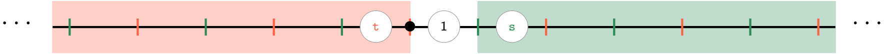
2-representations of Soergel bimodules—dihedral case

Or: Who colored my Dynkin diagrams?

Daniel Tubbenhauer



Joint with Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz and Xiaoting Zhang

September 2019

Let $A(\mathbf{\Gamma})$ be the adjacency matrix of a finite, connected, loopless graph $\mathbf{\Gamma}$. Let $\mathrm{U}_{e+1}(\mathtt{X})$ be the ▸ Chebyshev polynomial.

Classification problem (CP). Classify all $\mathbf{\Gamma}$ such that $\mathrm{U}_{e+1}(A(\mathbf{\Gamma}))=0$.

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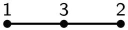
$$\mathrm{U}_3(\mathtt{X})=(\mathtt{X}-2\cos(\tfrac{\pi}{4}))\mathtt{X}(\mathtt{X}-2\cos(\tfrac{3\pi}{4}))$$

$\mathrm{A}_3 =$



$$\rightsquigarrow \quad A(\mathrm{A}_3)=\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix} \quad \rightsquigarrow \quad S_{\mathrm{A}_3}=\{2\cos(\tfrac{\pi}{4}),0,2\cos(\tfrac{3\pi}{4})\}$$

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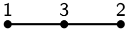
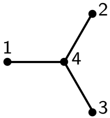


$$\rightsquigarrow \quad A(A_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \rightsquigarrow \quad S_{A_3} = \{2\cos(\tfrac{\pi}{4}), 0, 2\cos(\tfrac{3\pi}{4})\}$$

$$U_5(X) = (X - 2\cos(\tfrac{\pi}{6}))(X - 2\cos(\tfrac{2\pi}{6}))X(X - 2\cos(\tfrac{4\pi}{6}))(X - 2\cos(\tfrac{5\pi}{6}))$$

$D_4 =$



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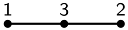

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✓ for $e=2$

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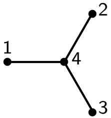

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Clas $= 0.$

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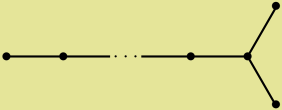
Smith ∼1969. The graphs solutions to (CP) are precisely ADE graphs for $e+2$ being (at most) the Coxeter number.

Type A_m:

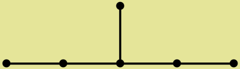
✓ for $e = m-1$

Type D_m:



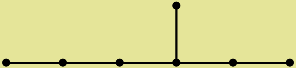
✓ for $e = 2m-4$

Type E_6:



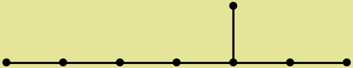
✓ for $e = 10$

Type E_7:



✓ for $e = 16$

Type E_8:



✓ for $e = 28$

1. **A bit of motivation**

2. **Dihedral (2-)representation theory**
   - Classical vs. $\mathbb{N}$-representation theory
   - Dihedral $\mathbb{N}$-representation theory
   - Categorified picture

3. **Non-semisimple fusion rings**
   - The asymptotic limit
   - The limit $\mathrm{v} \to 0$ of the $\mathbb{N}$-representations
   - Beyond

$\mathfrak{g}$ semisimple Lie algebra gives $\mathcal{O} \supset \mathcal{O}_0$.
Bernšteĭn–Gel'fand ∼1980. Projective functors $\mathcal{P}$ act on $\mathcal{O}_0$ and

$$\mathcal{O}_0 \curvearrowleft \mathcal{P} \xrightarrow{\text{decat.}} \mathbb{Z}[W] \curvearrowleft \mathbb{Z}[W]$$

categorifies the regular representation of the associated Weyl group W.
Aside. Add grading and get Hecke algebra.

List of properties.

- $\mathcal{P}$ is additive, Krull–Schmidt, $\mathbb{C}$-linear and monoidal, has finitely many indecomposables, and Hom-spaces are finite-dimensional. An adjoint of a projective functor is a projective functor. "Finitary/fiat acting 2-category"
- $\mathcal{O} \cong \mathrm{A}\text{-}p\mathcal{M}\mathrm{od}$ for A a finite-dimensional algebra. "Finitary 2-module"

Question. What kind of theory governs such actions? Our answer. Finitary 2-**representation theory.**

Goal. Classify the "simplest" such actions. "Simple transitive 2-modules or 2-simples"

$\mathfrak{g}$ semisimple
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Example/Theorem (Bernšteĭn–Gel'fand ∼1980).

$$\mathfrak{g} = \mathfrak{sl}_m.$$

2-simples are in 1:1 correspondence with simples of $W = S_m$.
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2-**repr**

Appearance of Soergel bimodules and there 2-representations in the wild.

$\mathcal{O}$, Hecke algebra, Kazhdan–Lusztig theory, braid group actions,
link homologies, modular representation theory, 3-manifold invariants,
tensor and fusion categories *etc.*

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Today: Dihedral 2-representation theory.

But keep in mind that we have a more general machinery to study such questions. (More tomorrow.)

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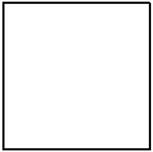
The dihedral groups are of ▸ Coxeter type $\mathrm{I}_2(e+2)$:

$$W_{e+2} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \overline{\mathtt{s}}_{e+2} = \underbrace{\ldots \mathtt{sts}}_{e+2} = w_0 = \underbrace{\ldots \mathtt{tst}}_{e+2} = \overline{\mathtt{t}}_{e+2} \rangle,$$

$$e.g.\text{:}\ \ W_4 = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \mathtt{tsts} = w_0 = \mathtt{stst} \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, *e.g.* for $e = 2$:



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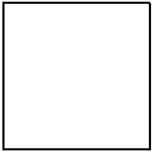

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$$W_{e+2} = \langle \mathtt{s}$$

Fact. The symmetries are given by exchanging flags.

$$= \overline{\mathtt{t}}_{e+2}\rangle,$$

$e+2$ $e+2$

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Fix a flag F.

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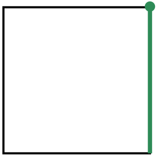

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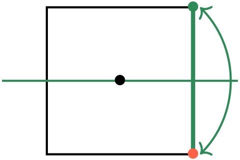

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Fix a flag F.

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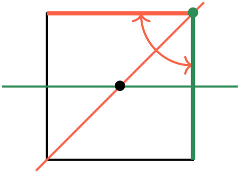
Fix a hyperplane H_1 permuting the adjacent 1-cells of F, *etc.*



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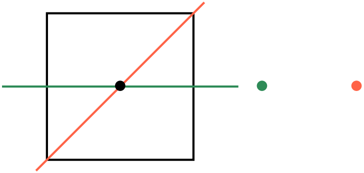

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This gives a generator-relation presentation.

$$W_{e+2} = \langle \mathtt{s},\mathtt{t} \mid \mathtt{s}^2=\mathtt{t}^2=1,\ \overline{\mathtt{s}}_{e+2} = \dots \mathtt{sts} = w_0 = \dots \mathtt{tst} = \overline{\mathtt{t}}_{e+2}\rangle,$$

And the braid relation measures the angle between hyperplanes.

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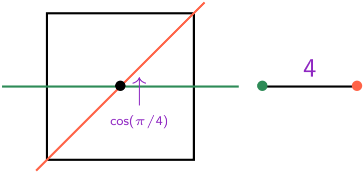
Connect i,j by an n-edge for H_i,H_j having angle $\cos(\pi/n)$.



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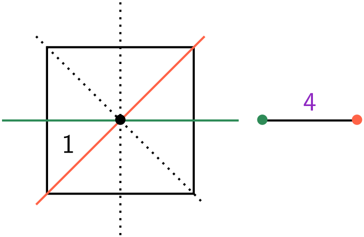

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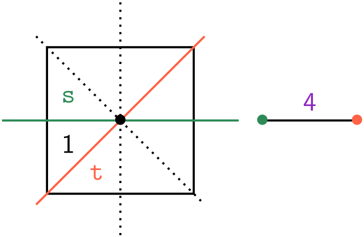

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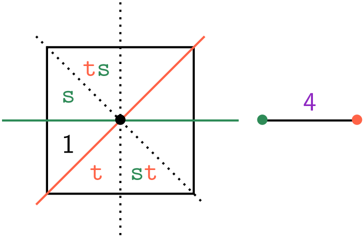

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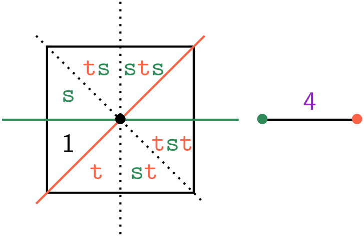

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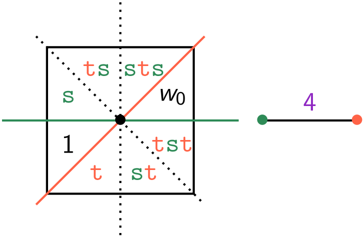

Dihedral representation theory on one slide.

The Bott–Samelson (BS) generators $b_{\mathtt{s}} = \mathtt{s} + 1, b_{\mathtt{t}} = \mathtt{t} + 1$.
There is also a Kazhdan–Lusztig (KL) basis c_w. We will nail it down later.

