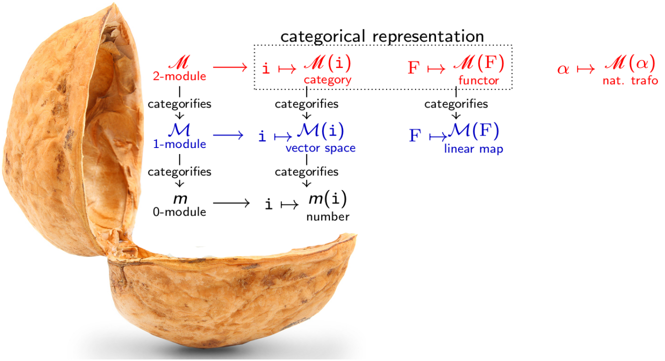
What is...finitary 2-representation theory?

Or: A (fairy) tale of matrices and functors

Daniel Tubbenhauer



February 2019

1. $\mathbb{C}$-**representation theory**
   - Main ideas
   - Some examples

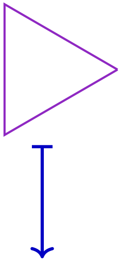
2. $\mathbb{N}$-**representation theory**
   - Main ideas
   - Some examples

3. 2-**representation theory**
   - Main ideas
   - Some examples

Slogan. Representation theory is group theory in vector spaces.

symmetries of n-gons $\subset \mathcal{A}\mathrm{ut}(\mathbb{R}^2)$

Idea (Coxeter ∼1934++).



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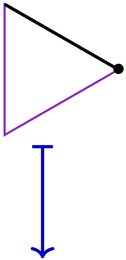

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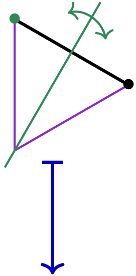

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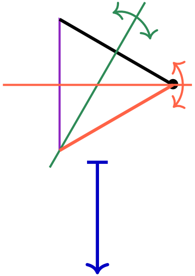

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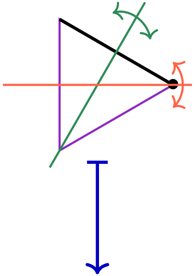
This gives a generator-relation presentation.



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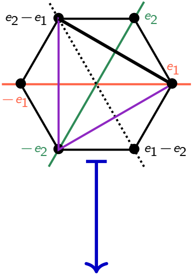
The reflection representation.



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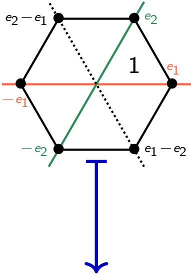

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \right\}$$

1

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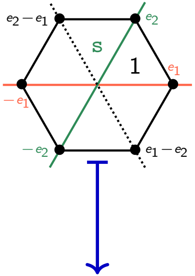

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \right\}$$

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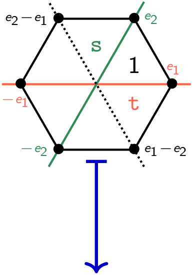

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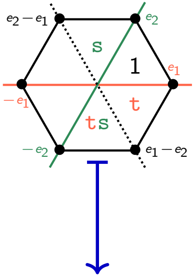

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$1 \quad \mathtt{s} \quad \mathtt{t} \quad \mathtt{ts}$

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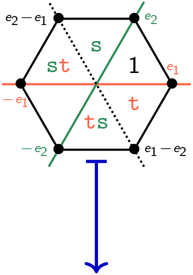

$$\left\{ \underset{1}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{s}{\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}}, \underset{t}{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}}, \underset{ts}{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}}, \underset{st}{\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}}, \qquad \right\}$$

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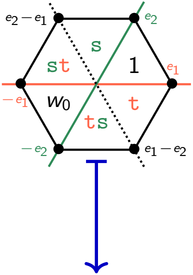


$$\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}-1&0\\1&1\end{pmatrix},\begin{pmatrix}1&1\\0&-1\end{pmatrix},\begin{pmatrix}0&1\\-1&-1\end{pmatrix},\begin{pmatrix}-1&-1\\1&0\end{pmatrix},\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\right\}$$

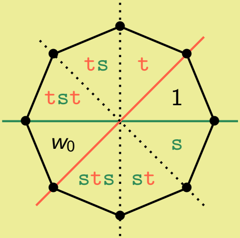
1   s   t   ts   st   sts=tst (w_0)

These symmetry groups of the regular n-gons are the so-called dihedral groups

$$\mathrm{D}_{2n} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1, \underbrace{\ldots \mathtt{tsts}}_{n} = w_0 = \underbrace{\ldots \mathtt{stst}}_{n} \rangle$$

which are the easiest examples of Coxeter groups.

Example $n = 4$; its Coxeter complex.



1 s t ts st sts=tst
w_0

Pioneers of representation theory

Let G be a finite group.

Frobenius ∼1895++, Burnside ∼1900++. Representation theory is the ▸ useful? study of linear group actions

$$\mathcal{M}\colon \mathrm{G} \longrightarrow \mathcal{A}\mathrm{ut}(\mathtt{V}), \quad \text{“}\mathcal{M}(g) = \text{a matrix in } \mathcal{A}\mathrm{ut}(\mathtt{V})\text{”}$$

with $\mathtt{V}$ being some vector space. (Called modules or representations.)

The “atoms” of such an action are called simple. A module is called semisimple if it is a direct sum of simples.

Maschke ∼1899. All modules are built out of simples (“Jordan–Hölder” filtration).

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We want to have a categorical version of this!

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Let A be a finite-dimensional algebra.

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We want to have a categorical version of this.

I am going to explain what we can do at present.

collection ("category") of modules $\leftrightsquigarrow$ the world

---

modules $\leftrightsquigarrow$ chemical compounds

---

simples $\leftrightsquigarrow$ elements

---

semisimple $\leftrightsquigarrow$ only trivial compounds

---

non-semisimple $\leftrightsquigarrow$ non-trivial compounds

Main goal of representation theory. Find the periodic table of simples.

colle

mod

simp

semi

non-

Example.

