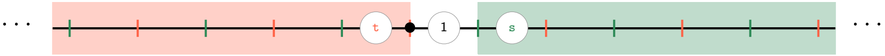
A tale of dihedral groups, $\mathrm{SL}(2)_q$, and beyond

Or: Who colored my Dynkin diagrams?

Daniel Tubbenhauer



Joint work with Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz and Xiaoting Zhang

February 2019

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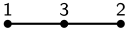
$\mathrm{A}_3 =$ 

$\rightsquigarrow \quad A(\mathrm{A}_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \rightsquigarrow \quad S_{\mathrm{A}_3} = \{2\cos(\tfrac{\pi}{4}), 0, 2\cos(\tfrac{3\pi}{4})\}$

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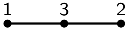
$\mathrm{A}_3 =$ 

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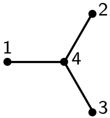
$\mathrm{D}_4 =$ 

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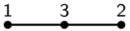
$\mathrm{A}_3 =$ 

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✓ for $e = 2$

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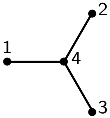
$\mathrm{D}_4 =$ 

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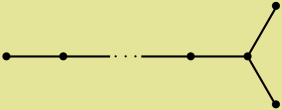
Clas … $= 0$.

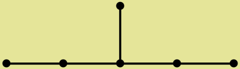
$\mathrm{A}_3 =$ 1 … $\cos(\frac{3\pi}{4})\}$

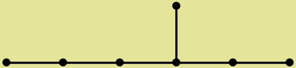
$\mathrm{D}_4 =$ 1 … $\cos(\frac{5\pi}{6})\}$

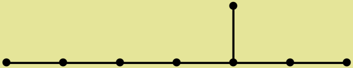
Smith ∼1969. The graphs solutions to (CP) are precisely ADE graphs for $e+2$ being (at most) the Coxeter number.

Type A_m:  ✓ for $e = m-1$

Type D_m:  ✓ for $e = 2m-4$

Type E_6:  ✓ for $e = 10$

Type E_7:  ✓ for $e = 16$

Type E_8:  ✓ for $e = 28$

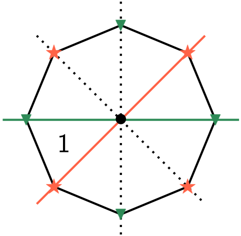
1. **Dihedral representation theory**
   - The classical representation theory
   - The $\mathbb{N}_0$-representation theory
   - Dihedral $\mathbb{N}_0$-representation theory
2. **Non-semisimple fusion rings**
   - The asymptotic limit
   - Cell modules
   - The dihedral example
3. **Beyond**

The dihedral groups are of Coxeter type $I_2(e+2)$:

$$W_{e+2} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \overline{\mathtt{s}}_{e+2} = \underbrace{\dots \mathtt{sts}}_{e+2} = w_0 = \underbrace{\dots \mathtt{tst}}_{e+2} = \overline{\mathtt{t}}_{e+2} \rangle,$$

$$e.g.\ :\ W_4 = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \mathtt{tsts} = w_0 = \mathtt{stst} \rangle$$

Example. These are the symmetry groups of regular $e+2$-gons, *e.g.* for $e=2$ the Coxeter complex is:



I will sneak in the Hecke case, later on.

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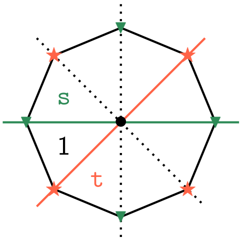

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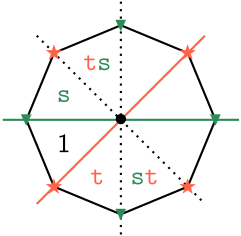

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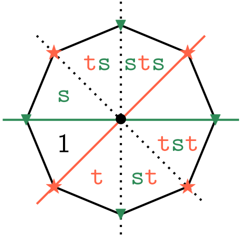

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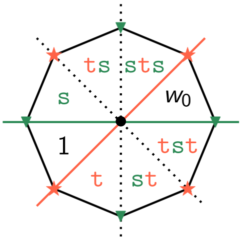

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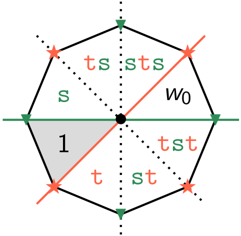

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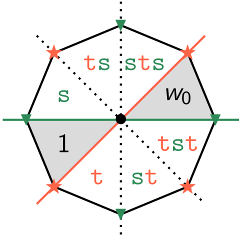

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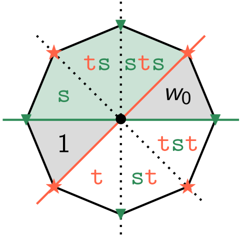

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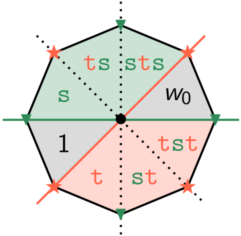

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t-cell.

Dihedral representation theory on one slide.

The Bott–Samelson (BS) generators $\theta_{s} = s + 1, \theta_{t} = t + 1$.
There is also a Kazhdan–Lusztig (KL) bases. Explicit formulas do not matter today.

One-dimensional modules. $\mathrm{M}_{\lambda_s,\lambda_t}, \lambda_s, \lambda_t \in \mathbb{C}, \theta_s \mapsto \lambda_s, \theta_t \mapsto \lambda_t$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_{0,0}, \mathrm{M}_{2,0}, \mathrm{M}_{0,2}, \mathrm{M}_{2,2}$	$\mathrm{M}_{0,0}, \mathrm{M}_{2,2}$

Two-dimensional modules. $\mathrm{M}_z, z \in \mathbb{C}, \theta_s \mapsto \begin{pmatrix} 2 & z \\ 0 & 0 \end{pmatrix}, \theta_t \mapsto \begin{pmatrix} 0 & 0 \\ \bar{z} & 2 \end{pmatrix}$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm} - \{0\}$	$\mathrm{M}_z, z \in \mathrm{V}_e^{\pm}$

$\mathrm{V}_e = \mathrm{roots}(\mathsf{U}_{e+1}(\mathtt{X}))$ and $\mathrm{V}_e^{\pm}$ the $\mathbb{Z}/2\mathbb{Z}$-orbits under $z \mapsto -z$.

Dihedral representation theory on one slide.

One-dimension ... $\mathtt{t}$.

Proposition (Lusztig?).

The list of one- and two-dimensional W_{e+2}-modules is a complete, irredundant list of simple modules.

$\mathrm{M}_{0,0}, \mathrm{M}_{2,0}, \mathrm{M}_{0,2}, \mathrm{M}_{2,2}$	$\mathrm{M}_{0,0}, \mathrm{M}_{2,2}$

I learned this construction from Mackaay in 2017.

Two-dimensional modules. $\mathrm{M}_z, z \in \mathbb{C}, \theta_{\mathtt{s}} \mapsto \left(\begin{smallmatrix} 2 & z \\ 0 & 0 \end{smallmatrix}\right), \theta_{\mathtt{t}} \mapsto \left(\begin{smallmatrix} 0 & 0 \\ \bar{z} & 2 \end{smallmatrix}\right)$.

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Example.

$\mathrm{M}_{0,0}$ is the sign representation and $\mathrm{M}_{2,2}$ is the trivial representation.

In case e is odd, $\mathrm{U}_{e+1}(\mathtt{X})$ has a constant term, so $\mathrm{M}_{2,0}$, $\mathrm{M}_{0,2}$ are not representations.

T

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Example.

M_{z} for z being a root of the Chebyshev polynomial is a representation because the braid relation in terms of BS generators involves the coefficients of the Chebyshev polynomial.

Two-dim

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Dihed

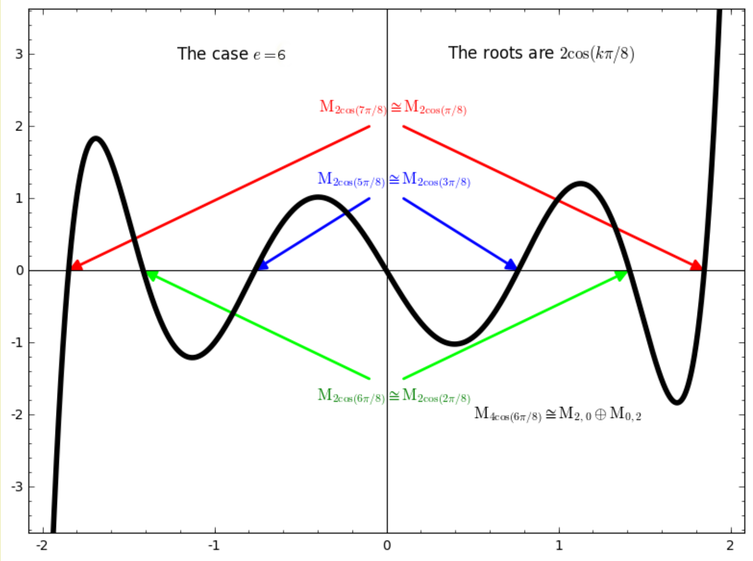
One-c

Two-c

$V_e =$

Example.