One-dimensional modules. $\mathrm{M}_{\lambda_{\mathtt{s}},\lambda_{\mathtt{t}}}, \lambda_{\mathtt{s}}, \lambda_{\mathtt{t}} \in \mathbb{C}, b_{\mathtt{s}} \mapsto \lambda_{\mathtt{s}}, b_{\mathtt{t}} \mapsto \lambda_{\mathtt{t}}$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_{0,0}, \mathrm{M}_{2,0}, \mathrm{M}_{0,2}, \mathrm{M}_{2,2}$	$\mathrm{M}_{0,0}, \mathrm{M}_{2,2}$

Two-dimensional modules. $\mathrm{M}_z, z \in \mathbb{C}, b_{\mathtt{s}} \mapsto \left(\begin{smallmatrix} 2 & z \\ 0 & 0 \end{smallmatrix}\right), b_{\mathtt{t}} \mapsto \left(\begin{smallmatrix} 0 & 0 \\ z & 2 \end{smallmatrix}\right)$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm} - \{0\}$	$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm}$

$\mathrm{V}_e = \mathrm{roots}(\mathsf{U}_{e+1}(\mathtt{X}))$ and $\mathrm{V}_e^{\pm}$ the $\mathbb{Z}/2\mathbb{Z}$-orbits under $z \mapsto -z$.

Dihedral representation theory on one slide.

One-dimension $_{\mathtt{t}}$.

Proposition (Lusztig?).

The list of one- and two-dimensional W_{e+2}-modules is a complete, irredundant list of simple modules.

$\mathrm{M}_{0,0}, \mathrm{M}_{2,0}, \mathrm{M}_{0,2}, \mathrm{M}_{2,2}$	$\mathrm{M}_{0,0}, \mathrm{M}_{2,2}$

I learned this construction in 2017.

Two-dimensional modules. $\mathrm{M}_z, z \in \mathbb{C}, b_{\mathtt{s}} \mapsto \left(\begin{smallmatrix} 2 & z \\ 0 & 0 \end{smallmatrix}\right), b_{\mathtt{t}} \mapsto \left(\begin{smallmatrix} 0 & 0 \\ \bar{z} & 2 \end{smallmatrix}\right)$.

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Example.

$\mathrm{M}_{0,0}$ is the sign representation and $\mathrm{M}_{2,2}$ is the trivial representation.

In case e is odd, $\mathrm{U}_{e+1}(\mathtt{X})$ has a constant term, so $\mathrm{M}_{2,0}, \mathrm{M}_{0,2}$ are not representations.

$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm} - \{0\}$	$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm}$

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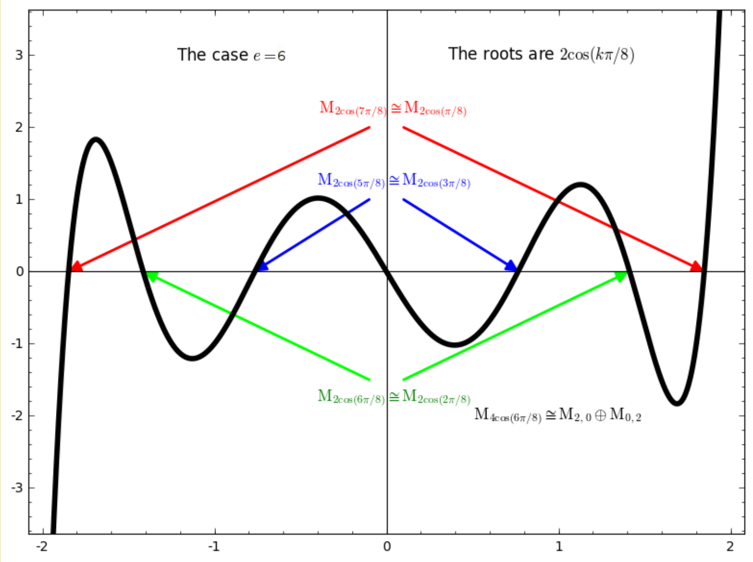
Dihedral representation theory on one slide

Example.

These representations are indexed by $\mathbb{Z}/2\mathbb{Z}$-orbits of the Chebyshev roots:



An algebra A with a fixed basis B^{A} is called a (multi) $\mathbb{N}$-algebra if

$$\mathrm{xy} \in \mathbb{N}\mathrm{B}^{\mathrm{A}} \quad (\mathrm{x}, \mathrm{y} \in \mathrm{B}^{\mathrm{A}}).$$

A A-module M with a fixed basis B^{M} is called a $\mathbb{N}$-module if

$$\mathrm{xm} \in \mathbb{N}\mathrm{B}^{\mathrm{M}} \quad (\mathrm{x} \in \mathrm{B}^{\mathrm{A}}, \mathrm{m} \in \mathrm{B}^{\mathrm{M}}).$$

These are $\mathbb{N}$-equivalent if there is a $\mathbb{N}$-valued change of basis matrix.

Example. $\mathbb{N}$-algebras and $\mathbb{N}$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}$-equivalence comes from 2-equivalence.

Example (group like).

Group algebras of finite groups with basis given by group elements are $\mathbb{N}$-algebras.

The regular module is an $\mathbb{N}$-module.

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Key example: $K_0(\mathcal{R}\mathrm{ep}(G,\mathbb{C}))$ (easy $\mathbb{N}$-representation theory).

Key example: $K_0(\mathcal{R}\mathrm{ep}_q^{ss}(\mathrm{U}_q(\mathfrak{g}))=\mathrm{G}_q)$ (intricate $\mathbb{N}$-representation theory).

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Exam $\mathbb{N}$-algebras and $\mathbb{N}$-modules arise naturally as the decategorification of 2-categories and 2 quivalence.

Example (semigroup like).

Hecke algebras of (finite) Coxeter groups with their KL basis are $\mathbb{N}$-algebras.

Their $\mathbb{N}$-representation theory is non-semisimple.

Clifford, Munn, Ponizovskiĭ, Green ∼1942++, Kazhdan–Lusztig ∼1979.

$\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ if y appears in zx with non-zero coefficient for $\mathrm{z} \in \mathrm{B}^{\mathrm{A}}$. $\mathrm{x} \sim_{\mathsf{L}} \mathrm{y}$ if $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ and $\mathrm{y} \leq_{\mathsf{L}} \mathrm{x}$.

$\sim_{\mathsf{L}}$ partitions A into left cells L. Similarly for right R, two-sided cells LR or $\mathbb{N}$-modules.

A $\mathbb{N}$-module M is transitive if all basis elements belong to the same $\sim_{\mathsf{L}}$ equivalence class. An apex of M is a maximal two-sided cell not killing it.

Fact. Each transitive $\mathbb{N}$-module has a unique apex.

Hence, one can study them cell-wise.

Example. Transitive $\mathbb{N}$-modules arise naturally as the decategorification of 2-simples.

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Example (group like).

Group algebras with the group element basis have only one cell, G itself.

Transitive ℕ-modules are $\mathbb{C}[G/H]$ for $H \subset G$ subgroup/conjugacy. The apex is G.

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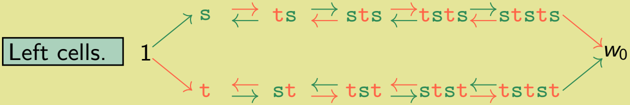
Hecke algebras for the dihedral group with KL basis have the following cells:



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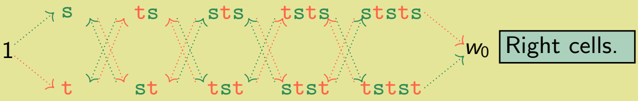

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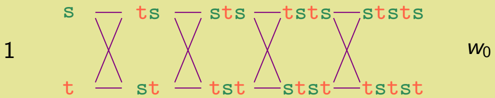


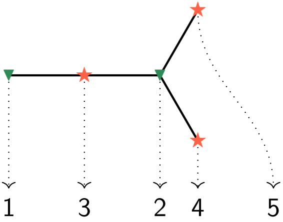
Two-sided cells.

We will see the transitive ℕ-modules in a second.

$\mathbb{N}$-modules via graphs.

Construct a W_∞-module M associated to a bipartite graph $\boldsymbol{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5\rangle$$



$$b_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad b_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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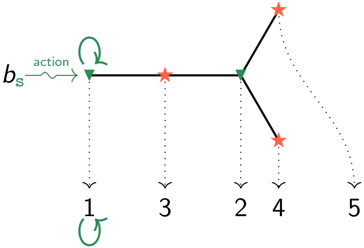

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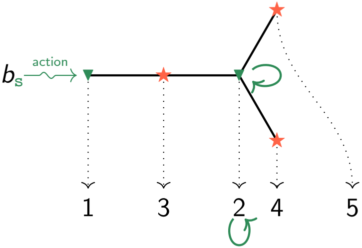

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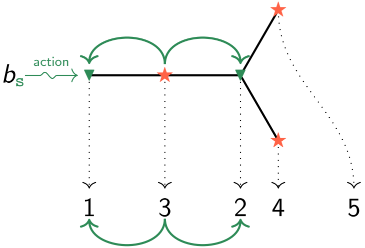

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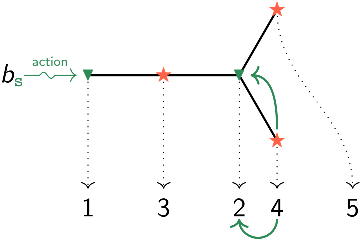

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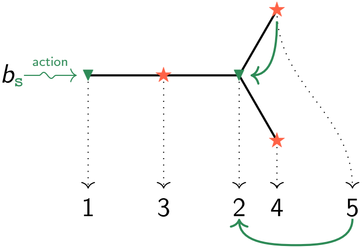