Back to the dihedral group, an invariant of the module is the ▸ character χ which only remembers the traces of the acting matrices:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}$$

1	s	t	ts	st	sts=tst (w_0)
$\chi = 2$	$\chi = 0$	$\chi = 0$	$\chi = -1$	$\chi = -1$	$\chi = 0$

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Main goal of repres … le of simples.

Fact.

Semisimple case:
the character determines the module
$\leftrightsquigarrow$
mass determines the chemical compound.

collection (

modules ⇜

simples ⇜

semisimple

Example.

$$\mathbb{Z}/2\mathbb{Z} \to \mathcal{A}\mathrm{ut}(\mathbb{C}^2),\ 0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ \&\ 1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Common eigenvectors: $(1,1)$ and $(1,-1)$ and base change gives

$$0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ \&\ 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the module decomposes.

non-semisimple ⇜⇝ non-trivial compounds

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and the module is non-simple, yet does not decompose.

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Morally: representation theory over $\mathbb{Z}$ is never semisimple.

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Dihedral representation theory on one slide.

One-dimensional modules. $\mathcal{M}_{\lambda_s,\lambda_t}, \lambda_s, \lambda_t \in \mathbb{C}, s \mapsto \lambda_s, t \mapsto \lambda_t$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathcal{M}_{-1,-1}, \mathcal{M}_{1,-1}, \mathcal{M}_{-1,1}, \mathcal{M}_{1,1}$	$\mathcal{M}_{-1,-1}, \mathcal{M}_{1,1}$

Two-dimensional modules. $\mathcal{M}_z, z \in \mathbb{C}, s \mapsto \begin{pmatrix} 1 & z \\ 0 & -1 \end{pmatrix}, t \mapsto \begin{pmatrix} -1 & 0 \\ \bar{z} & 1 \end{pmatrix}$.

$n \equiv 0 \bmod 2$	$n \not\equiv 0 \bmod 2$
$\mathcal{M}_z, z \in V(n) - \{0\}$	$\mathcal{M}_z, z \in V(n)$

$V(n) = \{2\cos(\pi k/n-1) \mid k = 1, \ldots, n-2\}$.

Dihedral representation theory on one slide.

One-dimensiona... $\mathtt{t}$.

Proposition (Lusztig?).

The list of one- and two-dimensional D_{2n}-modules is a complete, irredundant list of simples.

$\mathcal{M}_{-1,-1}, \mathcal{M}_{1,-1}, \mathcal{M}_{-1,1}, \mathcal{M}_{1,1}$	$\mathcal{M}_{-1,-1}, \mathcal{M}_{1,1}$

I learned this construction from Mackaay in 2017.

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Note that this requires complex parameters.
This does not work over $\mathbb{Z}$.

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$V(n) = \{2\cos(\pi k/n-1) \mid k = 1, \dots, n-2\}$.

An algebra P with a fixed, finite basis B^{P} with $1 \in \mathrm{B}^{\mathrm{P}}$ is called a $\mathbb{N}$-algebra if

$$\mathrm{xy} \in \mathbb{N}\mathrm{B}^{\mathrm{P}} \quad (\mathrm{x}, \mathrm{y} \in \mathrm{B}^{\mathrm{P}}).$$

A P-module M with a fixed, finite basis B^{M} is called a $\mathbb{N}$-module if

$$\mathrm{xm} \in \mathbb{N}\mathrm{B}^{\mathrm{M}} \quad (\mathrm{x} \in \mathrm{B}^{\mathrm{P}}, \mathrm{m} \in \mathrm{B}^{\mathrm{M}}).$$

These are $\mathbb{N}$-equivalent if there is a $\mathbb{N}$-valued change of basis matrix.

Example. $\mathbb{N}$-algebras and $\mathbb{N}$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}$-equivalence comes from 2-equivalence.

Example (semisimple world).

Group algebras of finite groups with basis given by group elements are $\mathbb{N}$-algebras.

The regular module is a $\mathbb{N}$-module, which decomposes over $\mathbb{C}$ into simples, but almost never over $\mathbb{N}$. (I will come back to this in a second.)

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Example. $\mathbb{N}$-algebras and $\mathbb{N}$-modules arise naturally as the decategorification of 2-categories and 2- ... equivalence.

Example (non-semisimple world).

Hecke algebras of (finite) Coxeter groups with their KL basis are $\mathbb{N}$-algebras.

Clifford, Munn, Ponizovskiĭ ∼1942++, Kazhdan–Lusztig ∼1979. $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ if y appears in zx with non-zero coefficient for $\mathrm{z} \in \mathrm{B}^{\mathrm{P}}$. $\mathrm{x} \sim_{\mathsf{L}} \mathrm{y}$ if $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ and $\mathrm{y} \leq_{\mathsf{L}} \mathrm{x}$. $\sim_{\mathsf{L}}$ partitions P into left cells L. Similarly for right R, two-sided cells J or $\mathbb{N}$-modules.

A $\mathbb{N}$-module M is transitive if all basis elements belong to the same $\sim_{\mathsf{L}}$ equivalence class.
Fact. $\mathbb{N}$-modules have transitive Jordan–Hölder filtrations. (The "atoms".)

Main goal of $\mathbb{N}$-representation theory. Find the periodic table of transitives.

Example. Transitive $\mathbb{N}$-modules arise naturally as the decategorification of simple 2-modules.

Clifford, Munn, Ponizovskiĭ ∼1942++, Kazhdan–Lusztig ∼1979. $x \leq_L y$ if y appears in zx with non-zero coefficient for $z \in B^P$. $x \sim_L y$ if $x \leq_L y$ and $y \leq_L x$. $\sim_L$ partitions P into left cells L. Similarly for right R, two-sided cells J or $\mathbb{N}$-modules.

A $\mathbb{N}$-module ... $\sim_L$ equivalence c... Fact. $\mathbb{N}$-mod... oms".)

Main goal o... transitives.

Example. Transitive $\mathbb{N}$-modules arise naturally as the decategorification of simple 2-modules.

Proposition ∼2016.

Fixing the KL basis, there is a one-to-one correspondence

$$\{\text{(non-trivial) } \mathbb{N}\text{-transitive } D_{2n}\text{-modules}\}/\mathbb{N}\text{-iso} \xleftrightarrow{1:1} \{\text{bicolored ADE Dynkin diagrams with Coxeter number } n\}.$$

Thus, its easy to write down a list.