These representations are indexed by $\mathbb{Z}/2\mathbb{Z}$-orbits of the Chebyshev roots:



An algebra P with a fixed basis B^{P} is called a (multi) $\mathbb{N}_0$-algebra if

$$\mathrm{xy} \in \mathbb{N}_0\mathrm{B}^{\mathrm{P}} \quad (\mathrm{x}, \mathrm{y} \in \mathrm{B}^{\mathrm{P}}).$$

A P-module M with a fixed basis B^{M} is called a $\mathbb{N}_0$-module if

$$\mathrm{xm} \in \mathbb{N}_0\mathrm{B}^{\mathrm{M}} \quad (\mathrm{x} \in \mathrm{B}^{\mathrm{P}}, \mathrm{m} \in \mathrm{B}^{\mathrm{M}}).$$

These are $\mathbb{N}_0$-equivalent if there is a $\mathbb{N}_0$-valued change of basis matrix.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}_0$-equivalence comes from 2-equivalence.

Example.

Group algebras of finite groups with basis given by group elements are $\mathbb{N}_0$-algebras.

The regular module is a $\mathbb{N}_0$-module.

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Key example: $K_0(\mathcal{R}\mathrm{ep}(G))$ (easy $\mathbb{N}_0$-representation theory).

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Example.

Hecke algebras of (finite) Coxeter groups with their KL basis are $\mathbb{N}_0$-algebras.

Their $\mathbb{N}_0$-representation theory is mostly widely open.

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The

Example. $\mathbb{N}_0$-a ... tegorification of 2-categories and ... uivalence.

Clifford, Munn, Ponizovskiĭ ∼1942++, Kazhdan–Lusztig ∼1979. $x \leq_{L} y$ if y appears in zx with non-zero coefficient for $z \in B^{P}$. $x \sim_{L} y$ if $x \leq_{L} y$ and $y \leq_{L} x$. $\sim_{L}$ partitions P into left cells L. Similarly for right R, two-sided cells LR or $\mathbb{N}_0$-modules.

A $\mathbb{N}_0$-module M is transitive if all basis elements belong to the same $\sim_{L}$ equivalence class. An apex of M is a maximal two-sided cell not killing it.

Fact. Each transitive $\mathbb{N}_0$-module has a unique apex.

Hence, one can study them cell-wise.

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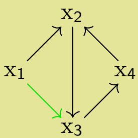

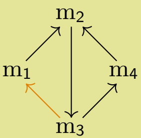

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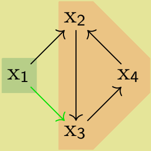

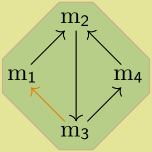

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Group algebras with the group element basis have only one cell, G itself.

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Hecke algebras for the dihedral group with KL basis have the following cells:

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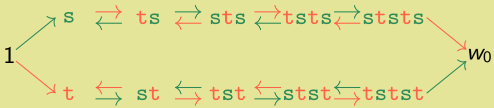

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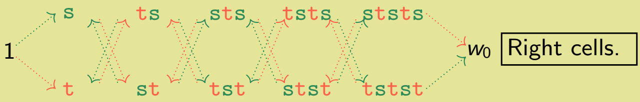

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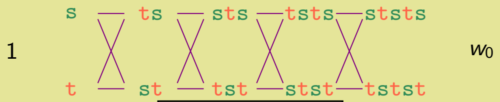

Two-sided cells.

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Clifford, Munn, Ponizovskiĭ ∼1942++, Kazhdan–Lusztig ∼1979. $x \leq_L y$ if y appears in zx with non-zero coefficient for $z \in B^P$. $x \sim_L y$ if $x \leq_L y$ and $y \leq_L x$. $\sim_L$ partitions P into left cells L. Similarly for right R, two-sided cells LR or $\mathbb{N}_0$-modules.

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Morally.

The further away an $\mathbb{N}_0$-algebra is from being semisimple, the more useful and interesting is its cell structure.

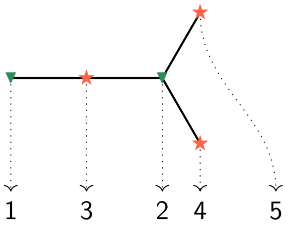
Hence, one ca

Example. Tr ation of simple 2-modules.

$\mathbb{N}_0$-modules via graphs.

Construct a W_∞-module M associated to a bipartite graph $\boldsymbol{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5 \rangle$$



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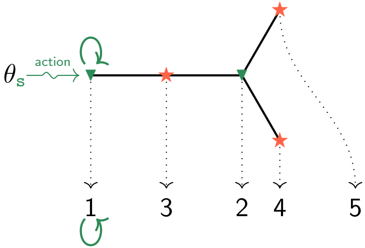

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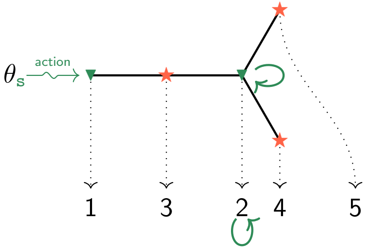

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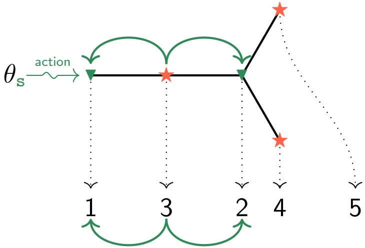

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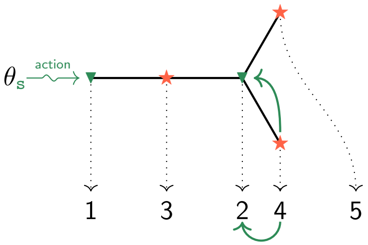

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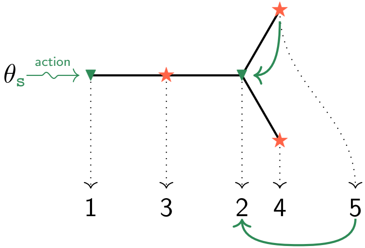

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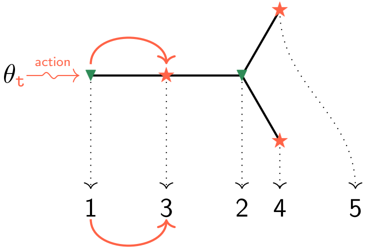

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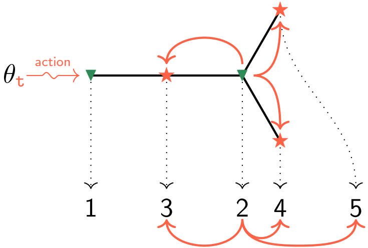

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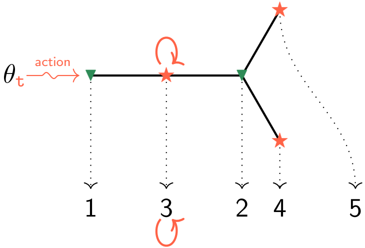

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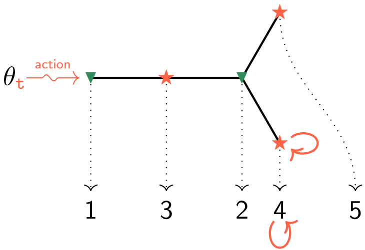

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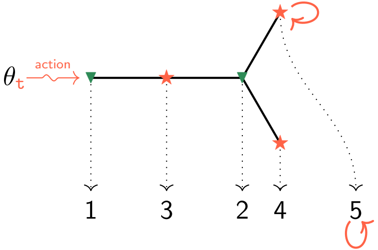

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Construct a W_∞-module M associated to a bipartite graph Γ:

The adjacency matrix $A(\Gamma)$ of Γ is

$$A(\Gamma) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

These are W_{e+2}-modules for some e
only if $A(\Gamma)$ is killed by the Chebyshev polynomial $\mathrm{U}_{e+1}(\mathtt{X})$.

Morally speaking: These are constructed as the simples
but with integral matrices having the Chebyshev-roots as eigenvalues.

It is not hard to see that the Chebyshev–braid-like relation can not hold otherwise.

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} \cdots \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} \cdots \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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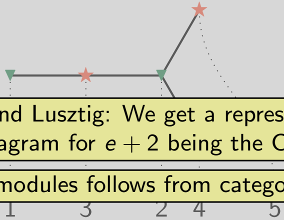

Hence, by Smith's (CP) and Lusztig: We get a representation of W_{e+2} if $\boldsymbol{\Gamma}$ is a ADE Dynkin diagram for $e+2$ being the Coxeter number.

That these are $\mathbb{N}_0$-modules follows from categorification.

'Smaller solutions' are never $\mathbb{N}_0$-modules.

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Construct a W_∞-module M associated to a bipartite graph Γ:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5 \rangle$$

Classification.

Complete, irredundant list of transitive $\mathbb{N}_0$-modules of W_{e+2}:

apex	1 cell	$s - t$ cell	w_0 cell
$\mathbb{N}_0$-reps.	$\mathrm{M}_{0,0}$	$\mathrm{M}_{\text{ADE+bicolering}}$ for $e+2 =$ Cox. num.	$\mathrm{M}_{2,2}$

I learned this from/with Kildetoft–Mackaay–Mazorchuk–Zimmermann ∼2016.

$$\theta_s \rightsquigarrow \mathrm{M}_s = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_t \rightsquigarrow \mathrm{M}_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Example ($e = 2$).

The Weyl group of type B_2. Number of elements: 8. Number of cells: 3, named 0 (trivial) to 2 (top).