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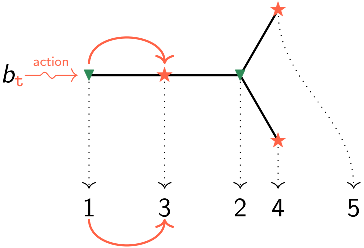

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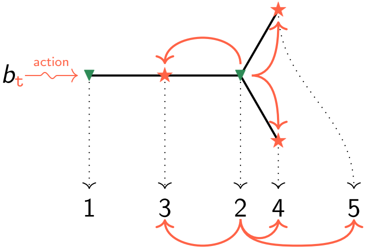

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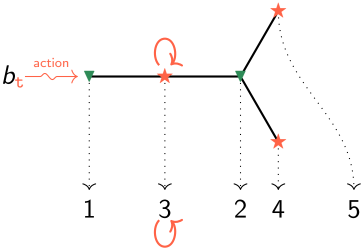

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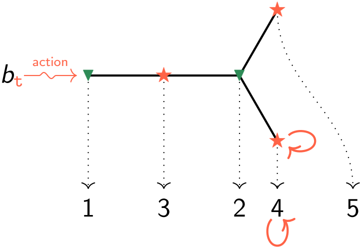

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$\mathbb{N}$-**modules via graphs.**

Construct a W_∞-module M associated to a bipartite graph $\mathbf{\Gamma}$:

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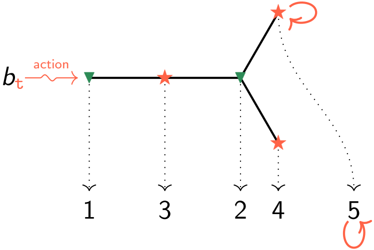

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ℕ-modules via graphs.

Construct a W_{∞}-module M associated to a bipartite graph Γ:

The adjacency matrix $A(\mathbf{\Gamma})$ of $\mathbf{\Gamma}$ is

$$A(\mathbf{\Gamma}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

These are W_{e+2}-modules for some e
only if $A(\mathbf{\Gamma})$ is killed by the Chebyshev polynomial $\mathrm{U}_{e+1}(\mathtt{X})$.

Morally speaking: These are constructed like the simples
but with integral matrices having the Chebyshev-roots as eigenvalues.

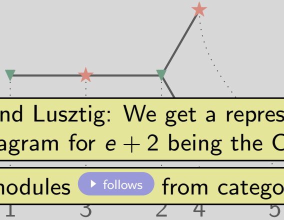
It is not hard to see that the Chebyshev–braid-like relation can not hold otherwise.

$$b_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad b_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}$-modules via graphs.

Construct a W_∞-module M associated to a bipartite graph $\mathbf{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5\rangle$$



Hence, by Smith's (CP) and Lusztig: We get a representation of W_{e+2} if $\mathbf{\Gamma}$ is a ADE Dynkin diagram for $e+2$ being the Coxeter number.

That these are $\mathbb{N}$-modules ▸ follows from categorification.

'Smaller solutions' are never $\mathbb{N}$-modules.

$$b_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad b_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{N}$-modules via graphs.

Construct a W_∞-module M associated to a bipartite graph Γ:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5 \rangle$$

Classification.

Complete, irredundant ▸ list of transitive $\mathbb{N}$-modules of W_{e+2}:

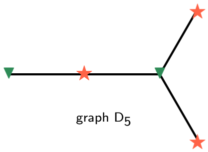
apex	① cell	s – t cell	w_0 cell
$\mathbb{N}$-reps.	$\mathrm{M}_{0,0}$	$\mathrm{M}_{\mathrm{ADE+bicolering}}$ for $e+2 =$ Cox. num.	$\mathrm{M}_{2,2}$

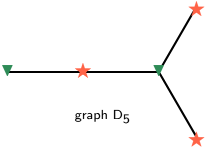
1 3 2 4 5

I learned this from Kildetoft–Mackaay–Mazorchuk–Zimmermann ∼2016.

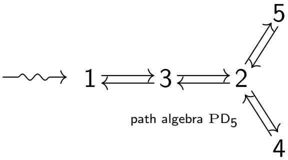
Fun fact about associated simples: ▸ Click.

$$b_s \rightsquigarrow \mathrm{M}_s = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad b_t \rightsquigarrow \mathrm{M}_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

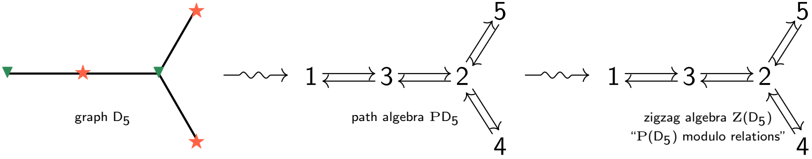




graph D_5



path algebra PD_5

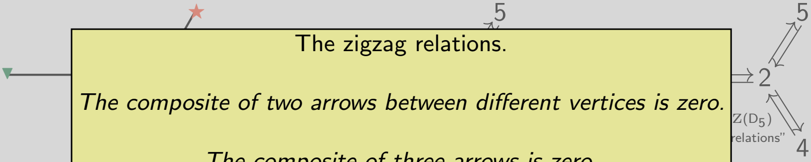


graph D_5

path algebra $\mathrm{P}\mathrm{D}_5$

zigzag algebra $\mathrm{Z}(\mathrm{D}_5)$

“$\mathrm{P}(\mathrm{D}_5)$ modulo relations”



The zigzag relations.

The composite of two arrows between different vertices is zero.

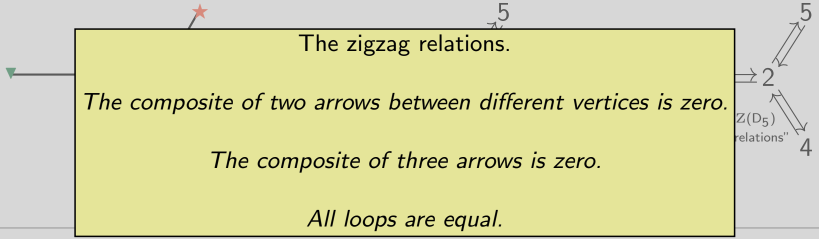
The composite of three arrows is zero.

All loops are equal.

Example.

There are two path from 2 to itself:
2 and $2|3|2 = 2|4|2 = 2|5|2$.

▷ A coherent choice of natural transformations can be made. (Skipped today.)



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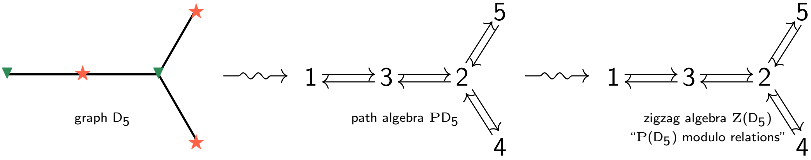
Example.

Projective left module $P_i = \mathrm{Z}(\mathrm{D}_5)i$.
Projective right module ${}_iP = i\mathrm{Z}(\mathrm{D}_5)$.
Bi-projective bimodule $P_i \otimes {}_iP$.



We get a categorical action of W_7:

- ▷ The category to act on is $Z(D_5)$-$p\mathcal{M}\mathrm{od}$.
- ▷ We have endofunctors $B_s = \bigoplus_{\blacktriangledown} P_i \otimes {}_iP \otimes \_$ and $B_t = \bigoplus_{\bigstar} P_j \otimes {}_jP \otimes \_$.
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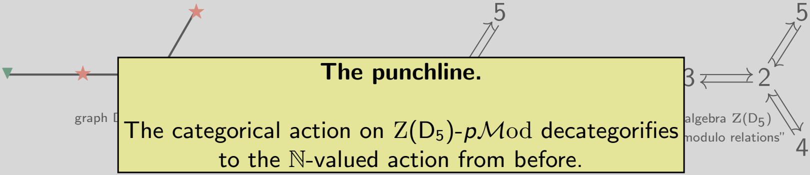

The punchline.

The categorical action on $\mathrm{Z}(\mathrm{D}_5)\text{-}p\mathcal{M}\mathrm{od}$ decategorifies to the $\mathbb{N}$-valued action from before.

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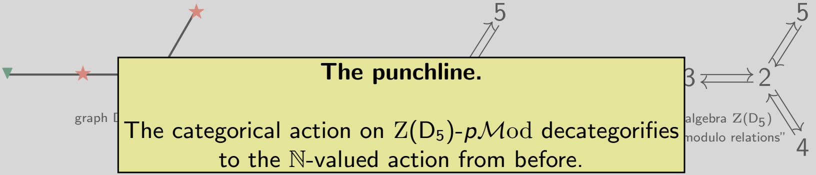

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One checks that $\mathrm{B}_{t}(P_2) \cong P_3 \oplus P_4 \oplus P_5$.

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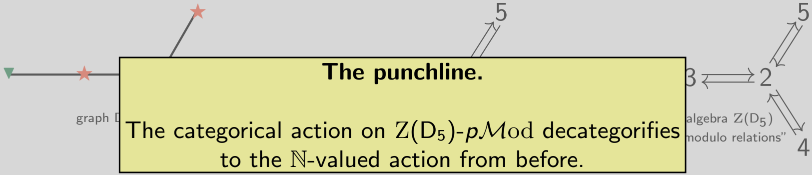

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- ▷ A today.)

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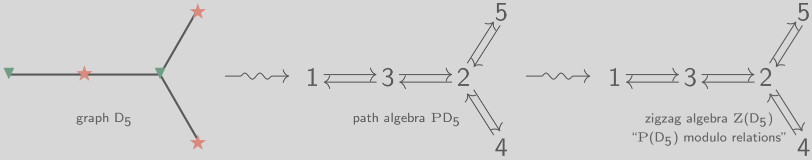
Example.