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$\sim_{\mathsf{L}}$ par

$\mathbb{N}$-mod

Example (semisimple world).

Group algebras with the group element basis have only one cell, G itself.

Transitive $\mathbb{N}$-modules are $\mathbb{C}[G/H]$ for $H \subset G$ subgroup/conjugacy.

A $\mathbb{N}$-m

equival

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Main goal of $\mathbb{N}$-representation theory. Find the periodic table of transitives.

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Clifford, Munn, Ponizovskiĭ ∼1942++, Kazhdan–Lusztig ∼1979. $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ if y appears in zx with non-zero coefficient for $\mathrm{z} \in \mathrm{B}^{\mathrm{P}}$. $\mathrm{x} \sim_{\mathsf{L}} \mathrm{y}$ if $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ and $\mathrm{y} \leq_{\mathsf{L}} \mathrm{x}$.
$\sim_{\mathsf{L}}$ par
$\mathbb{N}$-mod

Example (semisimple world).

Group algebras with the group element basis have only one cell, G itself.

Transitive $\mathbb{N}$-modules are $\mathbb{C}[G/H]$ for $H \subset G$ subgroup/conjugacy.

A $\mathbb{N}$-m
equival

Fact. $\mathbb{N}$)

Example (semisimple world).

Fusion algebras have only one cell.

The transitive $\mathbb{N}$-modules are known in special cases, *e.g.* for $g = \mathrm{SL}_2$ and I "basically know" the classification more generally.

Main g tives.

Exampl f simple
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ℕ-mod…

A ℕ-m…

equival…

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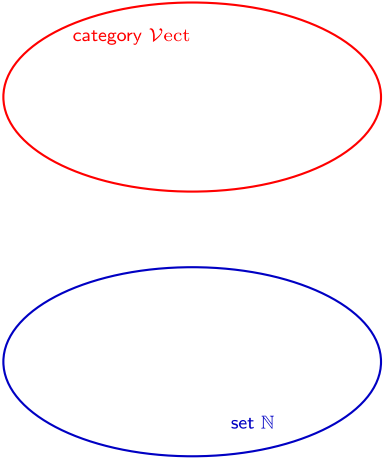
Example (non-semisimple world).

Hecke algebras with KL basis have a very rich cell theory.

The transitive ℕ-modules are only known in special cases.

Categorification in a nutshell



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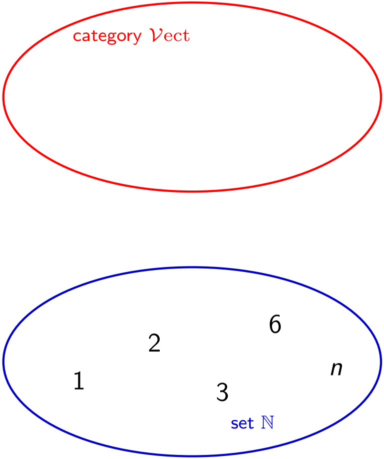

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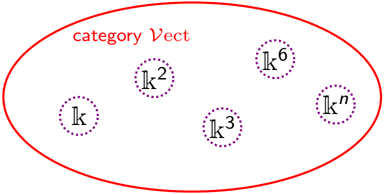

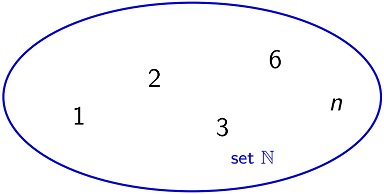

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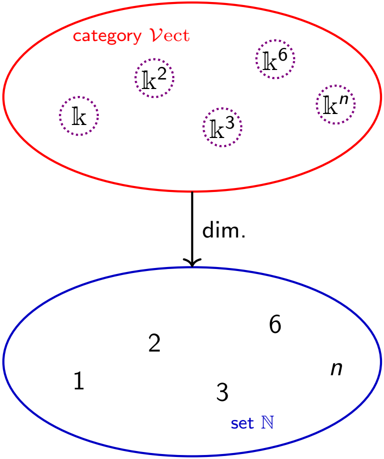

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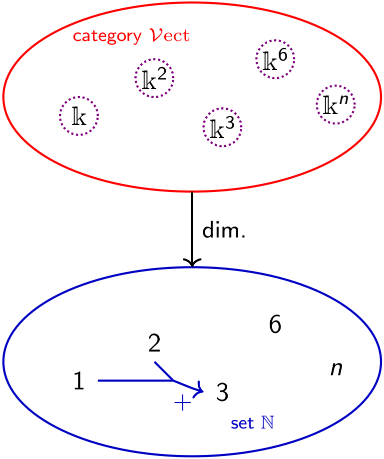

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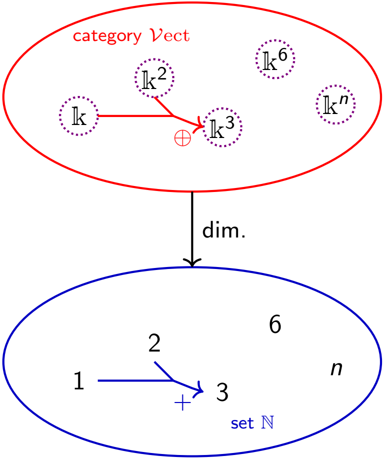

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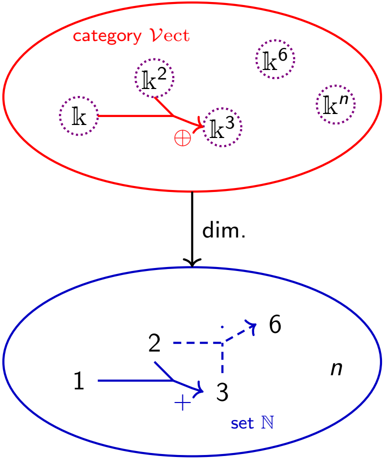