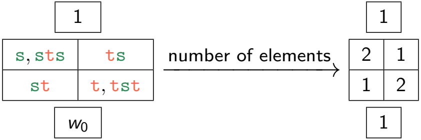
Cell order:

$$0 - 1 - 2$$

Size of the cells:

cell	0	1	2
size	1	6	1

Cell structure:



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Example (SAGE).

$$1 \cdot 1 = \mathtt{v}^0 1.$$

($\mathtt{v}$ is the Hecke parameter deforming the reflection equations $\mathtt{s}^2 = \mathtt{t}^2 = 1$.)

Cell o…

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size	1	6	1

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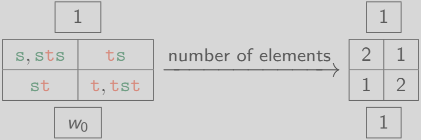

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Size of the cell

Example (SAGE).

$$\theta_{\mathtt{s}} \cdot \theta_{\mathtt{s}} = (\mathtt{v}^1 + \text{lower powers})\theta_{\mathtt{s}}.$$
$$\theta_{\mathtt{sts}} \cdot \theta_{\mathtt{s}} = (\mathtt{v}^1 + \text{lower powers})\theta_{\mathtt{sts}}.$$
$$\theta_{\mathtt{sts}} \cdot \theta_{\mathtt{sts}} = (\mathtt{v}^1 + \text{lower powers})\theta_{\mathtt{s}} + \text{higher cell elements}.$$
$$\theta_{\mathtt{sts}} \cdot \theta_{\mathtt{tst}} = (\text{lower powers})\theta_{\mathtt{st}} + \text{higher cell elements}.$$

Cell structure:

s, sts	ts
st	t, tst

w_0

number of elements →

2	1
1	2

1

Example ($e = 2$).

The W ... med 0 (trivia ...

Cell o...

$0 — 1 — 2$

Size of the cell

Cell structure:

s, sts	ts	number of elements	2	1
				2

Example (SAGE).

$$1 \cdot 1 = \mathrm{v}^0 1.$$

(v is the Hecke parameter deforming the reflection equations $\mathrm{s}^2 = \mathrm{t}^2 = 1$.)

Example (SAGE).

$$\theta_{\mathrm{s}} \cdot \theta_{\mathrm{s}} = (\mathrm{v}^1 + \text{lower powers})\theta_{\mathrm{s}}.$$
$$\theta_{\mathrm{sts}} \cdot \theta_{\mathrm{s}} = (\mathrm{v}^1 + \text{lower powers})\theta_{\mathrm{sts}}.$$
$$\theta_{\mathrm{sts}} \cdot \theta_{\mathrm{sts}} = (\mathrm{v}^1 + \text{lower powers})\theta_{\mathrm{s}} + \text{higher cell elements}.$$
$$\theta_{\mathrm{sts}} \cdot \theta_{\mathrm{tst}} = (\text{lower powers})\theta_{\mathrm{st}} + \text{higher cell elements}.$$

Example (SAGE).

$$\theta_{w_0} \cdot \theta_{w_0} = (\mathrm{v}^4 + \text{lower powers})\theta_{w_0}.$$

Example ($e = 2$).

Fact (Lusztig ∼1984++).

For any Coxeter group W
there is a well-defined function

$$a\colon \mathrm{W} \to \mathbb{N}_0$$

which is constant on two-sided cells.

▸ Big example

The Weyl group of type … ber of cells: 3, named 0 (trivial) to 2 (top).

Cell order:

Size of the cells:

cell	0	1	2
size	1	6	1

Cell structure:

1	
s, sts	ts
st	t, tst
w_0	

$\xrightarrow{\text{number of elements}}$

1	
2	1
1	2
1	

Example ($e = 2$).

The Weyl group of type ber of cells: 3, named 0 (trivial) to 2 (top).

Cell order:

Size of the cells:

Cell structure:

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Idea (Lusztig ∼1984).

Ignore everything except the leading coefficient a(two-sided cell).

s, sts	ts
st	t, tst

w_0

number of elements →

2	1
1	2

1

Example ($e = 2$).

The Weyl group of type … ber of cells: 3, named 0 (trivial) to 2 (top).

Cell order:

Size of the cells:

Cell structure:

Fact (Lusztig ∼1984++).

For any Coxeter group W
there is a well-defined function

$$a\colon \mathrm{W} \to \mathbb{N}_0$$

which is constant on two-sided cells.

Big example

Idea (Lusztig ∼1984).

Ignore everything except the leading coefficient
a(two-sided cell).

Why isn't that stupid?
Because a is also turns up as the leading coefficients
of traces of standard generators acting on simple modules.

Upshot. One can associate an apex to simples,
and the simples should be uniquely determent by the leading coefficients.

Let $\mathrm{H_v(W)}$ be the Hecke algebra associated to W. The asymptotic limit $\mathrm{J_\infty(W)}$ of $\mathrm{H_v(W)}$ is defined as follows.

As a free $\mathbb{Z}$-module:

$$\mathrm{J_\infty(W)} = \bigoplus_{\mathsf{LR}} \mathbb{Z}\{t_w \mid w \in \mathsf{LR}\}. \qquad \text{Compare: } \mathrm{H_v(W)} = \mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]\{\theta_w \mid \mathrm{W}\}.$$

Multiplication.

$$t_x t_y = \sum_{z \in \mathsf{LR}} \gamma_{x,y}^z t_z. \qquad \text{Compare: } \theta_x \theta_y = \sum_{z \in \mathsf{LR}} h_{x,y}^z \theta_z + \text{bigger friends.}$$

where $\gamma_{x,y}^z \in \mathbb{N}_0$ is the leading coefficient of $h_{x,y}^z \in \mathbb{N}_0[\mathrm{v}, \mathrm{v}^{-1}]$.

Example ($e = 2$).

The multiplication tables (empty entries are 0 and $[2] = v + v^{-1}$) in 1:

	t_s	t_{sts}	t_{st}	t_t	t_{tst}	t_{ts}
t_s	t_s	t_{sts}	t_{st}			
t_{sts}	t_{sts}	t_s	t_{st}			
t_{ts}	t_{ts}	t_{ts}	$t_t + t_{tst}$			
t_t				t_t	t_{tst}	t_{ts}
t_{tst}				t_{tst}	t_t	t_{ts}
t_{st}				t_{st}	t_{st}	$t_s + t_{sts}$

	θ_s	θ_{sts}	θ_{st}	θ_t	θ_{tst}	θ_{ts}
θ_s	$[2]\theta_s$	$[2]\theta_{sts}$	$[2]\theta_{st}$	θ_{st}	$\theta_{st} + \theta_{w_0}$	$\theta_s + \theta_{sts}$
θ_{sts}	$[2]\theta_{sts}$	$[2]\theta_s + [2]^2\theta_{w_0}$	$[2]\theta_{st} + [2]\theta_{w_0}$	$\theta_s + \theta_{sts}$	$\theta_s + [2]^2\theta_{w_0}$	$\theta_s + \theta_{sts} + [2]\theta_{w_0}$
θ_{ts}	$[2]\theta_{ts}$	$[2]\theta_{ts} + [2]\theta_{w_0}$	$[2]\theta_t + [2]\theta_{tst}$	$\theta_t + \theta_{tst}$	$\theta_t + \theta_{tst} + [2]\theta_{w_0}$	$2\theta_{ts} + \theta_{w_0}$
θ_t	θ_{ts}	$\theta_{ts} + \theta_{w_0}$	$\theta_t + \theta_{tst}$	$[2]\theta_t$	$[2]\theta_{tst}$	$[2]\theta_{ts}$
θ_{tst}	$\theta_t + \theta_{tst}$	$\theta_t + [2]^2\theta_{w_0}$	$\theta_t + \theta_{tst} + [2]\theta_{w_0}$	$[2]\theta_{tst}$	$[2]\theta_t + [2]^2\theta_{w_0}$	$[2]\theta_{ts} + [2]\theta_{w_0}$
θ_{st}	$\theta_s + \theta_{sts}$	$\theta_s + \theta_{sts} + [2]\theta_{w_0}$	$2\theta_{st} + \theta_{w_0}$	$[2]\theta_{st}$	$[2]\theta_{st} + [2]\theta_{w_0}$	$[2]\theta_s + [2]\theta_{sts}$

(Note the "subalgebras".)

The asymptotic algebra is much simpler!

Big example

Fact (Lusztig ∼1984++).

$\mathrm{J}_\infty(\mathrm{W}) = \bigoplus_{\mathsf{LR}} \mathrm{J}_\infty^{\mathsf{LR}}(\mathrm{W})$ with the t_w basis and all its summands $\mathrm{J}_\infty^{\mathsf{LR}}(\mathrm{W}) = \mathbb{Z}\{t_w \mid w \in \mathsf{LR}\}$ are multifusion algebras.

(Meaning semisimple $\mathbb{N}_0$-algebras with a certain nice trace form.)

Let $\mathrm{H}_\mathrm{v}(\mathrm{W}$ ... it $\mathrm{J}_\infty(\mathrm{W})$ of $\mathrm{H}_\mathrm{v}(\mathrm{W})$

As a free ...

$\mathrm{J}_\infty(\mathrm{W}) = \bigoplus_{\mathsf{LR}} \mathbb{Z}\{t_w \mid w \in \mathsf{LR}\}$. Compare: $\mathrm{H}_\mathrm{v}(\mathrm{W}) = \mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]\{\theta_w \mid \mathrm{W}\}$.

Multiplication.