One easily checks that $\mathrm{B}_{s} \circ \mathrm{B}_{s} \cong \mathrm{B}_{s} \oplus \mathrm{B}_{s}$ and $\mathrm{B}_{t} \circ \mathrm{B}_{t} \cong \mathrm{B}_{t} \oplus \mathrm{B}_{t}$. This ensures a categorical action of W_∞.

Checking the braid-like relation for $n = 7$ is a bit harder, but not much.



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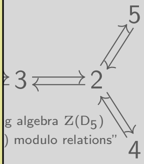

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Upshot of this approach.

Very explicit and one can get further consequences, *e.g.* a characterization of Dynkin diagrams.

Γ is a finite type ADE graph
if and only if
entries of $U_e(A)$ do not grow when $e \to \infty$.

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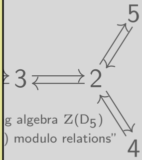

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Problem with this approach.

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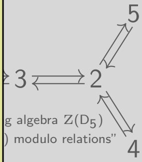

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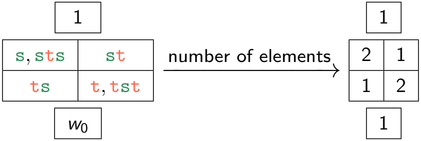
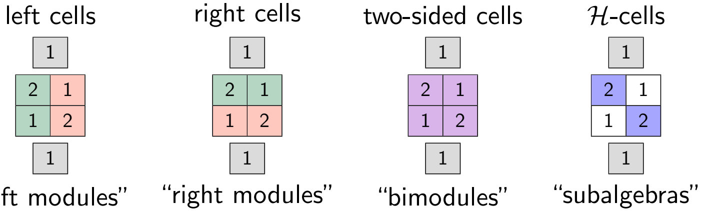
For the rest of today, I show you the
decategorification of something that does work in general.

Example ($I_2(4), e = 2$).

Cell structure:





Example ($\mathrm{I}_2(4), e = 2$).

Cell structure:

Example.

$$1 \cdot 1 = \mathrm{v}^0 1.$$

(v is the Hecke parameter deforming *e.g.* $\mathrm{s}^2 = 1$ to $T_{\mathrm{s}}^2 = (\mathrm{v}^{-1} - \mathrm{v})T_{\mathrm{s}} + 1$.)

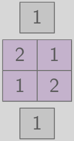
left cells

"left modules"

right cells

"right modules"

two-sided cells



"bimodules"

$\mathcal{H}$-cells

"subalgebras"

Example ($I_2(4)$, $e=2$).

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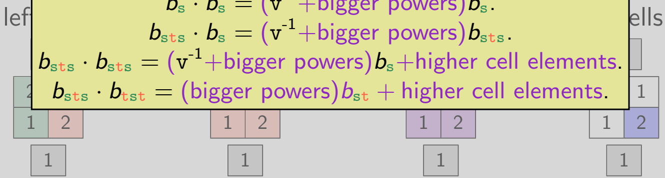
Example.

$$b_{\mathrm{s}} \cdot b_{\mathrm{s}} = (\mathrm{v}^{-1} + \text{bigger powers}) b_{\mathrm{s}}.$$
$$b_{\mathrm{sts}} \cdot b_{\mathrm{s}} = (\mathrm{v}^{-1} + \text{bigger powers}) b_{\mathrm{sts}}.$$
$$b_{\mathrm{sts}} \cdot b_{\mathrm{sts}} = (\mathrm{v}^{-1} + \text{bigger powers}) b_{\mathrm{s}} + \text{higher cell elements}.$$
$$b_{\mathrm{sts}} \cdot b_{\mathrm{tst}} = (\text{bigger powers}) b_{\mathrm{st}} + \text{higher cell elements}.$$



“left modules” “right modules” “bimodules” “subalgebras”

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lef

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$$b_s \cdot b_s = (v^{-1} + \text{bigger powers}) b_s.$$
$$b_{sts} \cdot b_s = (v^{-1} + \text{bigger powers}) b_{sts}.$$
$$b_{sts} \cdot b_{sts} = (v^{-1} + \text{bigger powers}) b_s + \text{higher cell elements}.$$
$$b_{sts} \cdot b_{tst} = (\text{bigger powers}) b_{st} + \text{higher cell elements}.$$

ells

2 1

1 2 1 2 1 2 1 2

1 1

Example.

$$b_{w_0} \cdot b_{w_0} = (v^{-4} + \text{bigger powers}) b_{w_0}.$$

“left modules”

“subalgebras”

Examp

Cell str

Fact (Lusztig $\sim$1980++).

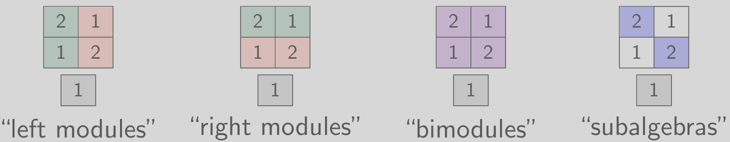
For any Coxeter group W
there is a well-defined function

$$\mathbf{a}\colon W \to \mathbb{N}$$

which is constant on two-sided cells such that
$\mathrm{v}^{\mathbf{a}} b_w$ has structure constants in $\mathbb{Z}[\mathrm{v}]$ up to higher cells.

Asymptotic limit $\mathrm{v} \to 0$ "$=$" kill non-leading terms of $c_w = \mathrm{v}^{\mathbf{a}} b_w$,
e.g. $c_s = \mathrm{v}^1 b_s$ and $c_s^2 = (1+\mathrm{v}^2) c_s$.

Think: Positively graded, and asymptotic limit is taking degree 0 part.



Compare multiplication tables. Example ($e = 2$).

a=asymptotic element and $[2] = 1 + \mathrm{v}^2$. (Note the “subalgebras”.)

	a_{s}	a_{sts}	a_{st}	a_{t}	a_{tst}	a_{ts}
a_{s}	a_{s}	a_{sts}	a_{st}			
a_{sts}	a_{sts}	a_{s}	a_{st}			
a_{ts}	a_{ts}	a_{ts}	$a_{t} + a_{tst}$			
a_{t}				a_{t}	a_{tst}	a_{ts}
a_{tst}				a_{tst}	a_{t}	a_{ts}
a_{st}				a_{st}	a_{st}	$a_{s} + a_{sts}$

	c_{s}	c_{sts}	c_{st}	c_{t}	c_{tst}	c_{ts}
c_{s}	$[2]c_{s}$	$[2]c_{sts}$	$[2]c_{st}$	c_{st}	$c_{st} + c_{w_0}$	$c_{s} + c_{sts}$
c_{sts}	$[2]c_{sts}$	$[2]c_{s} + [2]^2 c_{w_0}$	$[2]c_{st} + [2]c_{w_0}$	$c_{s} + c_{sts}$	$c_{s} + [2]^2 c_{w_0}$	$c_{s} + c_{sts} + [2]c_{w_0}$
c_{ts}	$[2]c_{ts}$	$[2]c_{ts} + [2]c_{w_0}$	$[2]c_{t} + [2]c_{tst}$	$c_{t} + c_{tst}$	$c_{t} + c_{tst} + [2]c_{w_0}$	$2c_{ts} + c_{w_0}$
c_{t}	c_{ts}	$c_{ts} + c_{w_0}$	$c_{t} + c_{tst}$	$[2]c_{t}$	$[2]c_{tst}$	$[2]c_{ts}$
c_{tst}	$c_{t} + c_{tst}$	$c_{t} + [2]^2 c_{w_0}$	$c_{t} + c_{tst} + [2]c_{w_0}$	$[2]c_{tst}$	$[2]c_{t} + [2]^2 c_{w_0}$	$[2]c_{ts} + [2]c_{w_0}$
c_{st}	$c_{s} + c_{sts}$	$c_{s} + c_{sts} + [2]c_{w_0}$	$2c_{st} + c_{w_0}$	$[2]c_{st}$	$[2]c_{st} + [2]c_{w_0}$	$[2]c_{s} + [2]c_{sts}$

The limit $\mathrm{v} \to 0$ is much simpler! Have you seen this ▸ before?

Back to graphs. Example ($e = 2$).

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3\rangle$$

$$c_{\mathtt{s}} \rightsquigarrow \begin{pmatrix} 1+\mathrm{v}^2 & 0 & \mathrm{v} \\ 0 & 1+\mathrm{v}^2 & \mathrm{v} \\ 0 & 0 & 0 \end{pmatrix} \qquad c_{\mathtt{t}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{v} & \mathrm{v} & 1+\mathrm{v}^2 \end{pmatrix}$$

$$c_{\mathtt{sts}} \rightsquigarrow \begin{pmatrix} 0 & 1+\mathrm{v}^2 & \mathrm{v} \\ 1+\mathrm{v}^2 & 0 & \mathrm{v} \\ 0 & 0 & 0 \end{pmatrix} \qquad c_{\mathtt{tst}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{v} & \mathrm{v} & 1+\mathrm{v}^2 \end{pmatrix}$$

$$c_{\mathtt{ts}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1+\mathrm{v}^2 & 1+\mathrm{v}^2 & \mathrm{v} \end{pmatrix} \qquad c_{\mathtt{st}} \rightsquigarrow \begin{pmatrix} \mathrm{v} & \mathrm{v} & 1+\mathrm{v}^2 \\ \mathrm{v} & \mathrm{v} & 1+\mathrm{v}^2 \\ 0 & 0 & 0 \end{pmatrix}$$

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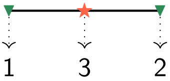

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Back to graphs. Example ($e = 2$).