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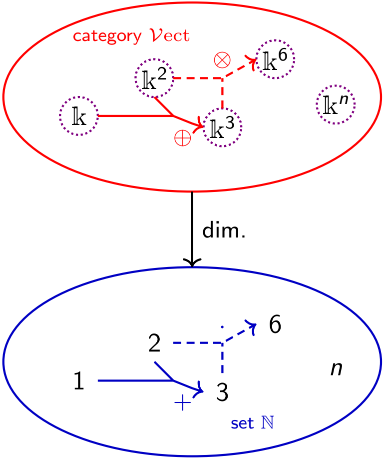

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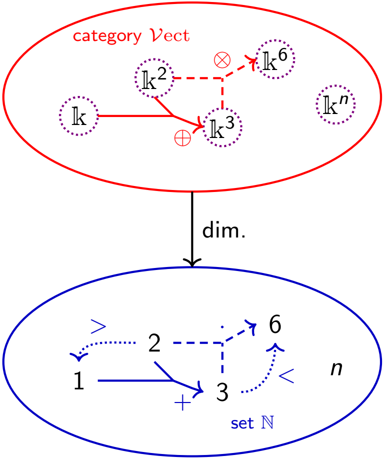

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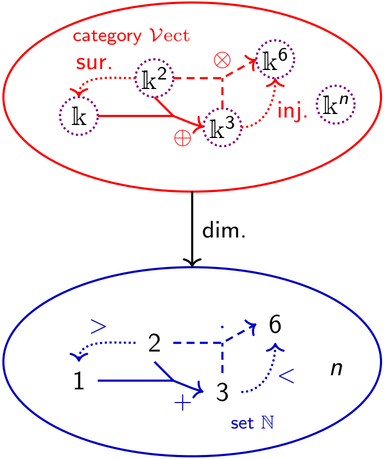

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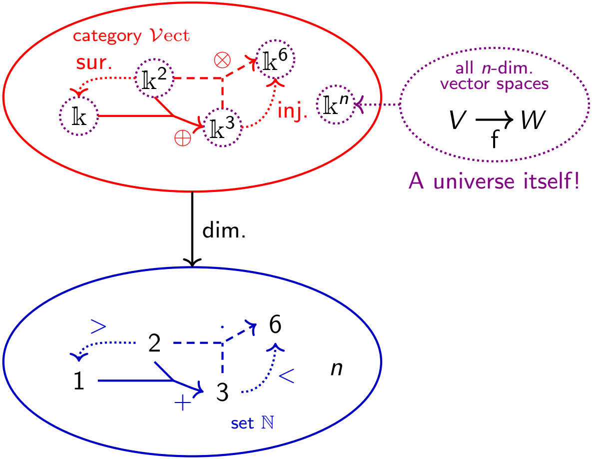

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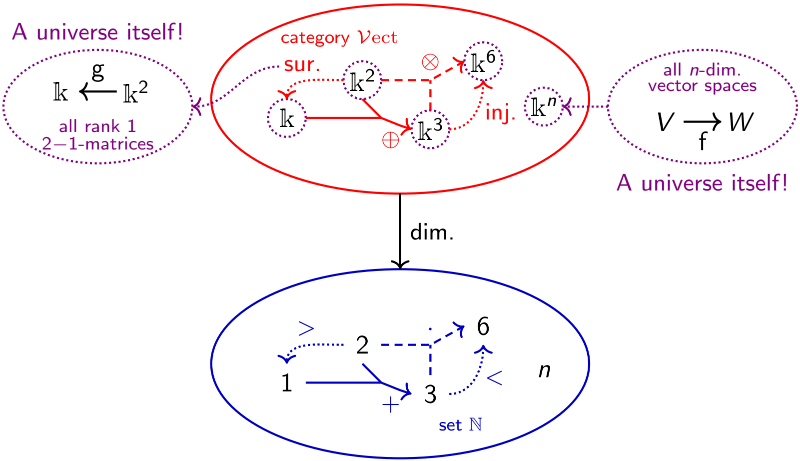
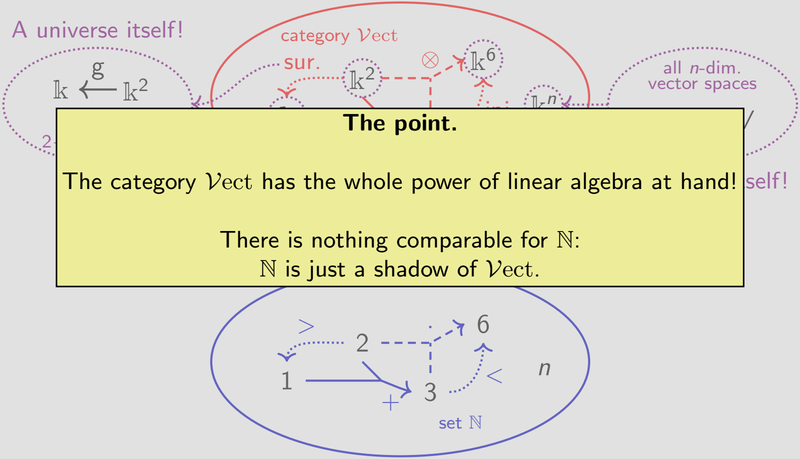


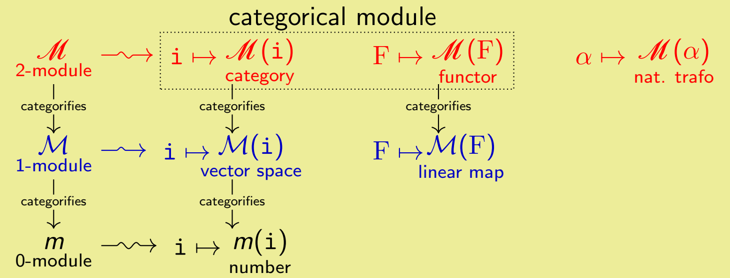


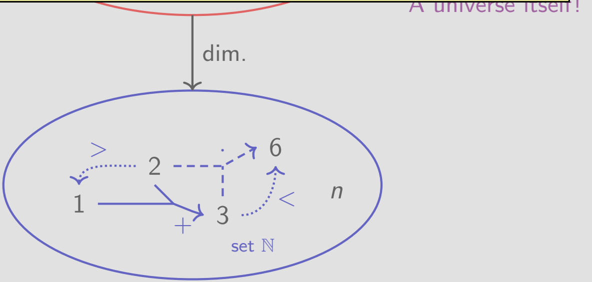
The point.