$t_x t_y = \sum_{z \in \mathsf{LR}} \gamma_{x,y}^z t_z$. Compare: $\theta_x\theta_y = \sum_{z \in \mathsf{LR}} h_{x,y}^z \theta_z$ + bigger friends.

where $\gamma_{x,y}^z \in \mathbb{N}_0$ is the leading coefficient of $h_{x,y}^z \in \mathbb{N}_0[\mathrm{v}, \mathrm{v}^{-1}]$.

Let $\mathrm{H}_{\mathrm{v}}(\mathrm{W}$ … it $\mathrm{J}_{\infty}(\mathrm{W})$
of $\mathrm{H}_{\mathrm{v}}(\mathrm{W})$ …

Fact (Lusztig ∼1984++).

$$\mathrm{J}_{\infty}(\mathrm{W}) = \bigoplus_{\mathsf{LR}} \mathrm{J}_{\infty}^{\mathsf{LR}}(\mathrm{W}) \text{ with the } t_w \text{ basis}$$

and all its summands $\mathrm{J}_{\infty}^{\mathsf{LR}}(\mathrm{W}) = \mathbb{Z}\{t_w \mid w \in \mathsf{LR}\}$
are multifusion algebras.

(Meaning semisimple $\mathbb{N}_0$-algebras with a certain nice trace form.)

As a free …

$\mathrm{J}_{\infty}(\mathrm{W}) = \bigoplus$ … $^{-1}]\{\theta_w \mid \mathrm{W}\}$.

Surprising fact 1 (Lusztig ∼1984++).
It seems one throws almost away everything, but:

There is an explicit embedding

$$\mathrm{H}_{\mathrm{v}}(\mathrm{W}) \hookrightarrow \mathrm{J}_{\infty}(\mathrm{W}) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$$

which is an isomorphism after scalar extension to $\mathbb{Q}(\mathrm{v})$.

Multiplication.

$t_x t_y = \sum_z$ … ger friends.

where $\gamma_{x,y}^{z} \in \mathbb{N}_0$ is the leading coefficient of $h_{x,y}^{z} \in \mathbb{N}_0[\mathrm{v}, \mathrm{v}^{-1}]$.

Let $\mathrm{H_v(W}$ … it $\mathrm{J_\infty(W)}$
of $\mathrm{H_v(W)}$

Fact (Lusztig ∼1984++).

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and all its summands $\mathrm{J}_\infty^{\mathsf{LR}}(\mathrm{W}) = \mathbb{Z}\{t_w \mid w \in \mathsf{LR}\}$
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which is an isomorphism after scalar extension to $\mathbb{Q}(\mathtt{v})$.

Multiplication.

$t_x t_y = \sum_z$ … ger friends.

where $\gamma_{x,y}^{z} \in$ …

Surprising fact 2 (Lusztig ∼1984++).

There is an explicit 1:1 correspondence

$$\{\text{simples of } \mathrm{H_v(W)} \text{ with apex } \mathsf{LR}\} \xleftrightarrow{1:1} \{\text{simples of } \mathrm{J}_{\mathtt{v}}^{\mathsf{LR}}(\mathrm{W})\}.$$

"Induced" transitive $\mathbb{N}_0$-algebras and -modules.

Fix a left cell L. Let $\mathrm{M}(\geq_{\mathsf{L}})$, respectively $\mathrm{M}(>_{\mathsf{L}})$, be the $\mathbb{N}_0$-modules spanned by all $\mathrm{x} \in \mathrm{B}^{\mathrm{P}}$ in the union $\mathsf{L}' \geq_{\mathsf{L}} \mathsf{L}$, respectively $\mathsf{L}' >_{\mathsf{L}} \mathsf{L}$. Similarly for right R, two-sided LR and diagonal $\mathsf{H} = \mathsf{L} \cap \mathsf{R}$ cells.

Left cell module $\mathrm{C}_{\mathsf{L}} = \mathrm{M}(\geq_{\mathsf{L}})/\mathrm{M}(>_{\mathsf{L}})$. (Left $\mathbb{N}_0$-module.)

Right cell module $\mathrm{C}_{\mathsf{R}} = \mathrm{M}(\geq_{\mathsf{R}})/\mathrm{M}(>_{\mathsf{R}})$. (Right $\mathbb{N}_0$-module.)

Two-sided cell module $\mathrm{C}_{\mathsf{LR}} = \mathrm{M}(\geq_{\mathsf{LR}})/\mathrm{M}(>_{\mathsf{LR}})$. ($\mathbb{N}_0$-bimodule.)

The diagonal cell $\mathrm{C}_{\mathsf{H}} = \mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W}) = (\mathrm{M}(\geq_{\mathsf{LR}})/\mathrm{M}(>_{\mathsf{LR}})) \cap \mathbb{K}\mathrm{B}^{\mathrm{P}}(\mathsf{H})$. ($\mathbb{N}_0$-subalgebra.)

Big example

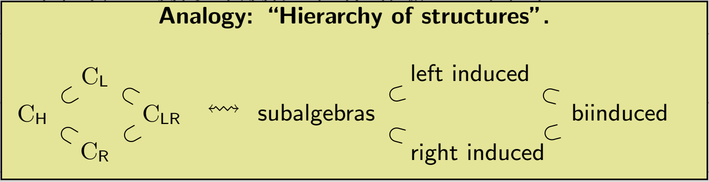
“Induced” transitive $\mathbb{N}_0$-algebras and -modules.

Fix a left cell L. Let $\mathrm{M}(\geq_{\mathsf{L}})$, respectively $\mathrm{M}(>_{\mathsf{L}})$, be the $\mathbb{N}_0$-modules spanned by all $x \in \mathrm{B}^{\mathrm{P}}$ in the union $\mathsf{L}' \geq_{\mathsf{L}} \mathsf{L}$, respectively $\mathsf{L}' >_{\mathsf{L}} \mathsf{L}$. Similarly for right R, two-sided LR and diagonal $\mathsf{H} = \mathsf{L} \cap \mathsf{R}$ cells.

Left cell

Right cel

Two-side



The diagonal cell $\mathrm{C}_{\mathsf{H}} = \mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W}) = (\mathrm{M}(\geq_{\mathsf{LR}})/\mathrm{M}(>_{\mathsf{LR}})) \cap \mathbb{K}\mathrm{B}^{\mathrm{P}}(\mathsf{H})$. ($\mathbb{N}_0$-subalgebra.)

Big example

“Induced” transitive $\mathbb{N}_0$-algebras and -modules

Example.

$\mathbb{C}[G]$ with the group element basis has only one cell module, the regular module.

Similarly for any fusion algebra.

Fix ... by all ... two ...

Left cell module $\mathrm{C}_{\mathsf{L}} = \mathrm{M}(\geq_{\mathsf{L}})/\mathrm{M}(>_{\mathsf{L}})$. (Left $\mathbb{N}_0$-module.)

Right cell module $\mathrm{C}_{\mathsf{R}} = \mathrm{M}(\geq_{\mathsf{R}})/\mathrm{M}(>_{\mathsf{R}})$. (Right $\mathbb{N}_0$-module.)

Two-sided cell module $\mathrm{C}_{\mathsf{LR}} = \mathrm{M}(\geq_{\mathsf{LR}})/\mathrm{M}(>_{\mathsf{LR}})$. ($\mathbb{N}_0$-bimodule.)

The diagonal cell $\mathrm{C}_{\mathsf{H}} = \mathrm{J}_{\infty}^{\mathsf{H}}(\mathrm{W}) = (\mathrm{M}(\geq_{\mathsf{LR}})/\mathrm{M}(>_{\mathsf{LR}})) \cap \mathbb{K}\mathrm{B}^{\mathrm{P}}(\mathsf{H})$. ($\mathbb{N}_0$-subalgebra.)

Big example

"Induced" transitive $\mathbb{N}_0$-algebras and -modules

Fix ... by all ... two ...

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$\mathbb{C}[G]$ with the group element basis has only one cell module, the regular module.

Similarly for any fusion algebra.

Left cell module $C_L = M(\geq_L)/M(>_L)$. (Left $\mathbb{N}_0$-module.)

Right ...

Two- ...

Example (Kazhdan–Lusztig ∼1979, Lusztig ∼1983++).

For Hecke algebras of the symmetric group with KL basis the cell modules are Lusztig's cell modules studied in connection with reductive groups in characteristic p.

The diagonal cell $C_H = J^H_\infty(W) = (M(\geq_{LR})/M(>_{LR})) \cap \mathbb{K}B^P(H)$. ($\mathbb{N}_0$-subalgebra.)

Big example

“Induced” transitive $\mathbb{N}_0$-algebras and -modules

Fix ... by all ... two

Example.

$\mathbb{C}[G]$ with the group element basis has only one cell module, the regular module.

Similarly for any fusion algebra.

Left cell module $C_L = M(\geq_L)/M(>_L)$. (Left $\mathbb{N}_0$-module.)

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For Hecke algebras of the symmetric group with KL basis
the cell modules are Lusztig's
cell modules studied in connection with reductive groups in characteristic p.

Right

Two-

Example (dihedral case).

Cells:

cell	0	1	2
size	1	$2n-2$	1
a	0	1	n

1 for n even:

$\frac{n}{2}$	$\frac{n-2}{2}$
$\frac{n-2}{2}$	$\frac{n}{2}$

1 for n odd:

$\frac{n-1}{2}$	$\frac{n-1}{2}$
$\frac{n-1}{2}$	$\frac{n-1}{2}$

n **even.** Two left cell modules $\leftrightsquigarrow$ Two bicolorings of the type A graph.
n **odd.** One left cell module $\leftrightsquigarrow$ One bicoloring of the type A graph.

