrix} 2 \cdot \cos^2(\frac{\pi k}{n}) & \sin(\frac{2\pi k}{n}) \\ \sin(\frac{2\pi k}{n}) & 2 \cdot \sin^2(\frac{\pi k}{n}) \end{pmatrix}; \theta_t \rightsquigarrow \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \end{cases} \cong \mathrm{V}_k.$$

Most of these do not "categorify".

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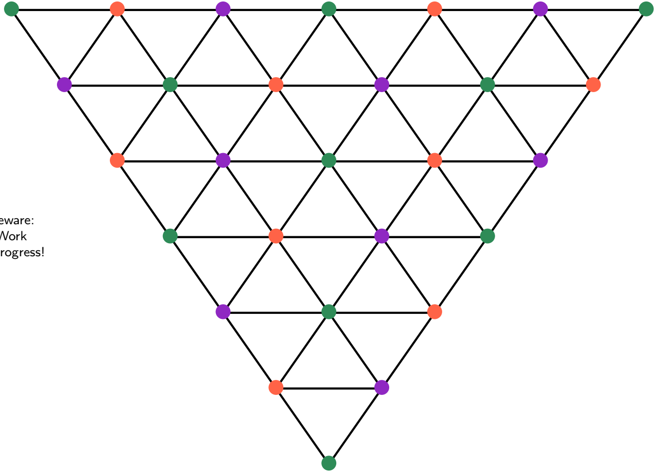
“Cell gives type A”



Beware:
Work
in progress!

“The case of $\mathrm{U}_q(\mathfrak{sl}_3)$ for $q^{18}=1$ which equals $G(18,18,3)$”

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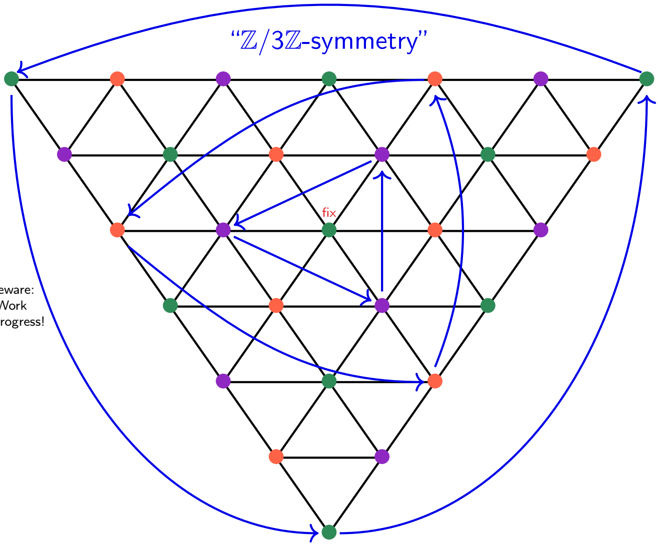
“$\mathbb{Z}/3\mathbb{Z}$-symmetry”



Beware:
Work
in progress!

“The case of $U_q(\mathfrak{sl}_3)$ for $q^{18} = 1$ which equals $G(18, 18, 3)$”

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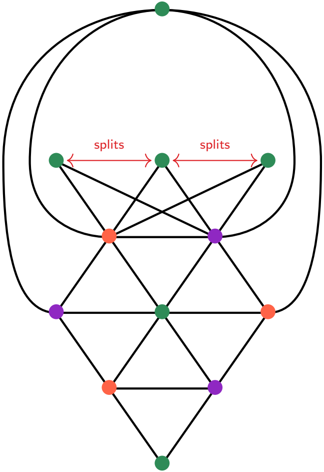
“Orbit gives type D”



Beware:
Work
in progress!

“The case of $\mathrm{U}_q(\mathfrak{sl}_3)$ for $q^{18}=1$ which equals $G(18,18,3)$”

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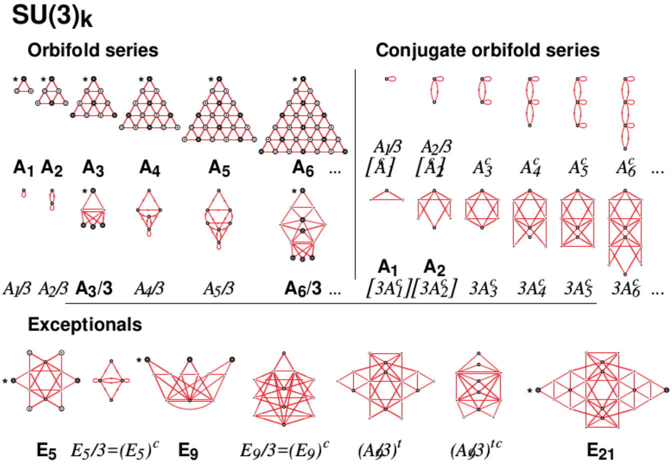

Figure: “Subgroups” of quantum $\mathrm{SU}(3)$.

(Picture from “The classification of subgroups of quantum $\mathrm{SU}(N)$” by Ocneanu $\sim$2000.)

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