The category $\mathcal{V}\mathrm{ect}$ has the whole power of linear algebra at hand!

There is nothing comparable for $\mathbb{N}$:
$\mathbb{N}$ is just a shadow of $\mathcal{V}\mathrm{ect}$.

Slogan. 2-representation theory is group theory in categories.





Slogan. 2-representation theory is group theory in categories.

categorical module

$\mathscr{M}$ 2-module $\rightsquigarrow$	$\mathtt{i} \mapsto \mathscr{M}(\mathtt{i})$ category	$\mathrm{F} \mapsto \mathscr{M}(\mathrm{F})$ functor	$\alpha \mapsto \mathscr{M}(\alpha)$ nat. trafo
categorifies	categorifies	categorifies	
$\mathcal{M}$ 1-module $\rightsquigarrow$	$\mathtt{i} \mapsto \mathcal{M}(\mathtt{i})$ vector space	$\mathrm{F} \mapsto \mathcal{M}(\mathrm{F})$ linear map	
categorifies	categorifies		
m 0-module $\rightsquigarrow$	$\mathtt{i} \mapsto m(\mathtt{i})$ number		

A universe itself!

What one can hope for.

Problem involving a group action $\mathrm{G} \circlearrowright \mathtt{X}$ ···"lift"···> Problem involving a categorical group action ···> Decomposition of the problem into 2-simples ···new insights?···> $\mathrm{G} \circlearrowright \mathtt{X}$

Pioneers of 2-representation theory.

Let $\mathscr{C}$ be a finitary 2-category.

Slogan (finitary).
Everything that could be finite is finite.

Etingof–Ostrik, Chuang–Rouquier, many others ∼2000++. Higher representation theory is the useful? study of actions of 2-categories:

$$\mathscr{M}\colon \mathscr{C} \longrightarrow \mathscr{E}\mathrm{nd}(\mathcal{V}), \quad \text{“}\mathscr{M}(\mathrm{F}) = \text{a functor in } \mathscr{E}\mathrm{nd}(\mathcal{V})\text{”}$$

with $\mathcal{V}$ being some finitary category. (Called 2-modules or 2-representations.)

The “atoms” of such an action are called 2-simple.

Mazorchuk–Miemietz ∼2014. All (suitable) 2-modules are built out of 2-simples (“weak 2-Jordan–Hölder filtration”).

Pioneers of 2-representation theory.

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$$\mathscr{M} : \mathscr{C} \longrightarrow \mathscr{E}\mathrm{nd}(\mathcal{V}),$$

with $\mathcal{V}$ being some fi **A main goal of 2-representation theory.** representations.)

Classify 2-simples.

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Pioneers of 2-representation theory.

Let $\mathscr{C}$ be

Example. $\mathscr{C} = \mathcal{V}\mathrm{ec}_G$ or $\mathcal{R}\mathrm{ep}(G)$.
Features. Semisimple, classification of 2-simples well-understood.
Comments. I will (try to) discuss the classification "in real time".

Etingof–Ostrik, Chuang–Rouquier, many others ∼2000++. Higher representation theory is the useful? study of actions of 2-categories:

Example. $\mathscr{C} = \mathcal{R}\mathrm{ep}_q^{sesi}(g)_{\text{level } n}$.
Features. Semisimple, finitely many 2-simples,
classification of 2-simples only known for $g = \mathrm{Sl}_2$, some guesses for general g.
Comments. The classification of 2-simples is related to Dynkin diagrams.

with

The "atoms" of such an action are called 2-simple

Example. $\mathscr{C} =$ Hecke category.
Features. Non-semisimple, not known whether there are finitely many 2-simples,
classification of 2-simples only known in special cases.
Comments. Hopefully, by the end of the year we have a classification
by reducing the problem to the above examples.

Ma

2-s

An additive, $\mathbb{k}$-linear, idempotent complete, Krull–Schmidt category $\mathcal{C}$ is called finitary if it has only finitely many isomorphism classes of indecomposable objects and the morphism sets are finite-dimensional. A 2-category $\mathscr{C}$ with finitely many objects is finitary if its hom-categories are finitary, $\circ_h$-composition is additive and linear, and identity 1-morphisms are indecomposable.

A simple transitive 2-module (2-simple) of $\mathscr{C}$ is an additive, $\mathbb{k}$-linear 2-functor

$$\mathscr{M}\colon \mathscr{C} \to \mathscr{A}^{\mathrm{f}}(= \text{2-cat of finitary cats}),$$

such that there are no non-zero proper $\mathscr{C}$-stable ideals.

There is also the notion of 2-equivalence.

Example. $\mathbb{N}$-algebras and $\mathbb{N}$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}$-equivalence comes from 2-equivalence.

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Mazorchuk–Miemietz ∼2014.

2-Simples $\leftrightsquigarrow$ simples (*e.g.* weak 2-Jordan–Hölder filtration),

but their decategorifications are transitive $\mathbb{N}$-modules and usually not simple.

A si… …or

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A si[...] or

$$\mathscr{M}\colon \mathscr{C} \to \mathscr{A}^{\mathrm{f}} (= \text{2-cat of finitary cats}),$$

such tha[...]

There is [...]

Example.

$\mathrm{B}\text{-p}\mathcal{M}\mathrm{od}$ (with B finite-dimensional) is a prototypical object of $\mathscr{A}^{\mathrm{f}}$.

A 2-module usually is given by endofunctors on $\mathrm{B}\text{-p}\mathcal{M}\mathrm{od}$.

Example[...]ation of 2-categories and 2-modules, and $\mathbb{N}$-equivalence comes from 2-equivalence.

Example (semisimple).

G can be (naively) categorified using G-graded vector spaces $\mathcal{V}\mathrm{ec}_G \in \mathscr{A}^{\mathrm{f}}$.

The 2-simples are indexed by (conjugacy classes of) subgroups H and $\phi \in \mathrm{H}^2(H, \mathbb{C}^*)$.

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Example (semisimple).