▸ Big example

Example ($e = 2$).

The fusion ring $K_0(\mathrm{SL}(2)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_1], [L_2]$. The fusion ring $\mathrm{J}_{\infty}^{\mathsf{LR}}(\mathrm{W})$ has simple objects $t_{\mathtt{s}}, t_{\mathtt{sts}}, t_{\mathtt{st}}, t_{\mathtt{t}}, t_{\mathtt{tst}}, t_{\mathtt{ts}}$.

Comparison of multiplication tables:

	$[L_0]$	$[L_2]$	$[L_1]$
$[L_0]$	$[L_0]$	$[L_2]$	$[L_1]$
$[L_2]$	$[L_2]$	$[L_0]$	$[L_1]$
$[L_1]$	$[L_1]$	$[L_1]$	$[L_0] + [L_2]$

&

	$t_{\mathtt{s}}$	$t_{\mathtt{sts}}$	$t_{\mathtt{st}}$	$t_{\mathtt{t}}$	$t_{\mathtt{tst}}$	$t_{\mathtt{ts}}$
$t_{\mathtt{s}}$	$t_{\mathtt{s}}$	$t_{\mathtt{sts}}$	$t_{\mathtt{st}}$			
$t_{\mathtt{sts}}$	$t_{\mathtt{sts}}$	$t_{\mathtt{s}}$	$t_{\mathtt{st}}$			
$t_{\mathtt{ts}}$	$t_{\mathtt{ts}}$	$t_{\mathtt{ts}}$	$t_{\mathtt{t}} + t_{\mathtt{tst}}$			
$t_{\mathtt{t}}$				$t_{\mathtt{t}}$	$t_{\mathtt{tst}}$	$t_{\mathtt{ts}}$
$t_{\mathtt{tst}}$				$t_{\mathtt{tst}}$	$t_{\mathtt{t}}$	$t_{\mathtt{ts}}$
$t_{\mathtt{st}}$				$t_{\mathtt{st}}$	$t_{\mathtt{st}}$	$t_{\mathtt{s}} + t_{\mathtt{sts}}$

$\mathrm{J}_{\infty}^{\mathsf{LR}}(\mathrm{W})$ is a bicolored version of $K_0(\mathrm{SL}(2)_q)$:

$$t_{\mathtt{s}} \& t_{\mathtt{t}} \leftrightsquigarrow [L_0], \quad t_{\mathtt{sts}} \& t_{\mathtt{tst}} \leftrightsquigarrow [L_2], \quad t_{\mathtt{st}} \& t_{\mathtt{ts}} \leftrightsquigarrow [L_1].$$

Example ($e = 2$).

The fusion ring $K_0(\mathrm{SO}(3)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_2]$. The fusion ring $\mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W})$ ($\mathsf{H} = \mathsf{L}_{\mathtt{s}} \cap \mathsf{R}_{\mathtt{s}}$) has simple objects $t_{\mathtt{s}}, t_{\mathtt{sts}}$.

Comparison of multiplication tables:

	$[L_0]$	$[L_2]$
$[L_0]$	$[L_0]$	$[L_2]$
$[L_2]$	$[L_2]$	$[L_0]$

&

	$t_{\mathtt{s}}$	$t_{\mathtt{sts}}$
$t_{\mathtt{s}}$	$t_{\mathtt{s}}$	$t_{\mathtt{sts}}$
$t_{\mathtt{sts}}$	$t_{\mathtt{sts}}$	$t_{\mathtt{s}}$

$\mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W})$ is $K_0(\mathrm{SO}(3)_q)$:

$$t_{\mathtt{s}} \leftrightsquigarrow [L_0], \quad t_{\mathtt{sts}} \leftrightsquigarrow [L_2].$$

This is the slightly nicer statement.

Example ($e = 2$)

Fact.

Both connections are always true (*i.e.* for any e).

The fusion ring K … $[L_2]$. The fusion ring $\mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W})$ ($\mathsf{H} = \mathsf{L_s} \cap \mathsf{R_s}$) has simple objects $t_{\mathtt{s}}, t_{\mathtt{sts}}$.

Comparison of multiplication tables:

	$[L_0]$	$[L_2]$
$[L_0]$	$[L_0]$	$[L_2]$
$[L_2]$	$[L_2]$	$[L_0]$

&

	$t_{\mathtt{s}}$	$t_{\mathtt{sts}}$
$t_{\mathtt{s}}$	$t_{\mathtt{s}}$	$t_{\mathtt{sts}}$
$t_{\mathtt{sts}}$	$t_{\mathtt{sts}}$	$t_{\mathtt{s}}$

$\mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W})$ is $K_0(\mathrm{SO}(3)_q)$:

$$t_{\mathtt{s}} \leftrightsquigarrow [L_0], \quad t_{\mathtt{sts}} \leftrightsquigarrow [L_2].$$

Example $(e = 2)$

The fusion ring K … $[L_2]$. The fusion ring $J^{H}_{\infty}(W)$ $(H = L_s \cap R_s)$ has simple objects t_s, t_{sts}.

Fact.

Both connections are always true (*i.e.* for any e).

C

H-**cell-theorem.**

There are 1:1 correspondences

$$\{\text{transitives of } H_v(W) \text{ with apex LR}\} \xleftrightarrow{1:1} \{\text{transitives of } J^{LR}_v(W)\} \xleftrightarrow{1:1} \{\text{transitives of } J^{H}_v(W)\},$$

$$\{\text{transitives of } H_v(W) \text{ with apex LR}\} \xleftrightarrow{1:1} \{\text{transitives of } K_0(SL(2)^{s,t}_q)\} \xleftrightarrow{1:1} \{\text{transitives of } K_0(SO(3)_q)\}.$$

Example

J

$$t_s \leftrightsquigarrow [L_0], \quad t_{sts} \leftrightsquigarrow [L_2].$$

Example $(e = 2)$

The fusion ring K … $[L_2]$. The fusion ring $J_\infty^{H}(W)$ $(H = L_s \cap R_s)$ has simple objects t_s, t_{sts}.

C

J

$$t_s \leftrightsquigarrow [L_0], \quad t_{sts} \leftrightsquigarrow [L_2].$$

Fact.

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H-**cell-theorem.**

There are 1:1 correspondences

$$\{\text{transitives of } H_v(W) \text{ with apex LR}\} \xleftrightarrow{1:1} \{\text{transitives of } J_v^{LR}(W)\} \xleftrightarrow{1:1} \{\text{transitives of } J_v^{H}(W)\},$$

$$\{\text{transitives of } H_v(W) \text{ with apex LR}\} \xleftrightarrow{1:1} \{\text{transitives of } K_0(SL(2)_q^{s,t})\} \xleftrightarrow{1:1} \{\text{transitives of } K_0(SO(3)_q)\}.$$

Example

Upshot.

$H_v(W)$ is a non-semisimple version of $K_0(SL(2)_q)$,

$H_v^{H}(W)$ is a non-semisimple version of $K_0(SO(3)_q)$.

In particular, the Hecke algebras have a v parameter.

Example ($e = 2$).

The fusion ring $K_0(\mathrm{SO}(3)_q)$ for $q^{2e} = 1$ has simple objects $[L_0], [L_2]$. The fusion ring $\mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W})$ ($\mathsf{H} = \mathsf{L_s} \cap \mathsf{R_s}$) has simple objects $t_{\mathtt{s}}, t_{\mathtt{sts}}$.

Comparison of multiplication tables:

$\mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W})$ is $K_0(\mathrm{S}$

Fact.

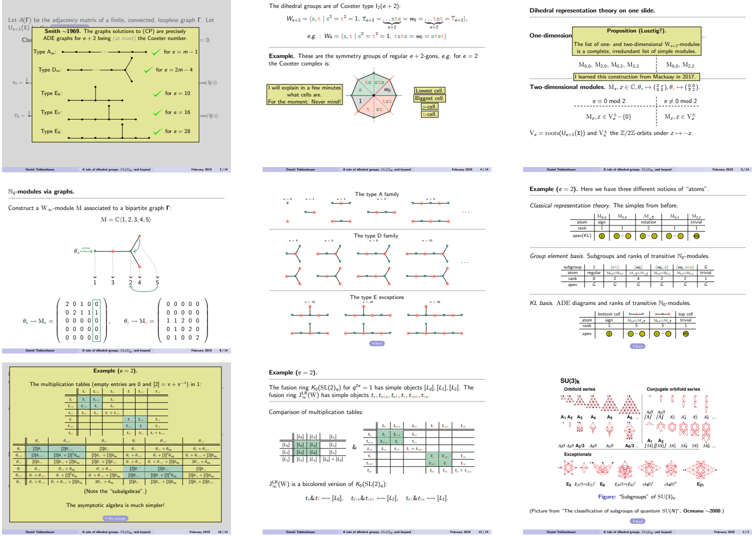
With a bit more care (with the H-cell-theorem) all the above generalizes to any Coxeter group W.

Thus, Hecke algebras are non-semisimple fusion rings.