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3\rangle$$

$$a_{s} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad a_{t} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Back

Example.

$$a_{\mathtt{st}}a_{\mathtt{ts}} = a_{\mathtt{s}} + a_{\mathtt{sts}}$$

$$\leftrightsquigarrow$$

$$[L_1][L_1] = [L_0] + [L_2]$$

$$\leftrightsquigarrow$$

$$\begin{pmatrix}0&0&1\\0&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\end{pmatrix} = \begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}.$$

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Back

Example.

$$a_{st}a_{ts} = a_s + a_{sts}$$

$$\leftrightsquigarrow$$

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This works in general and recovers the transitive $\mathbb{N}$-modules of $K_0(\mathrm{SL}(2)_q)$ found by Etingof–Khovanov ∼1995, Kirillov–Ostrik ∼2001 and Ostrik ∼2003, which are also ADE classified.

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However, at this point this was just an observation and it took a while until we understood its meaning.

(Cliffhanger: Wait for tomorrow.)

Back to graphs. Example ($e = 2$).

Classification.

Complete, irredundant list of graded 2-simples of dihedral Soergel bimodules:

apex	① cell	s – t cell	w_0 cell
2-reps.	$\mathbf{M}_{0,0}$	$\mathbf{M}_{\text{ADE+bicolering}}$ for $e + 2 =$ Cox. num.	$\mathbf{M}_{2,2}$

▸ Construction

$a_s \rightsquigarrow (0 \quad 1 \quad 0)$ $\qquad$ $a_t \rightsquigarrow (0 \quad 0 \quad 0)$

I learned this from Kildetoft–Mackaay–Mazorchuk–Zimmermann ∼2016.

$$a_{sts} \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad a_{tst} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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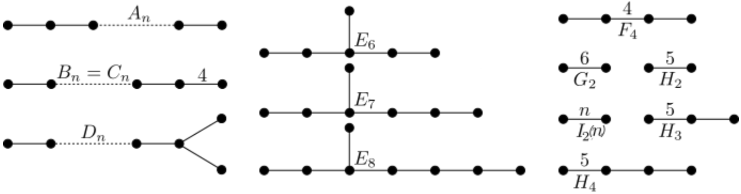
Proof?

The first proof was “brute force”.
Now we have a much better way of doing this.
(Again: cliffhanger.)

▸ Please stop!

Where to find $\mathrm{SL}(m)_q$?

First try: What are the asymptotic limits of finite types?



- ▶ No luck in finite Weyl type: $\mathrm{v} \to 0$ is (almost always) $\mathcal{R}\mathrm{ep}((\mathbb{Z}/2\mathbb{Z})^k)$.
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- ▷ Maybe generalize the dihedral case?

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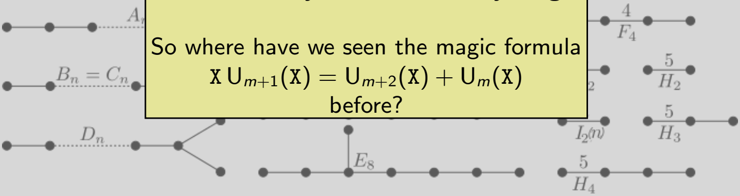

Idea 1: Chebyshev knows everything!

So where have we seen the magic formula
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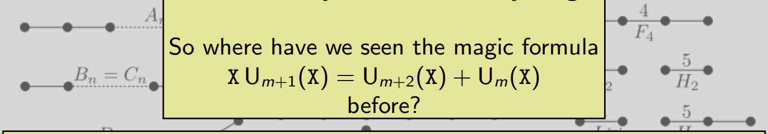

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$$[2] \cdot [e+1] = [e+2] + [e]$$

$$\mathrm{L}_1 \otimes \mathrm{L}_{e+1} \cong \mathrm{L}_{e+2} \oplus \mathrm{L}_e$$

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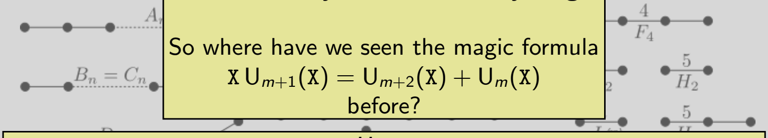

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- N
- No l
- No l
- ▷ Mayb

Idea 2: The dihedral type is
a quotient of affine type A_1.

Very vague philosophy I want to sell:

Fusion categories appear as degree 0 parts of Soergel bimodules.

Whe

Quantum Satake (Elias ∼2013, Mackaay–Mazorchuk–Miemietz ∼2018)
– rough version.

$\mathrm{SL}(m)_q$ is the semisimple version of
a subquotient of Soergel bimodules for affine type A_{m-1}.
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Beware: Only the cases $m = 2$ (dihedral) and $m = 3$ (trihedral) are proven,
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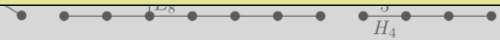

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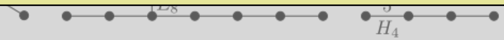


Summary of Nhedral.

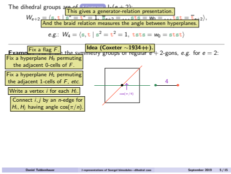
Most questions are still open, but nice patterns appear.

Leaves the realm of groups. (No associated Coxeter group; only a subquotient.)

Generalized zigzag algebras, Chebyshev polynomials and ADE diagrams appear.

ADE-type classification(?) of 2-representations.

Fusion: $\mathrm{SL}(m)_q$ appears.



The dihedral groups are of ▸ Coxeter type $I_2(e+2)$:

This gives a generator-relation presentation.

$W_{e+2} = \langle s,t \mid s^2 = t^2 = 1, \overline{s}_{e+2} = \dots sts = w_0 = \dots tst = \overline{t}_{e+2}\rangle$,

And the braid relation measures the angle between hyperplanes.

e.g.: $W_4 = \langle s,t \mid s^2 = t^2 = 1, tsts = w_0 = stst\rangle$

Idea (Coxeter ∼1934++).

Example. … the symmetry groups of regular $e+2$-gons, *e.g.* for $e = 2$:

Fix a flag F.

Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

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Write a vertex i for each H_i.

Connect i, j by an n-edge for H_i, H_j having angle $\cos(\pi/n)$.

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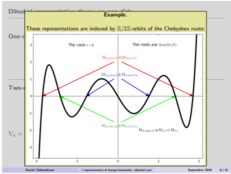

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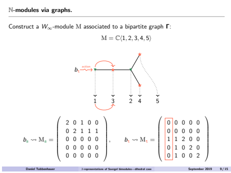

$\mathbb{N}$-modules via graphs.

Construct a W_∞-module M associated to a bipartite graph Γ:

$$\mathrm{M} = \mathbb{C}\langle 1,2,3,4,5\rangle$$

$$b_s \rightsquigarrow \mathrm{M}_s = \begin{pmatrix} 2&0&1&0&0\\ 0&2&1&1&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \qquad b_t \rightsquigarrow \mathrm{M}_t = \begin{pmatrix} \boxed{\begin{matrix}0\\0\\1\\0\\0\end{matrix}} & \begin{matrix}0&0&0&0\\0&0&0&0\\1&2&0&0\\1&0&2&0\\1&0&0&2\end{matrix} \end{pmatrix}$$

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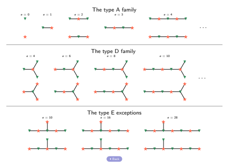
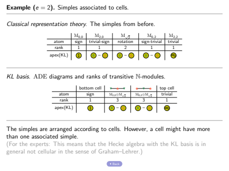


The type A family

The type D family

The type E exceptions



Example ($e=2$). Simples associated to cells.

Classical representation theory. The simples from before.

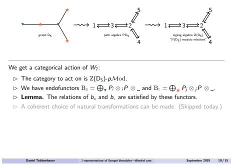
	$M_{0,0}$	$M_{2,0}$	$M_{\sqrt{2}}$	$M_{0,2}$	$M_{2,2}$
atom	sign	trivial-sign	rotation	sign-trivial	trivial
rank	1	1	2	1	1
apex(KL)	①	ⓢ – ⓣ	ⓢ – ⓣ	ⓢ – ⓣ	w_0

KL basis. ADE diagrams and ranks of transitive $\mathbb{N}$-modules.

	bottom cell			top cell
atom	sign	$M_{2,0} \oplus M_{\sqrt{2}}$	$M_{0,2} \oplus M_{\sqrt{2}}$	trivial
rank	1	3	3	1
apex(KL)	①	ⓢ – ⓣ	ⓢ – ⓣ	w_0

The simples are arranged according to cells. However, a cell might have more than one associated simple.
(For the experts: This means that the Hecke algebra with the KL basis is in general not cellular in the sense of Graham–Lehrer.)

◂ Back



We get a categorical action of W_7:

- ▷ The category to act on is $Z(D_5)$-$p\mathcal{M}od$.
- ▷ We have endofunctors $B_s = \bigoplus_{\blacktriangledown} P_i \otimes {}_iP \otimes \_$ and $B_t = \bigoplus_{\bigstar} P_j \otimes {}_jP \otimes \_$.
- ▷ **Lemma.** The relations of b_s and b_t are satisfied by these functors.
- ▷ A coherent choice of natural transformations can be made. (Skipped today.)

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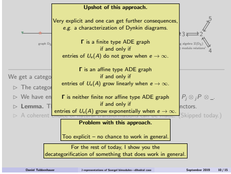

Upshot of this approach.