Fusion or modular categories are semisimple examples of finitary 2-categories, *e.g.* $\mathcal{R}\mathrm{ep}_q^{sesi}(g)_n$.
Their 2-modules play a prominent role in quantum algebra and topology.

A simp ... ctor

such th

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Example (non-semisimple).

Soergel bimodules for finite Coxeter groups are finitary 2-categories.

(Coxeter=Weyl: "Indecomposable projective functors on $\mathcal{O}_0$.")

Dihedral group: the ▸ 2-simples have an *ADE* classification.

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A simple ... unctor

such that ...

There is ...

On the categorical level the impact of the choice of basis is evident:

These are the indecomposable objects in some 2-category,
and different bases are categorified by
potentially non-equivalent 2-categories.

So, of course, the 2-representation theory differs!

Example. $\mathbb{N}$-algebras and $\mathbb{N}$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}$-equivalence comes from 2-equivalence.

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A simple transitiv ... near 2-functor

such that there a ...

There is also the ...

Philosophy to take away.

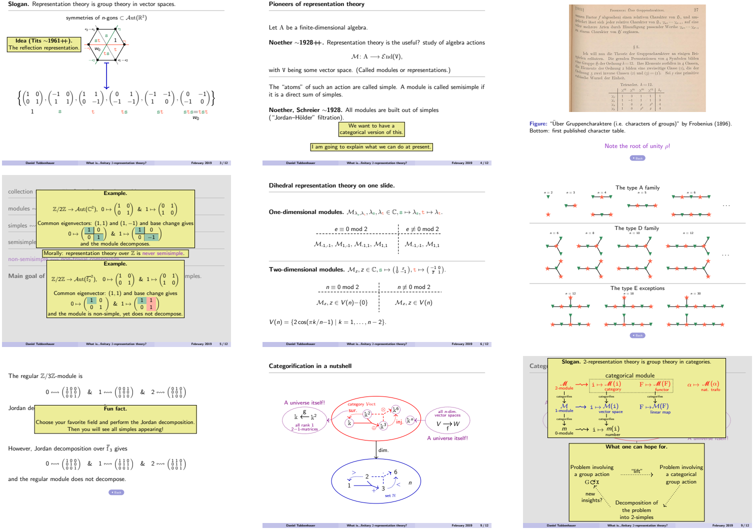
"Finitary 2-representation theory

$\Leftrightarrow$

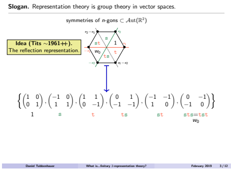
representation theory of finite-dimensional algebras

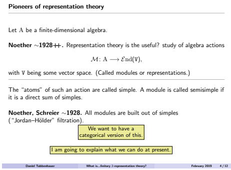
for all primes $p \geq 0$."

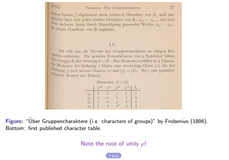
Example. $\mathbb{N}$-algebras and $\mathbb{N}$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}$-equivalence comes from 2-equivalence.

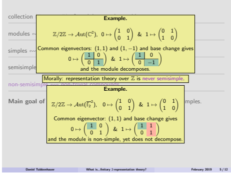


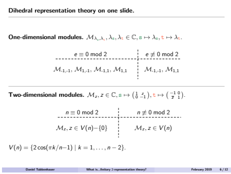
There is still much to do...

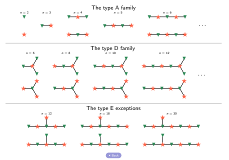


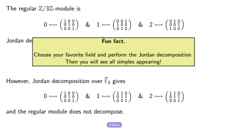


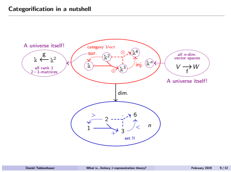


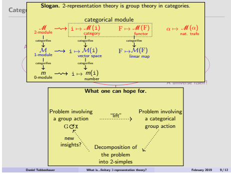












Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

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Nowadays representation theory is pervasive across mathematics, and beyond.

VERY considerable advances in the theory of groups of

But this wasn't clear at all when Frobenius started it.

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Back

[1011] Frobenius: Über Gruppencharaktere. 27

samen Factor f abgesehen) einen relativen Charakter von $\mathfrak{H}$, und umgekehrt lässt sich jeder relative Charakter von $\mathfrak{H}$, $\chi_0, \cdots \chi_{k-1}$, auf eine oder mehrere Arten durch Hinzufügung passender Werthe $\chi_k, \cdots \chi_{k'-1}$ zu einem Charakter von $\mathfrak{H}'$ ergänzen.

§ 8.

Ich will nun die Theorie der Gruppencharaktere an einigen Beispielen erläutern. Die geraden Permutationen von 4 Symbolen bilden eine Gruppe $\mathfrak{H}$ der Ordnung $h = 12$. Ihre Elemente zerfallen in 4 Classen, die Elemente der Ordnung 2 bilden eine zweiseitige Classe (1), die der Ordnung 3 zwei inverse Classen (2) und (3) = (2′). Sei ρ eine primitive cubische Wurzel der Einheit.

Tetraeder. $h = 12$.

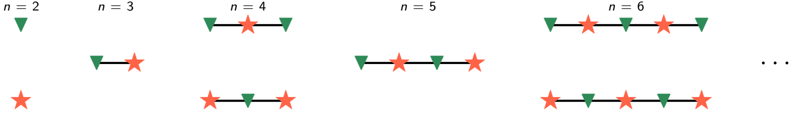
	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	h_α
χ_0	1	3	1	1	1
χ_1	1	−1	1	1	3
χ_2	1	0	ρ	ρ^2	4
χ_3	1	0	ρ^2	ρ	4

Figure: "Über Gruppencharaktere (i.e. characters of groups)" by Frobenius (1896). Bottom: first published character table.