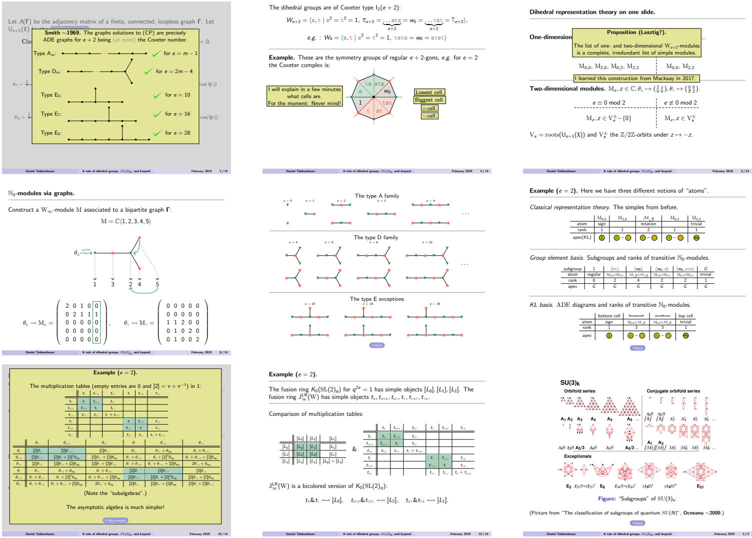
In general $\mathrm{J}_{\infty}(\mathrm{W})$ is not understood, but for W being a finite Weyl group $\mathrm{J}^{\mathsf{H}}_{\infty}(\mathrm{W})$ is very nice.

Beyond?

- Categorification?
  - Non-semisimple: Replace Hecke algebra by Soergel bimodules. ✓
  - Non-semisimple: Categorical $\mathbb{N}_0$-modules for dihedral groups. ✓ Zigzag algebras appear.
  - Fusion: Replace asymptotic Hecke algebra by asymptotic Soergel bimodules. ✓
  - Fusion: Categorical $\mathbb{N}_0$-modules for $\mathrm{SL}(2)_q$. ✓ Algebras are trivial.
  - H: Asymptotic Soergel bimodules are very nice, just remove K_0 everywhere. ✓
  - H-cell-theorem ? . Work in progress! Click
- $\mathrm{SL}(n)_q$?
  - Non-semisimple: Nhedral; leaves the realm of groups. ✓
  - Non-semisimple: Categorical $\mathbb{N}_0$-modules for Nhedral algebras have a Ncolored ADE-type classification. ✓ Generalized zigzag algebras and Chebyshev polynomials appear.
  - Fusion: One gets $\mathrm{SL}(N)_q$. ✓
  - Fusion: Categorical $\mathbb{N}_0$-modules of $\mathrm{SL}(N)_q$ have an ADE-type classification. ✓ Algebras are trivial. Click



There is still much to do...



Thanks for your attention!

$$\mathrm{U}_0(\mathtt{X}) = 1, \quad \mathrm{U}_1(\mathtt{X}) = \mathtt{X}, \quad \mathtt{X}\,\mathrm{U}_{e+1}(\mathtt{X}) = \mathrm{U}_{e+2}(\mathtt{X}) + \mathrm{U}_e(\mathtt{X})$$

$$\mathrm{U}_0(\mathtt{X}) = 1, \quad \mathrm{U}_1(\mathtt{X}) = 2\mathtt{X}, \quad 2\mathtt{X}\,\mathrm{U}_{e+1}(\mathtt{X}) = \mathrm{U}_{e+2}(\mathtt{X}) + \mathrm{U}_e(\mathtt{X})$$

Kronecker ∼1857. Any complete set of conjugate algebraic integers in $]-2,2[$ is a subset of $\mathrm{roots}(\mathrm{U}_{e+1}(\mathtt{X}))$ for some e.

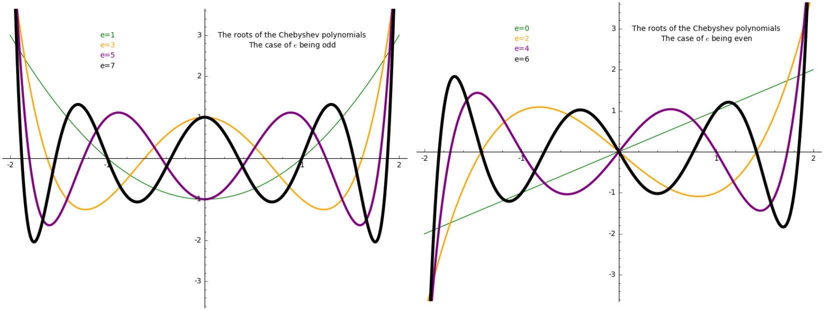


Figure: The roots of the Chebyshev polynomials (of the second kind).

In case you are wondering why this is supposed to be true, here is the main observation of **Smith ∼1969**:

$$\mathsf{U}_{e+1}(\mathtt{X},\mathtt{Y}) = \pm\det(\mathtt{X}\mathrm{Id} - A(\mathsf{A}_{e+1}))$$

Chebyshev poly. = char. poly. of the type A_{e+1} graph

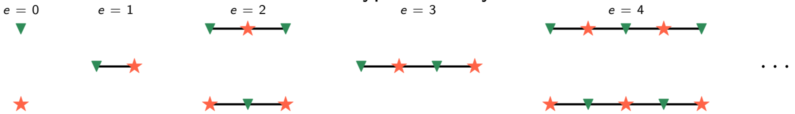
and

$$\mathtt{X}\mathsf{T}_{n-1}(\mathtt{X}) = \pm\det(\mathtt{X}\mathrm{Id} - A(\mathsf{D}_n)) \pm (-1)^{n \bmod 4}$$

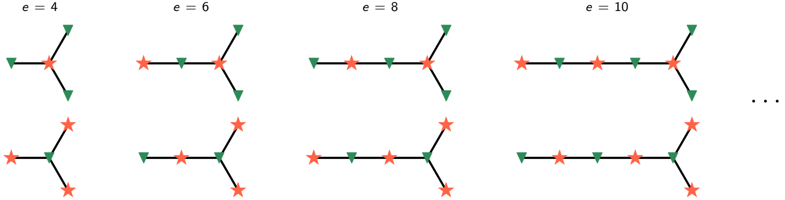
first kind Chebyshev poly. '=' char. poly. of the type D_n graph ($n = \frac{e+4}{2}$).

Back

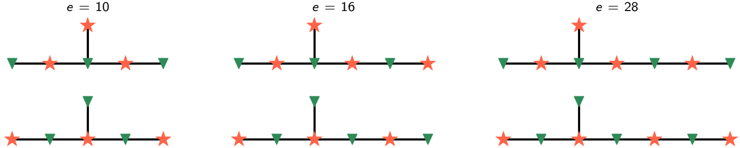
The type A family



The type D family

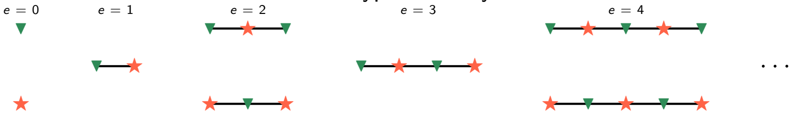


The type E exceptions

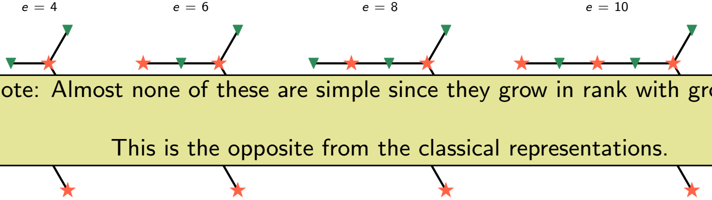


Back

The type A family



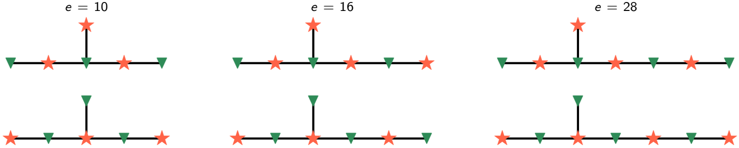
The type D family



Note: Almost none of these are simple since they grow in rank with growing e.

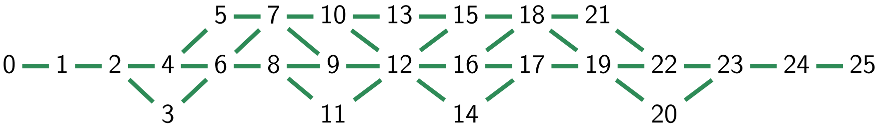
This is the opposite from the classical representations.

The type E exceptions



Example (SAGE). The Weyl group of type B_6. Number of elements: 46080. Number of cells: 26, named 0 (trivial) to 25 (top).

Cell order:



Size of the cells and *a*-value:

cell	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
size	1	62	342	576	650	3150	350	1600	2432	3402	900	2025	14500	600	2025	900	3402	2432	1600	350	576	3150	650	342	62	1
a	0	1	2	3	3	4	4	5	5	6	6	6	7	9	10	10	10	15	11	16	17	12	15	25	25	36

Back

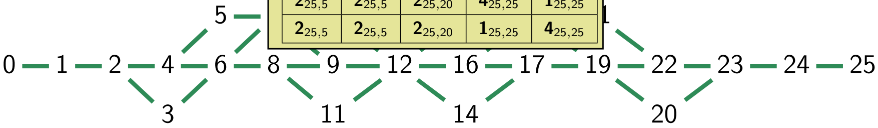
Example (cell 12).