Very explicit and one can get further consequences, *e.g.* a characterization of Dynkin diagrams.

Γ is a finite type ADE graph
if and only if
entries of $U_e(A)$ do not grow when $e \to \infty$.

Γ is an affine type ADE graph
if and only if
entries of $U_e(A)$ grow linearly when $e \to \infty$.

Γ is neither finite nor affine type ADE graph
if and only if
entries of $U_e(A)$ grow exponentially when $e \to \infty$.

Problem with this approach.

Too explicit – no chance to work in general.

For the rest of today, I show you the decategorification of something that does work in general.

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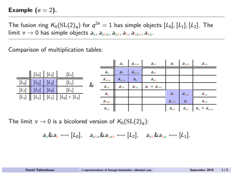

Example ($e=2$).

The fusion ring $K_0(\mathrm{SL}(2)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_1], [L_2]$. The limit $v \to 0$ has simple objects $a_s, a_{sts}, a_{st}, a_t, a_{tst}, a_{ts}$.

Comparison of multiplication tables:

	$[L_0]$	$[L_2]$	$[L_1]$
$[L_0]$	$[L_0]$	$[L_2]$	$[L_1]$
$[L_2]$	$[L_2]$	$[L_0]$	$[L_1]$
$[L_1]$	$[L_1]$	$[L_1]$	$[L_0] + [L_2]$

&

	a_s	a_{sts}	a_{st}	a_t	a_{tst}	a_{ts}
a_s	a_s	a_{sts}	a_{st}			
a_{sts}	a_{sts}	a_s	a_{st}			
a_{ts}	a_{ts}	a_{ts}	$a_t + a_{tst}$			
a_t				a_t	a_{tst}	a_{ts}
a_{tst}				a_{tst}	a_t	a_{ts}
a_{st}				a_{st}	a_{st}	$a_s + a_{sts}$

The limit $v \to 0$ is a bicolored version of $K_0(\mathrm{SL}(2)_q)$:

$$a_s \& a_t \leftrightsquigarrow [L_0], \quad a_{sts} \& a_{tst} \leftrightsquigarrow [L_2], \quad a_{st} \& a_{ts} \leftrightsquigarrow [L_1].$$

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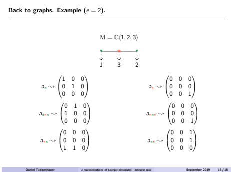

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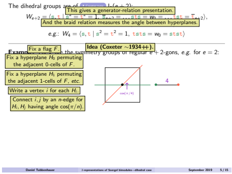
There is still much to do…



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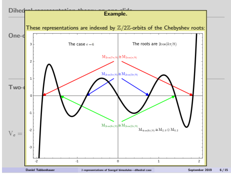

Dihedral representation theory on one slide

One-d

Two-

$V_e =$

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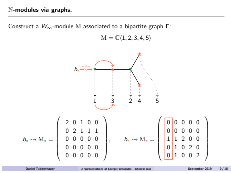

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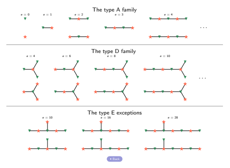

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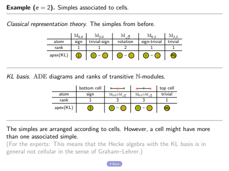

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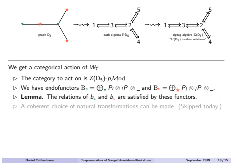
rank	1	3	3	1
apex(KL)	1	s – t	s – t	w_0

The simples are arranged according to cells. However, a cell might have more than one associated simple.

(For the experts: This means that the Hecke algebra with the KL basis is in general not cellular in the sense of Graham–Lehrer.)

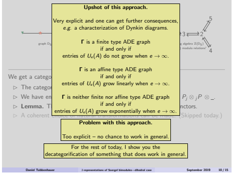
Back



We get a categorical action of W_7:

- ▷ The category to act on is $Z(D_5)$-$p\mathcal{M}od$.
- ▷ We have endofunctors $B_s = \bigoplus_{▼} P_i \otimes {}_iP \otimes \_$ and $B_t = \bigoplus_{★} P_j \otimes {}_jP \otimes \_$.
- ▷ **Lemma.** The relations of b_s and b_t are satisfied by these functors.
- ▷ A coherent choice of natural transformations can be made. (Skipped today.)

Daniel Tubbenhauer 2-representations of Soergel bimodules—dihedral case September 2019 10 / 15



Upshot of this approach.

Very explicit and one can get further consequences, *e.g.* a characterization of Dynkin diagrams.

Γ is a finite type ADE graph
if and only if
entries of $U_e(A)$ do not grow when $e \to \infty$.

Γ is an affine type ADE graph
if and only if
entries of $U_e(A)$ grow linearly when $e \to \infty$.

Γ is neither finite nor affine type ADE graph
if and only if
entries of $U_e(A)$ grow exponentially when $e \to \infty$.

Problem with this approach.

Too explicit – no chance to work in general.

For the rest of today, I show you the decategorification of something that does work in general.

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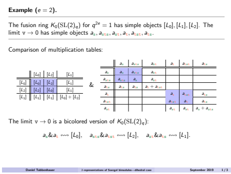

Example ($e = 2$).

The fusion ring $K_0(\mathrm{SL}(2)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_1], [L_2]$. The limit $v \to 0$ has simple objects $a_s, a_{sts}, a_{st}, a_t, a_{tst}, a_{ts}$.

Comparison of multiplication tables:

	$[L_0]$	$[L_2]$	$[L_1]$
$[L_0]$	$[L_0]$	$[L_2]$	$[L_1]$
$[L_2]$	$[L_2]$	$[L_0]$	$[L_1]$
$[L_1]$	$[L_1]$	$[L_1]$	$[L_0] + [L_2]$

&

	a_s	a_{sts}	a_{st}	a_t	a_{tst}	a_{ts}
a_s	a_s	a_{sts}	a_{st}			
a_{sts}	a_{sts}	a_s	a_{st}			
a_{ts}	a_{ts}	a_{ts}	$a_t + a_{tst}$			
a_t				a_t	a_{tst}	a_{ts}
a_{tst}				a_{tst}	a_t	a_{ts}
a_{st}				a_{st}	a_{st}	$a_s + a_{sts}$

The limit $v \to 0$ is a bicolored version of $K_0(\mathrm{SL}(2)_q)$:

$$a_s \& a_t \leftrightsquigarrow [L_0], \quad a_{sts} \& a_{tst} \leftrightsquigarrow [L_2], \quad a_{st} \& a_{ts} \leftrightsquigarrow [L_1].$$

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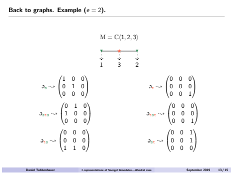

Back to graphs. Example ($e = 2$).

$$M = \mathbb{C}\langle 1, 2, 3 \rangle$$

$$a_s \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad a_t \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_{sts} \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad a_{tst} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_{ts} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad a_{st} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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Thanks for your attention!

$$\mathsf{U}_0(\mathtt{X}) = 1,\quad \mathsf{U}_1(\mathtt{X}) = \mathtt{X},\quad \mathtt{X}\,\mathsf{U}_{e+1}(\mathtt{X}) = \mathsf{U}_{e+2}(\mathtt{X}) + \mathsf{U}_e(\mathtt{X})$$

$$\mathsf{U}_0(\mathtt{X}) = 1,\quad \mathsf{U}_1(\mathtt{X}) = 2\mathtt{X},\quad 2\mathtt{X}\,\mathsf{U}_{e+1}(\mathtt{X}) = \mathsf{U}_{e+2}(\mathtt{X}) + \mathsf{U}_e(\mathtt{X})$$

Kronecker $\sim$1857. Any complete set of conjugate algebraic integers in $]-2,2[$ is a subset of $\mathrm{roots}(\mathsf{U}_{e+1}(\mathtt{X}))$ for some e.

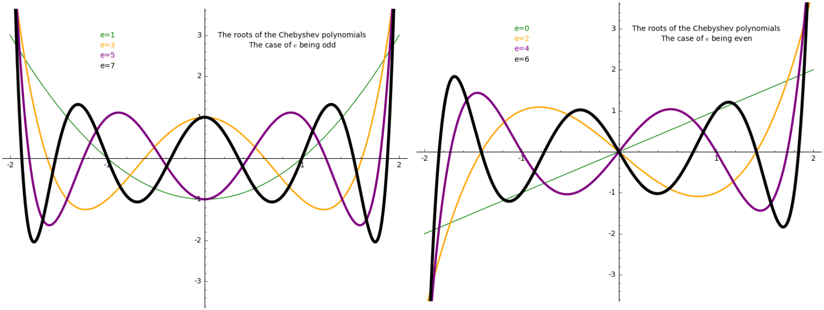


Figure: The roots of the Chebyshev polynomials (of the second kind).

◂ Back

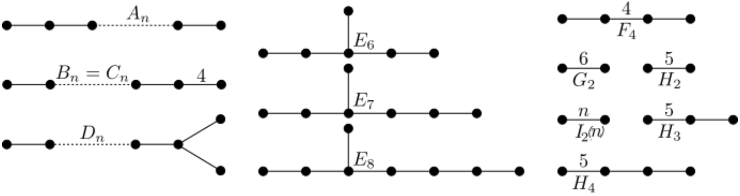


Figure: The connected Coxeter diagrams of finite type. Their numbers ordered by dimension: $1, \infty, 3, 5, 3, 4, 4, 4, 3, 3, 3, 3, 3, \ldots$.