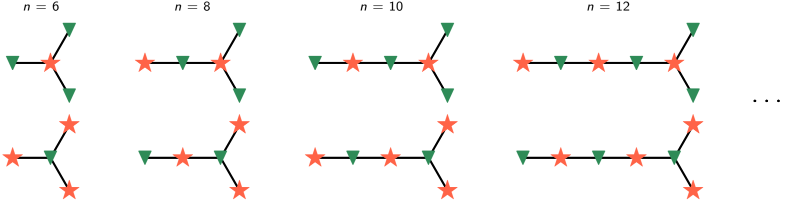
Note the root of unity ρ!

Back

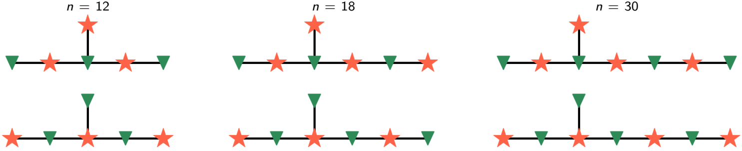
The type A family



The type D family

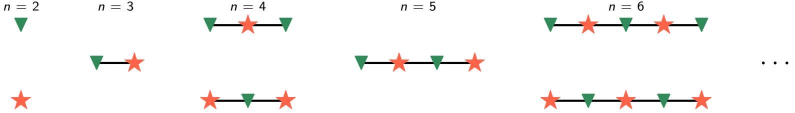


The type E exceptions



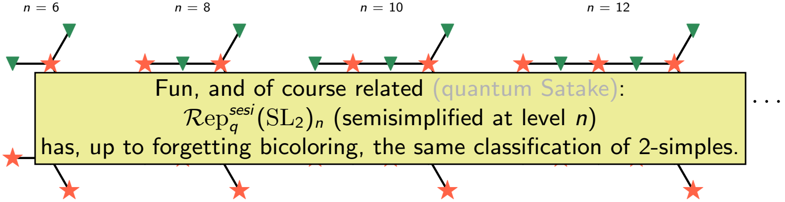
Back

The type A family



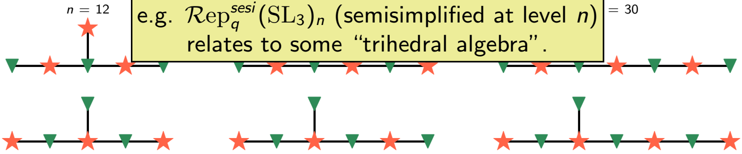
This is an unexpected ADE classification.

The type D family



Fun, and of course related (quantum Satake):
$\mathcal{R}\mathrm{ep}_q^{sesi}(\mathrm{SL}_2)_n$ (semisimplified at level n)
has, up to forgetting bicoloring, the same classification of 2-simples.

There is a similar story for all types,
e.g. $\mathcal{R}\mathrm{ep}_q^{sesi}(\mathrm{SL}_3)_n$ (semisimplified at level n)
relates to some "trihedral algebra".



The regular $\mathbb{Z}/3\mathbb{Z}$-module is

$$0 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad 1 \leftrightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \& \quad 2 \leftrightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Jordan decomposition over $\mathbb{C}$ with $\zeta^3 = 1$ gives

$$0 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad 1 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^{-1} \end{pmatrix} \quad \& \quad 2 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^{-1} & 0 \\ 0 & 0 & \zeta \end{pmatrix}$$

However, Jordan decomposition over $\overline{\mathbb{F}}_3$ gives

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and the regular module does not decompose.

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Fun fact.

Choose your favorite field and perform the Jordan decomposition.
Then you will see all simples appearing!

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Example ($G = \mathrm{D}_8$). Here we have three different notions of “atoms”.

Classical representation theory. The simples from before.

	$\mathcal{M}_{-1,-1}$	$\mathcal{M}_{1,-1}$	$\mathcal{M}_{\sqrt{2}}$	$\mathcal{M}_{-1,1}$	$\mathcal{M}_{1,1}$
atom	sign		rotation		trivial
rank	1	1	2	1	1

Group element basis. Subgroups and ranks of $\mathbb{N}$-modules.

subgroup	1	$\langle \mathtt{st} \rangle$	$\langle w_0 \rangle$	$\langle w_0, \mathtt{s} \rangle$	$\langle w_0, \mathtt{sts} \rangle$	G
atom	regular	$\mathcal{M}_{1,1} \oplus \mathcal{M}_{-1,-1}$	$\mathcal{M}_{\sqrt{2}} \oplus \mathcal{M}_{\sqrt{2}}$	$\mathcal{M}_{1,1} \oplus \mathcal{M}_{1,-1}$	$\mathcal{M}_{1,1} \oplus \mathcal{M}_{-1,1}$	trivial
rank	8	2	4	2	2	1

KL basis. ADE diagrams and ranks of $\mathbb{N}$-modules.

	bottom cell			top cell
atom	sign	$\mathcal{M}_{1,-1} \oplus \mathcal{M}_{\sqrt{2}}$	$\mathcal{M}_{-1,1} \oplus \mathcal{M}_{\sqrt{2}}$	trivial
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rank	8				2	1

“Knowing the transitive $\mathbb{N}$-modules

$\Leftrightarrow$

knowing the simples for all primes $p \geq 0$.”

KL basis. ADE diagrams and ranks of $\mathbb{N}$-modules.

	bottom cell			top cell
atom	sign	$\mathcal{M}_{1,-1} \oplus \mathcal{M}_{\sqrt{2}}$	$\mathcal{M}_{-1,1} \oplus \mathcal{M}_{\sqrt{2}}$	trivial
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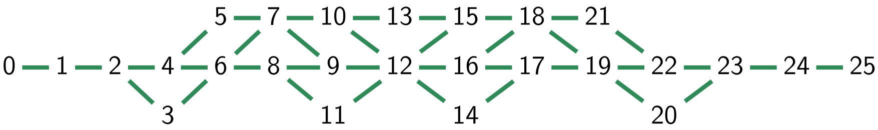
Back

Example (SAGE). The Weyl group of type B_6. Number of elements: 46080. Number of cells: 26, named 0 (trivial) to 25 (top).

Cell order:



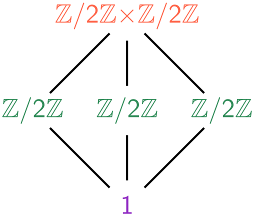
Size of the cells:

cell	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
size	1	62	342	576	650	3150	350	1600	2432	3402	900	2025	14500	600	2025	900	3402	2432	1600	350	576	3150	650	342	62	1

Back

Example ($G = \mathbb{Z}/2\times\mathbb{Z}/2$).