Cell 12 is a bit scary:

$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{5,5}$	$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{20,5}$	$\mathbf{1}_{20,5}$	$\mathbf{4}_{20,20}$	$\mathbf{2}_{20,25}$	$\mathbf{2}_{20,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{4}_{25,25}$	$\mathbf{1}_{25,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{1}_{25,25}$	$\mathbf{4}_{25,25}$

Example (SAGE). The W er of elements: 46080.
Number of cells: 26, nam

Cell order:



Size of the cells and *a*-value:

cell	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
size	1	62	342	576	650	3150	350	1600	2432	3402	900	2025	14500	600	2025	900	3402	2432	1600	350	576	3150	650	342	62	1
a	0	1	2	3	3	4	4	5	5	6	6	6	7	9	10	10	10	15	11	16	17	12	15	25	25	36

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Example ($e = 2$). Here we have three different notions of "atoms".

Classical representation theory. The simples from before.

	$\mathrm{M}_{0,0}$	$\mathrm{M}_{2,0}$	$\mathcal{M}_{\sqrt{2}}$	$\mathrm{M}_{0,2}$	$\mathrm{M}_{2,2}$
atom	sign		rotation		trivial
rank	1	1	2	1	1
apex(KL)	1	s – t	s – t	s – t	w_0

Group element basis. Subgroups and ranks of transitive $\mathbb{N}_0$-modules.

subgroup	1	$\langle st \rangle$	$\langle w_0 \rangle$	$\langle w_0, s \rangle$	$\langle w_0, sts \rangle$	G
atom	regular	$\mathrm{M}_{0,0} \oplus \mathrm{M}_{2,2}$	$\mathcal{M}_{\sqrt{2}} \oplus \mathcal{M}_{\sqrt{2}}$	$\mathrm{M}_{2,0} \oplus \mathrm{M}_{2,2}$	$\mathrm{M}_{0,2} \oplus \mathrm{M}_{2,2}$	trivial
rank	8	2	4	2	2	1
apex	G	G	G	G	G	G

KL basis. ADE diagrams and ranks of transitive $\mathbb{N}_0$-modules.

	bottom cell			top cell
atom	sign	$\mathrm{M}_{2,0} \oplus \mathcal{M}_{\sqrt{2}}$	$\mathrm{M}_{0,2} \oplus \mathcal{M}_{\sqrt{2}}$	trivial
rank	1	3	3	1
apex	1	s – t	s – t	w_0

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = \texttt{i}$):

$$d = d^{-1} = 132123565,\ u = u^{-1} = 12132123565.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$\begin{gathered}\theta_d\theta_d = \\ (\mathtt{v}^7 + 5\mathtt{v}^5 + 12\mathtt{v}^3 + 18\mathtt{v} + 18\mathtt{v}^{-1} + 12\mathtt{v}^{-3} + 5\mathtt{v}^{-5} + \mathtt{v}^{-7})\theta_d \\ +(\mathtt{v}^5 + 4\mathtt{v}^3 + 7\mathtt{v} + 7\mathtt{v}^{-1} + 4\mathtt{v}^{-3} + \mathtt{v}^{-5})\theta_u \\ +(\mathtt{v}^6 + 5\mathtt{v}^4 + 11\mathtt{v}^2 + 14 + 11\mathtt{v}^{-2} + 5\mathtt{v}^{-4} + \mathtt{v}^{-6})\theta_{121232123565}\end{gathered}$$

Graph:

$$1 \overset{4}{\text{—}} 2 \text{—} 3 \text{—} 4 \text{—} 5 \text{—} 6$$

Elements (shorthand $s_i = \mathtt{i}$):

$$d = d^{-1} = 132123565,\ u = u^{-1} = 12132123565.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$
\begin{gathered}
t_d t_d = \\
(\mathrm{v}^7 + 5\mathrm{v}^5 + 12\mathrm{v}^3 + 18\mathrm{v} + 18\mathrm{v}^{-1} + 12\mathrm{v}^{-3} + 5\mathrm{v}^{-5} + \mathrm{v}^{-7})\theta_d \\
+(\mathrm{v}^5 + 4\mathrm{v}^3 + 7\mathrm{v} + 7\mathrm{v}^{-1} + 4\mathrm{v}^{-3} + \mathrm{v}^{-5})\theta_u \\
+(\mathrm{v}^6 + 5\mathrm{v}^4 + 11\mathrm{v}^2 + 14 + 11\mathrm{v}^{-2} + 5\mathrm{v}^{-4} + \mathrm{v}^{-6})\theta_{121232123565}
\end{gathered}
$$

Graph:

$$1 \overset{4}{\text{—}} 2 \text{ — } 3 \text{ — } 4 \text{ — } 5 \text{ — } 6$$

Elements (shorthand $s_i = \mathtt{i}$):

$$d = d^{-1} = 132123565, \ u = u^{-1} = 12132123565.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$t_d t_d =$$
$$(v^7 + 5v^5 + 12v^3 + 18v + 18v^{-1} + 12v^{-3} + 5v^{-5} + v^{-7})\theta_d$$
$$+(v^5 + 4v^3 + 7v + 7v^{-1} + 4v^{-3} + v^{-5})\theta_u$$
$$+(v^6 + 5v^4 + 11v^2 + 14 + 11v^{-2} + 5v^{-4} + v^{-6})\theta_{121232123565}$$

Bigger friends.

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = \mathtt{i}$):

$$d = d^{-1} = 132123565, \; u = u^{-1} = 12132123565.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$
\begin{gathered}
t_d t_d = \\
(\mathtt{v}^7 + 5\mathtt{v}^5 + 12\mathtt{v}^3 + 18\mathtt{v} + 18\mathtt{v}^{-1} + 12\mathtt{v}^{-3} + 5\mathtt{v}^{-5} + \mathtt{v}^{-7})\theta_d \\
+(\mathtt{v}^5 + 4\mathtt{v}^3 + 7\mathtt{v} + 7\mathtt{v}^{-1} + 4\mathtt{v}^{-3} + \mathtt{v}^{-5})\theta_u
\end{gathered}
$$

Graph:

$$1 \overset{4}{\text{—}} 2 \text{—} 3 \text{—} 4 \text{—} 5 \text{—} 6$$

Elements (shorthand $s_i = \mathtt{i}$):

$$d = d^{-1} = \mathtt{132123565}, \; u = u^{-1} = \mathtt{12132123565}.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$t_d t_d =$$
$$(\mathtt{v}^7 + 5\mathtt{v}^5 + 12\mathtt{v}^3 + 18\mathtt{v} + 18\mathtt{v}^{-1} + 12\mathtt{v}^{-3} + 5\mathtt{v}^{-5} + \mathtt{v}^{-7})\theta_d$$
$$+(\mathtt{v}^5 + 4\mathtt{v}^3 + 7\mathtt{v} + 7\mathtt{v}^{-1} + 4\mathtt{v}^{-3} + \mathtt{v}^{-5})\theta_u$$

Killed in the limit $\mathtt{v} \to \infty$.

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = \mathtt{i}$):

$$d = d^{-1} = 132123565, \; u = u^{-1} = 12132123565.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$t_d t_d =$$
$$t_d$$

Looks much simpler.

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = \mathtt{i}$):

$$d = d^{-1} = 132123565, \; u = u^{-1} = 12132123565.$$

Back

Example (SAGE; Type B_6).

Up to $\mathbb{N}_0$-equivalence: five left cell modules, five right cell modules, one two-sided cell bimodule, three H subalgebras:

$L =$

$4_{5,5}$	$1_{5,5}$	$1_{5,20}$	$2_{5,25}$	$2_{5,25}$
$1_{5,5}$	$4_{5,5}$	$1_{5,20}$	$2_{5,25}$	$2_{5,25}$
$1_{20,5}$	$1_{20,5}$	$4_{20,20}$	$2_{20,25}$	$2_{20,25}$
$2_{25,5}$	$2_{25,5}$	$2_{25,20}$	$4_{25,25}$	$1_{25,25}$
$2_{25,5}$	$2_{25,5}$	$2_{25,20}$	$1_{25,25}$	$4_{25,25}$

$R =$

$4_{5,5}$	$1_{5,5}$	$1_{5,20}$	$2_{5,25}$	$2_{5,25}$
$1_{5,5}$	$4_{5,5}$	$1_{5,20}$	$2_{5,25}$	$2_{5,25}$
$1_{20,5}$	$1_{20,5}$	$4_{20,20}$	$2_{20,25}$	$2_{20,25}$
$2_{25,5}$	$2_{25,5}$	$2_{25,20}$	$4_{25,25}$	$1_{25,25}$
$2_{25,5}$	$2_{25,5}$	$2_{25,20}$	$1_{25,25}$	$4_{25,25}$

$LR =$

$4_{5,5}$	$1_{5,5}$	$1_{5,20}$	$2_{5,25}$	$2_{5,25}$
$1_{5,5}$	$4_{5,5}$	$1_{5,20}$	$2_{5,25}$	$2_{5,25}$
$1_{20,5}$	$1_{20,5}$	$4_{20,20}$	$2_{20,25}$	$2_{20,25}$
$2_{25,5}$	$2_{25,5}$	$2_{25,20}$	$4_{25,25}$	$1_{25,25}$
$2_{25,5}$	$2_{25,5}$	$2_{25,20}$	$1_{25,25}$	$4_{25,25}$

$H =$

$4_{5,5}$	$1_{5,5}$	$1_{5,20}$	$2_{5,25}$	$2_{5,25}$
$1_{5,5}$	$4_{5,5}$	$1_{5,20}$	$2_{5,25}$	$2_{5,25}$
$1_{20,5}$	$1_{20,5}$	$4_{20,20}$	$2_{20,25}$	$2_{20,25}$
$2_{25,5}$	$2_{25,5}$	$2_{25,20}$	$4_{25,25}$	$1_{25,25}$
$2_{25,5}$	$2_{25,5}$	$2_{25,20}$	$1_{25,25}$	$4_{25,25}$

Fact. The three $\mathbb{N}_0$-algebras $J_{\infty}^{H}(W)$ are all "categorical Morita equivalent". (They have the same number of transitive $\mathbb{N}_0$-modules.)