Examples.

Type A_3 $\leftrightsquigarrow$ tetrahedron $\leftrightsquigarrow$ symmetric group S_4.
Type B_3 $\leftrightsquigarrow$ cube/octahedron $\leftrightsquigarrow$ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$.
Type H_3 $\leftrightsquigarrow$ dodecahedron/icosahedron $\leftrightsquigarrow$ exceptional Coxeter group.

(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Back

The positivity on the KL basis is non-trivial.

Example ($e = 2$). What happens for a different graph? For example,

$$\Gamma =$$


$$A(\Gamma) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

$$b_1 \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$b_{s} \rightsquigarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}, b_{ts} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 2 & 5 \end{pmatrix}, b_{sts} \rightsquigarrow \begin{pmatrix} 8 & 4 & 10 \\ 4 & 2 & 5 \\ 0 & 0 & 0 \end{pmatrix},$$

$$b_{t} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 2 \end{pmatrix}, b_{st} \rightsquigarrow \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, b_{tst} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 10 & 5 & 10 \end{pmatrix},$$

$$b_{stst} \rightsquigarrow \begin{pmatrix} 20 & 10 & 20 \\ 10 & 5 & 10 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 20 & 10 & 25 \end{pmatrix} \leftsquigarrow b_{tsts}.$$

◂ Back

The positivity on

Example ($e = 2$)

mple,

KL basis.

$$c_1 = b_1,\ c_{s} = b_{s},\ c_{t} = b_{t},\ c_{ts} = b_{ts},\ c_{st} = b_{st},$$

but

$$c_{sts} = b_{sts} - b_{s} \text{ and } c_{tst} = b_{tst} - b_{t}$$

and $c_{stst} = b_{stst} - 2b_{st}$ and $c_{tsts} = b_{tsts} - 2b_{ts}$.

$$b_1 \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$b_{s} \rightsquigarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ b_{ts} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 2 & 5 \end{pmatrix},\ b_{sts} \rightsquigarrow \begin{pmatrix} 8 & 4 & 10 \\ 4 & 2 & 5 \\ 0 & 0 & 0 \end{pmatrix},$$

$$b_{t} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 2 \end{pmatrix},\ b_{st} \rightsquigarrow \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},\ b_{tst} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 10 & 5 & 10 \end{pmatrix},$$

$$b_{stst} \rightsquigarrow \begin{pmatrix} 20 & 10 & 20 \\ 10 & 5 & 10 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 20 & 10 & 25 \end{pmatrix} \leftsquigarrow b_{tsts}.$$

◂ Back

The positivity on

Example ($e=2$) mple,

KL basis.

$$c_{1}=b_{1},\ c_{\mathtt{s}}=b_{\mathtt{s}},\ c_{\mathtt{t}}=b_{\mathtt{t}},\ c_{\mathtt{ts}}=b_{\mathtt{ts}},\ c_{\mathtt{st}}=b_{\mathtt{st}},$$

but

$$c_{\mathtt{sts}}=b_{\mathtt{sts}}-b_{\mathtt{s}} \text{ and } c_{\mathtt{tst}}=b_{\mathtt{tst}}-b_{\mathtt{t}}$$

$$\text{and } c_{\mathtt{stst}}=b_{\mathtt{stst}}-2b_{\mathtt{st}} \text{ and } c_{\mathtt{tsts}}=b_{\mathtt{tsts}}-2b_{\mathtt{ts}}.$$

$$c_{1}\rightsquigarrow\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},$$

$$c_{\mathtt{s}}\rightsquigarrow\begin{pmatrix}2&0&2\\0&2&1\\0&0&0\end{pmatrix},c_{\mathtt{ts}}\rightsquigarrow\begin{pmatrix}0&0&0\\0&0&0\\4&2&5\end{pmatrix},c_{\mathtt{sts}}\rightsquigarrow\begin{pmatrix}6&4&8\\4&0&4\\0&0&0\end{pmatrix},$$

$$c_{\mathtt{t}}\rightsquigarrow\begin{pmatrix}0&0&0\\0&0&0\\2&1&2\end{pmatrix},c_{\mathtt{st}}\rightsquigarrow\begin{pmatrix}4&2&4\\2&1&2\\0&0&0\end{pmatrix},c_{\mathtt{tst}}\rightsquigarrow\begin{pmatrix}0&0&0\\0&0&0\\8&4&8\end{pmatrix},$$

$$c_{\mathtt{stst}}\rightsquigarrow\begin{pmatrix}12&6&12\\6&3&6\\0&0&0\end{pmatrix}\neq\begin{pmatrix}0&0&0\\0&0&0\\12&16&15\end{pmatrix}\leftsquigarrow c_{\mathtt{tsts}}.$$

◂ Back

The positivity on the KL basis is non-trivial.

Example ($e = 2$). What happens for a different graph? For example,

$$\Gamma = $$

$$, \quad A(\Gamma) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem.

For the infinite dihedral group
all except the ADE graphs work.
The only proof of this I know uses categorification.

$$c_1 \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},$$

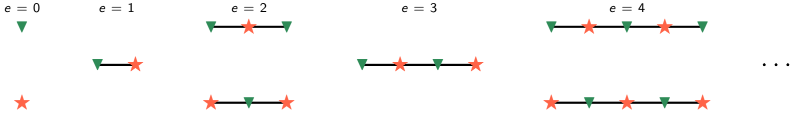
This implies that the Chebyshev polynomials evaluated at non-ADE graphs
stay positive for all e.
Note that this is much harder to prove
than the vanishing of the Chebyshev polynomials.

$$c_{t} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 2 \end{pmatrix}, c_{st} \rightsquigarrow \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, c_{tst} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 4 & 8 \end{pmatrix},$$

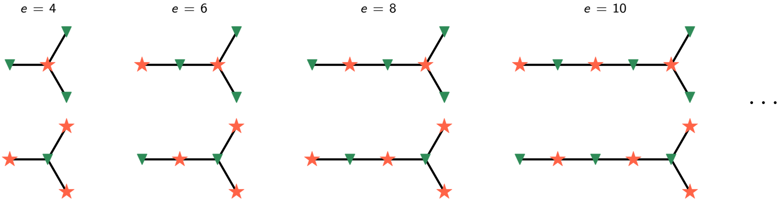
$$c_{stst} \rightsquigarrow \begin{pmatrix} 12 & 6 & 12 \\ 6 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 16 & 15 \end{pmatrix} \leftsquigarrow c_{tsts}.$$

Back

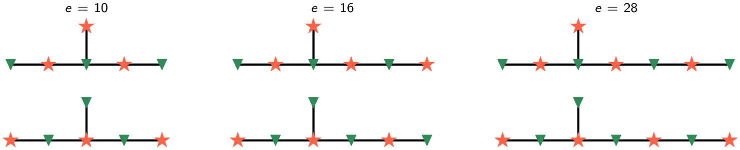
The type A family



The type D family



The type E exceptions



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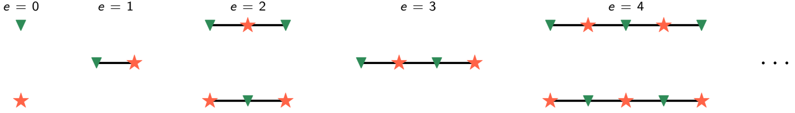
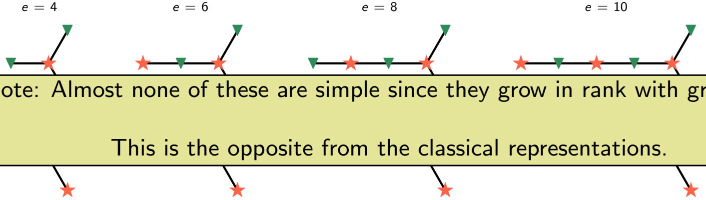
The type A family



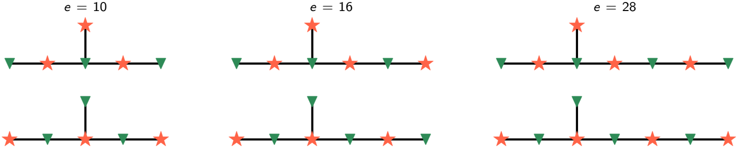
The type D family



Note: Almost none of these are simple since they grow in rank with growing e.

This is the opposite from the classical representations.

The type E exceptions



◀ Back

Example ($e = 2$). Simples associated to cells.

Classical representation theory. The simples from before.

	$\mathrm{M}_{0,0}$	$\mathrm{M}_{2,0}$	$\mathrm{M}_{\sqrt{2}}$	$\mathrm{M}_{0,2}$	$\mathrm{M}_{2,2}$
atom	sign	trivial-sign	rotation	sign-trivial	trivial
rank	1	1	2	1	1
apex(KL)	1	s – t	s – t	s – t	w_0

KL basis. ADE diagrams and ranks of transitive $\mathbb{N}$-modules.

	bottom cell	▼—★—▼	★—▼—★	top cell
atom	sign	$\mathrm{M}_{2,0}\oplus\mathrm{M}_{\sqrt{2}}$	$\mathrm{M}_{0,2}\oplus\mathrm{M}_{\sqrt{2}}$	trivial
rank	1	3	3	1
apex(KL)	1	s – t	s – t	w_0

The simples are arranged according to cells. However, a cell might have more than one associated simple.

(For the experts: This means that the Hecke algebra with the KL basis is in general not cellular in the sense of Graham–Lehrer.)

◂ Back

Example ($e = 2$).