Subgroups, Schur multipliers and 2-simples.



In particular, there are two categorifications of the trivial module, and the rank sequences read

$$\text{decat: } 1, 2, 2, 2, 4, \qquad \text{cat: } 1, 1, 2, 2, 2, 4.$$

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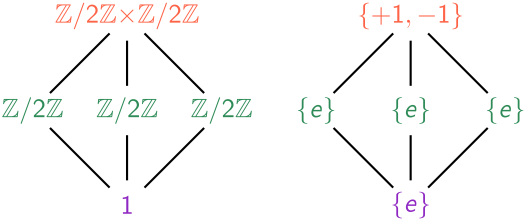

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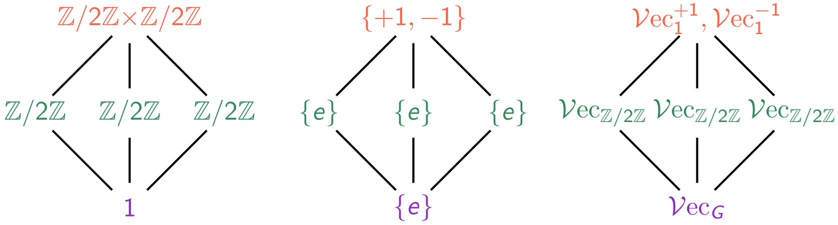

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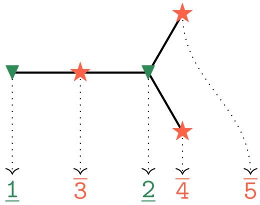
Construct a D_∞-module $\mathtt{V}$ associated to a bipartite graph G:

$$\mathtt{V} = \langle \underline{1}, \underline{2}, \overline{3}, \overline{4}, \overline{5} \rangle_{\mathbb{C}}$$



$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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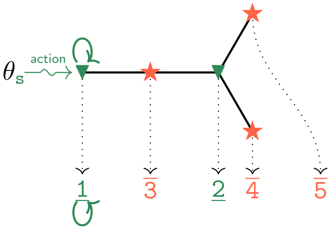

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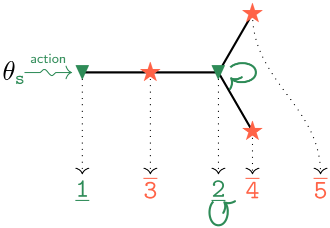

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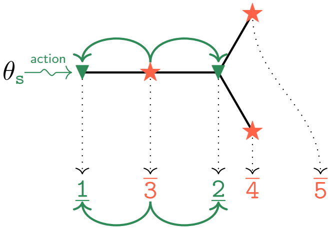

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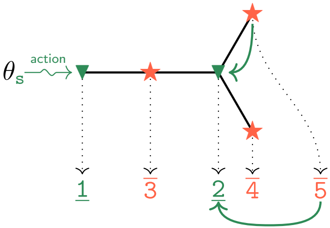

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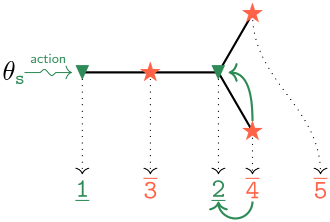

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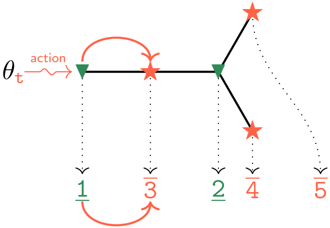

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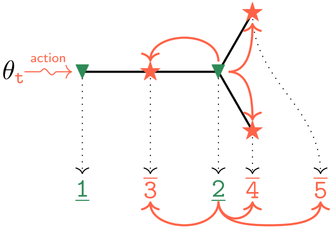

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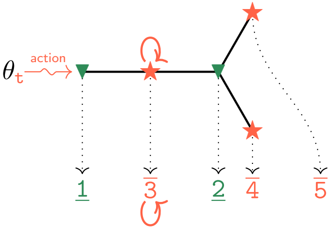

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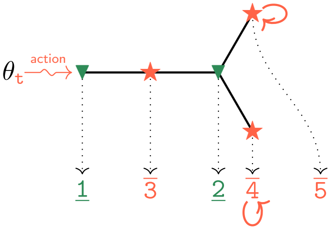

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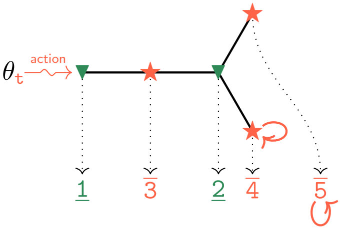

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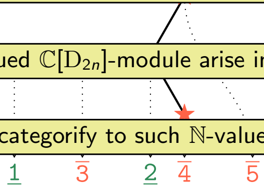
Lemma. For certain values of n these are $\mathbb{N}$-valued $\mathbb{C}[\mathrm{D}_{2n}]$-modules.

Lemma. All $\mathbb{N}$-valued $\mathbb{C}[\mathrm{D}_{2n}]$-module arise in this way.

Lemma. All 2-modules decategorify to such $\mathbb{N}$-valued $\mathbb{C}[\mathrm{D}_{2n}]$-module.



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Categorification.

Category $\rightsquigarrow \mathcal{V} = \mathrm{Z}\text{-}\mathcal{M}\mathrm{od}$,
Z quiver algebra with underlying graph G.

Endofunctors $\rightsquigarrow$ tensoring with Z-bimodules.

Lemma. These satisfy the relations of $\mathbb{C}[\mathrm{D}_{2n}]$.

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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Theorem ∼2016.

Fixing the Hecke category, there is a one-to-one correspondence

$$\{\text{(non-trivial) 2-simples } \mathrm{D}_{2n}\text{-modules}\}/\text{2-iso} \xleftrightarrow{1:1} \{\text{bicolored ADE Dynkin diagrams with Coxeter number } n\}.$$

Same as on the decategorified level.

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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