Back

Example ($e = 2$).

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3\rangle$$

$$\theta_{\mathtt{s}} \rightsquigarrow \begin{pmatrix} v+v^{-1} & 0 & 1 \\ 0 & v+v^{-1} & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \theta_{\mathtt{t}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & v+v^{-1} \end{pmatrix}$$

$$\theta_{\mathtt{sts}} \rightsquigarrow \begin{pmatrix} 0 & v+v^{-1} & 1 \\ v+v^{-1} & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \theta_{\mathtt{tst}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & v+v^{-1} \end{pmatrix}$$

$$\theta_{\mathtt{ts}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ v+v^{-1} & v+v^{-1} & 1 \end{pmatrix} \qquad \theta_{\mathtt{st}} \rightsquigarrow \begin{pmatrix} 1 & 1 & v+v^{-1} \\ 1 & 1 & v+v^{-1} \\ 0 & 0 & 0 \end{pmatrix}$$

Back

Example ($e = 2$).

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3 \rangle$$

$$\theta_{\mathtt{s}} \rightsquigarrow \begin{pmatrix} \mathrm{v}+\mathrm{v}^{-1} & 0 & 1 \\ 0 & \mathrm{v}+\mathrm{v}^{-1} & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \theta_{\mathtt{t}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & \mathrm{v}+\mathrm{v}^{-1} \end{pmatrix}$$

$$\theta_{\mathtt{sts}} \rightsquigarrow \begin{pmatrix} 0 & \mathrm{v}+\mathrm{v}^{-1} & 1 \\ \mathrm{v}+\mathrm{v}^{-1} & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \theta_{\mathtt{tst}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & \mathrm{v}+\mathrm{v}^{-1} \end{pmatrix}$$

$$\theta_{\mathtt{ts}} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{v}+\mathrm{v}^{-1} & \mathrm{v}+\mathrm{v}^{-1} & 1 \end{pmatrix} \qquad \theta_{\mathtt{st}} \rightsquigarrow \begin{pmatrix} 1 & 1 & \mathrm{v}+\mathrm{v}^{-1} \\ 1 & 1 & \mathrm{v}+\mathrm{v}^{-1} \\ 0 & 0 & 0 \end{pmatrix}$$

Back

Example ($e = 2$).

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3 \rangle$$

$$t_{s} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t_{t} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$t_{sts} \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t_{tst} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$t_{ts} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad t_{st} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Back

Example ($e = 2$).

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3\rangle$$

Example.

$$t_{st} t_{ts} = t_s + t_{sts}$$

$$\leftrightsquigarrow$$

$$[L_1][L_1] = [L_0] + [L_2]$$

$$\leftrightsquigarrow$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$t_{ts} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad t_{st} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Back

Example ($e = 2$).

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3 \rangle$$

This works in general and recovers the transitive $\mathbb{N}_0$-modules of $K_0(\mathrm{SL}(2)_q)$ found by Etingof–Khovanov ∼1995 and Kirillov–Ostrik ∼2001, which are also ADE classified.

(For the experts: the bicoloring kills the tadpole solutions.)

$$t_{ts} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad t_{st} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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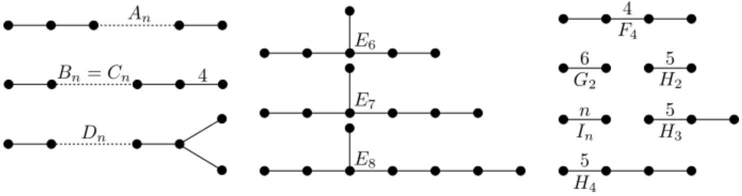

Figure: The connected Coxeter diagrams of finite type. The finite Weyl groups are of type A, B = C, D, E, F and G.

Example: Hecke algebras as non-semisimple fusion rings (Lusztig ∼1984).

type	A	B = C	D	E_6
worst case	$J^H_\infty \cong 1$	$J^H_\infty \cong K_0(\mathcal{V}\mathrm{ec}_{(\mathbb{Z}/2\mathbb{Z})^d})$	$J^H_\infty \cong K_0(\mathcal{V}\mathrm{ec}_{(\mathbb{Z}/2\mathbb{Z})^d})$	$J^H_\infty \cong K_0(\mathcal{R}\mathrm{ep}(S_3))$

type	E_7	E_8	F_4	G_2
worst case	$J^H_\infty \cong K_0(\mathcal{R}\mathrm{ep}(S_3))$	$J^H_\infty \cong K_0(\mathcal{R}\mathrm{ep}(S_5))$	$J^H_\infty \cong K_0(\mathcal{R}\mathrm{ep}(S_4))$	$J^H_\infty \cong K_0(\mathcal{R}\mathrm{ep}(S_2))$

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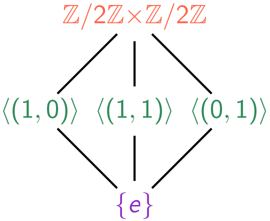
(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Example ($G = \mathbb{Z}/2\times\mathbb{Z}/2$).

Subgroups, Schur multipliers and 2-simples.

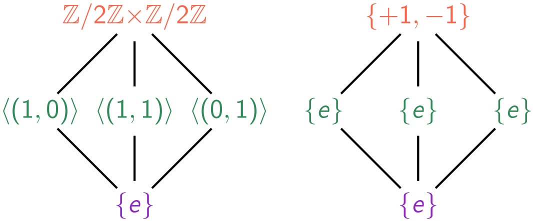


In particular, there are two categorifications of the trivial module, and the rank sequences read

$$\text{decat: } 1, 2, 2, 2, 4, \qquad \text{cat: } 1, 1, 2, 2, 2, 4.$$

Example ($G = \mathbb{Z}/2\times\mathbb{Z}/2$).

Subgroups, Schur multipliers and 2-simples.

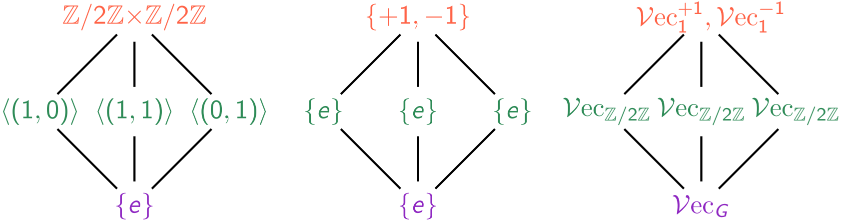


In particular, there are two categorifications of the trivial module, and the rank sequences read

$$\text{decat: } 1, 2, 2, 2, 4, \qquad \text{cat: } 1, 1, 2, 2, 2, 4.$$

Example ($G = \mathbb{Z}/2\times\mathbb{Z}/2$).

Subgroups, Schur multipliers and 2-simples.



In particular, there are two categorifications of the trivial module, and the rank sequences read

$$\text{decat: } 1, 2, 2, 2, 4, \qquad \text{cat: } 1, 1, 2, 2, 2, 4.$$

Example (SAGE; Type B_6).

Reducing from 46080 to 14500 to 4:

LR =

$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{5,5}$	$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{20,5}$	$\mathbf{1}_{20,5}$	$\mathbf{4}_{20,20}$	$\mathbf{2}_{20,25}$	$\mathbf{2}_{20,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{4}_{25,25}$	$\mathbf{1}_{25,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{1}_{25,25}$	$\mathbf{4}_{25,25}$

$\rightsquigarrow$ H =

$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{5,5}$	$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{20,5}$	$\mathbf{1}_{20,5}$	$\mathbf{4}_{20,20}$	$\mathbf{2}_{20,25}$	$\mathbf{2}_{20,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{4}_{25,25}$	$\mathbf{1}_{25,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{1}_{25,25}$	$\mathbf{4}_{25,25}$

$\mathscr{J}_{\infty}^{\mathsf{H}} = \mathcal{V}\mathrm{ec}_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$, rank sequence: 1, 1, 2, 2, 2, 4.

In particular, there is one non-cell 2-simple: one 2 is missing.

Back

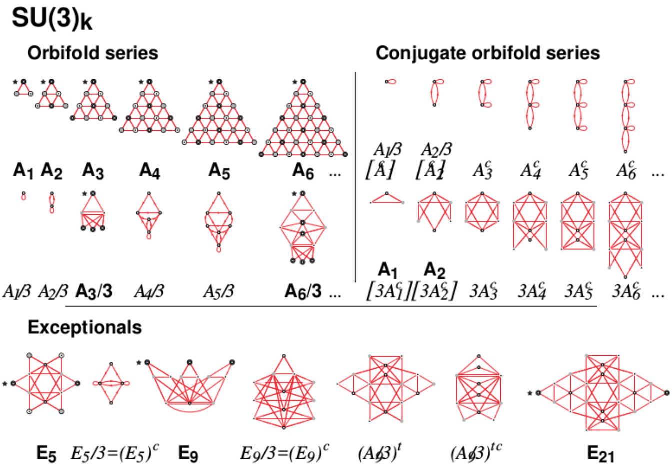


Figure: "Subgroups" of $\mathrm{SU}(3)_q$.

(Picture from "The classification of subgroups of quantum $\mathrm{SU}(N)$", **Ocneanu ∼2000**.)

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