The fusion ring $K_0(\mathrm{SL}(2)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_1], [L_2]$. The limit $\mathrm{v} \to 0$ has simple objects $a_{\mathtt{s}}, a_{\mathtt{sts}}, a_{\mathtt{st}}, a_{\mathtt{t}}, a_{\mathtt{tst}}, a_{\mathtt{ts}}$.

Comparison of multiplication tables:

	$[L_0]$	$[L_2]$	$[L_1]$
$[L_0]$	$[L_0]$	$[L_2]$	$[L_1]$
$[L_2]$	$[L_2]$	$[L_0]$	$[L_1]$
$[L_1]$	$[L_1]$	$[L_1]$	$[L_0] + [L_2]$

&

	$a_{\mathtt{s}}$	$a_{\mathtt{sts}}$	$a_{\mathtt{st}}$	$a_{\mathtt{t}}$	$a_{\mathtt{tst}}$	$a_{\mathtt{ts}}$
$a_{\mathtt{s}}$	$a_{\mathtt{s}}$	$a_{\mathtt{sts}}$	$a_{\mathtt{st}}$			
$a_{\mathtt{sts}}$	$a_{\mathtt{sts}}$	$a_{\mathtt{s}}$	$a_{\mathtt{st}}$			
$a_{\mathtt{ts}}$	$a_{\mathtt{ts}}$	$a_{\mathtt{ts}}$	$a_{\mathtt{t}} + a_{\mathtt{tst}}$			
$a_{\mathtt{t}}$				$a_{\mathtt{t}}$	$a_{\mathtt{tst}}$	$a_{\mathtt{ts}}$
$a_{\mathtt{tst}}$				$a_{\mathtt{tst}}$	$a_{\mathtt{t}}$	$a_{\mathtt{ts}}$
$a_{\mathtt{st}}$				$a_{\mathtt{st}}$	$a_{\mathtt{st}}$	$a_{\mathtt{s}} + a_{\mathtt{sts}}$

The limit $\mathrm{v} \to 0$ is a bicolored version of $K_0(\mathrm{SL}(2)_q)$:

$$a_{\mathtt{s}} \& a_{\mathtt{t}} \leftrightsquigarrow [L_0], \quad a_{\mathtt{sts}} \& a_{\mathtt{tst}} \leftrightsquigarrow [L_2], \quad a_{\mathtt{st}} \& a_{\mathtt{ts}} \leftrightsquigarrow [L_1].$$

Example ($e = 2$).

This is the slightly nicer statement.

The fusion ring $K_0(\mathrm{SO}(3)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_2]$. The $\mathcal{H}$-cell limit $\mathrm{v} \to 0$ has simple objects $a_{\mathtt{s}}, a_{\mathtt{sts}}$.

Comparison of multiplication tables:

	$[L_0]$	$[L_2]$
$[L_0]$	$[L_0]$	$[L_2]$
$[L_2]$	$[L_2]$	$[L_0]$

&

	$a_{\mathtt{s}}$	$a_{\mathtt{sts}}$
$a_{\mathtt{s}}$	$a_{\mathtt{s}}$	$a_{\mathtt{sts}}$
$a_{\mathtt{sts}}$	$a_{\mathtt{sts}}$	$a_{\mathtt{s}}$

The $\mathcal{H}$-cell limit $\mathrm{v} \to 0$ is $K_0(\mathrm{SO}(3)_q)$:

$$a_{\mathtt{s}} \leftrightsquigarrow [L_0], \quad a_{\mathtt{sts}} \leftrightsquigarrow [L_2].$$

Example ($e = 2$).

This is the slightly nicer statement.

The fusion ring $K_0(\mathrm{SO}(3)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_2]$. The $\mathcal{H}$-cell limit $\mathrm{v} \to 0$ has simple objects $a_{\mathtt{s}}, a_{\mathtt{sts}}$.

Comparison of multiplication tables:

	$[L_0]$	$[L_2]$
$[L_0]$	$[L_0]$	$[L_2]$
$[L_2]$	$[L_2]$	$[L_0]$

&

	$a_{\mathtt{s}}$	$a_{\mathtt{sts}}$
$a_{\mathtt{s}}$	$a_{\mathtt{s}}$	$a_{\mathtt{sts}}$
$a_{\mathtt{sts}}$	$a_{\mathtt{sts}}$	$a_{\mathtt{s}}$

The $\mathcal{H}$-cell limit v

$$a_{\mathtt{s}} \leftrightsquigarrow [L_0], \quad a_{\mathtt{sts}} \leftrightsquigarrow [L_2].$$

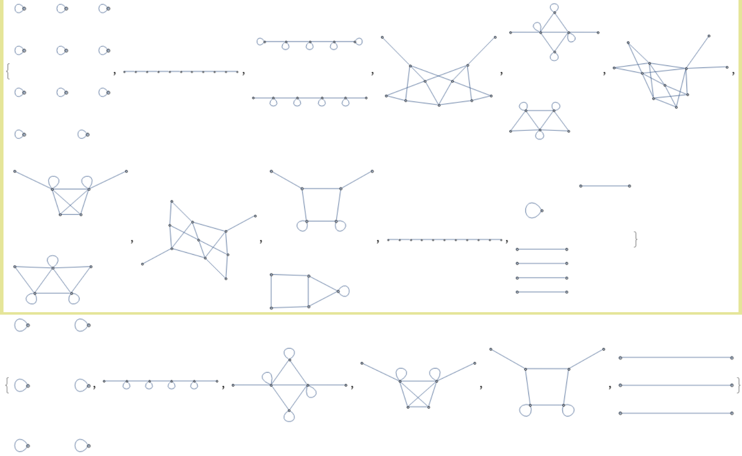
Fact.

Both connections are always true (*i.e.* for any e).

Example ($e = 2$).

The fusion ring $K_0(\mathrm{SO}(3)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_2]$. The $\mathcal{H}$-cell

The bicoloring is basically coming from slightly different fusion graphs *e.g.* for $e = 6$:



Back

The zigzag algebra $\mathrm{Z}(\Gamma)$

$$\begin{tikzcd} \blacktriangledown \arrow[r, shift left, "u"] & \bigstar \arrow[l, shift left, "d"] \arrow[r, shift left, "u"] & \blacktriangledown \arrow[l, shift left, "d"] \end{tikzcd}$$

$$uu = 0 = dd, ud = du$$

Apply the usual philosophy:

- Take projectives $P_{\mathtt{s}} = \bigoplus_{\blacktriangledown} P_i \otimes {}_iP \otimes \_$ and $P_{\mathtt{t}} = \bigoplus_{\bigstar} P_j \otimes {}_jP \otimes \_$.
- Get endofunctors $\mathrm{B}_{\mathtt{s}} = \mathrm{P}_{\mathtt{s}} \otimes_{\mathrm{Z}(\Gamma)} -$ and $\mathrm{B}_{\mathtt{t}} = \mathrm{P}_{\mathtt{t}} \otimes_{\mathrm{Z}(\Gamma)} -$.
- Check: These decategorify to $b_{\mathtt{s}}$ and $b_{\mathtt{t}}$. (Easy.)
- Check: These give a genuine 2-representation. (Bookkeeping.)
- Check: There are no $\boxed{\text{graded}}$ deformations. (Bookkeeping.)

Difference to $\mathrm{SL}(2)_q$: There is an honest quiver as this is non-semisimple.

Back

Example (type H_4).

cell	0	1	2	3	4	5	6=6′	5′	4′	3′	2′	1′	0′
size	1	32	162	512	625	1296	9144	1296	625	512	162	32	1
a	0	1	2	3	4	5	6	15	16	18	22	31	60
$v \to 0$	$\square$	$2\square$	$2\square$	$2\square$	$\square$	$\square$	big	$\square$	$\square$	$2\square$	$2\square$	$2\square$	$\square$

The big cell:

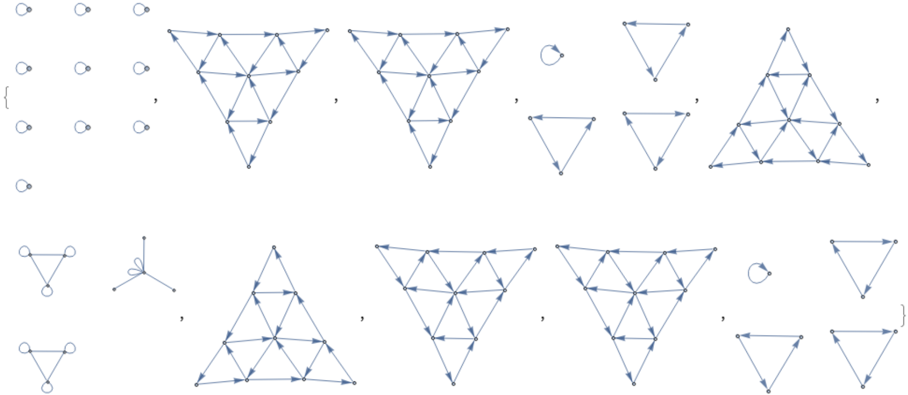
$14_{8,8}$	$13_{10,8}$	$14_{6,8}$
$13_{8,10}$	$18_{10,10}$	$18_{6,10}$
$14_{8,6}$	$18_{10,6}$	$24_{6,6}$

$\boxed{14_{8,8}}$:

$$\mathrm{PFdim}(gen) = 1 + \sqrt{5},$$
$$\mathrm{PFdim} = 120(9 + 4\sqrt{5}).$$

Back

Example (Fusion graphs for level 3).



In the non-semisimple case one gets quiver algebras supported on these graphs. ("Trihedral zigzag algebras".)

◂ Stop - you are annoying!