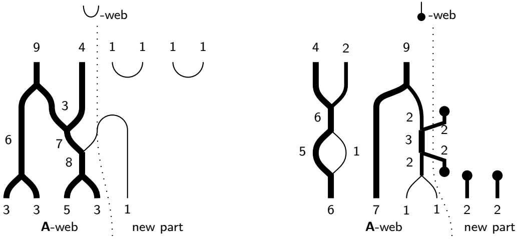
Webs and q-Howe dualities in types **BCD**

Or: “Howe” not to define link invariants

Daniel Tubbenhauer



Joint work with Antonio Sartori (and David Rose, Pedro Vaz and Paul Wedrich)

June 2017

The idea which started this project.

Type **A**:

Howe duality $\xrightarrow{\text{gives}}$ Web calculi $\xrightarrow{\text{gives}}$
Reshetikhin–Turaev polynomial
Khovanov–Rozansky homology
HOMFLY–PT homology

Types **BCD**:

Type **BCD** Howe duality $\xrightarrow[\text{give}]{\text{should}}$ Type **BCD** web calculi $\xrightarrow[\text{give}]{\text{should}}$
Type **BCD**
Reshetikhin–Turaev polynomial
Khovanov–Rozansky homology
Kauffman homology

However, this does not work (in any straightforward way).

The idea which started this project.

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Howe duality $\xrightarrow{\text{gives}}$ Web calculi $\xrightarrow{\text{gives}}$ Reshetikhin–Turaev polynomial
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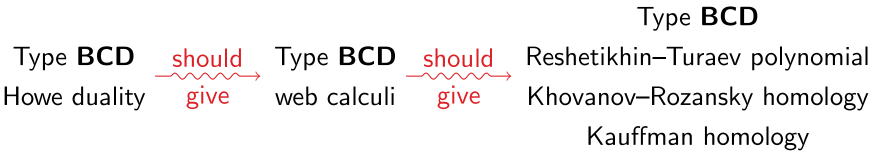

Type **BCD** Howe duality $\xrightarrow[\text{give}]{\text{should}}$ Type **BCD** web calculi $\xrightarrow[\text{give}]{\text{should}}$ Type **BCD**
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1. The type **A** story
    - Classical Schur–Weyl duality
    - Howe's dualities in type **A**
2. The type **BCD** story
    - Classical Schur–Weyl–Brauer duality
    - Howe's dualities in types **BCD**
3. The quantum story
    - Various quantizations
    - Some concluding remarks

A pioneer of representation theory

▸ Schur's remarkable relationship between $\mathfrak{gl}_n$ and the symmetric group S_k:

Schur ∼1901. Let $\mathrm{V} = \mathrm{V}^{\mathfrak{gl}} = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{gl}_n) \circlearrowright \underbrace{\mathrm{V} \otimes \cdots \otimes \mathrm{V}}_{k \text{ times}} \circlearrowleft \mathbb{C}[S_k]$$

generating each other's centralizer. The $\mathbf{U}(\mathfrak{gl}_n)$-$\mathbb{C}[S_k]$-bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} \mathrm{L}(\mathfrak{gl}_n, \lambda) \otimes \mathrm{D}(S_k, \lambda^{\mathrm{T}}).$$

The λ's are partitions (Young diagrams) of k with at most n rows.

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The precise form does not matter for today. It is only important that one can make it explicit.

The λ's are partitions (Young diagrams) of k with at most n rows.

The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowright \underbrace{\mathrm{V} \otimes \cdots \otimes \mathrm{V}}_{k \text{ times}} \circlearrowleft \overset{\text{use}}{\mathbb{C}[S_k]}$$

Schur's first statement gives a functor

$$\boxed{\text{Categorical version of the symmetric group}} \quad \mathcal{S} \xrightarrow{\Phi} \mathcal{R}\mathbf{ep}(\mathfrak{gl}_n)$$

The diagrammatic presentation machine

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Schur's second statement gives a full functor

$$\mathcal{S} \xrightarrow[\text{full}]{\Phi} \mathcal{R}\mathbf{ep}(\mathfrak{gl}_n)$$

The diagrammatic presentation machine

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Schur's third statement gives a full functor

$$\mathcal{S} \xrightarrow[\text{full}]{\Phi} \mathcal{R}\mathbf{ep}(\mathfrak{gl}_n)$$

$$\mathcal{S}/\text{“}\ker(\Phi)\text{”} \xrightarrow[\text{fully faithful}]{\Phi} \mathcal{R}\mathbf{ep}(\mathfrak{gl}_n)$$

whose “kernel $\ker(\Phi)$” can be calculated.

Hence, up to taking duals and additive/Karoubi closures, Schur gave us a ▸ diagrammatic presentation of the representation category $\mathcal{R}\mathbf{ep}(\mathfrak{gl}_n)$ of $\mathfrak{gl}_n$.

“Thickened” Schur–Weyl duality

▸ One of Howe’s remarkable relationships between $\mathfrak{gl}_n$ and $\mathfrak{gl}_k$:

Howe ∼1975. Let $\mathrm{V} = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{gl}_n) \circlearrowright \underbrace{\textstyle\bigwedge^{\bullet}\mathrm{V} \otimes \cdots \otimes \bigwedge^{\bullet}\mathrm{V}}_{k \text{ times}} \circlearrowleft \mathbf{U}(\mathfrak{gl}_k)$$

generating each other’s centralizer, and $\bigwedge^{a_1}\mathrm{V} \otimes \cdots \otimes \bigwedge^{a_k}\mathrm{V}$ is the $(a_1, \dots, a_k)$th weight space as regards $\mathbf{U}(\mathfrak{gl}_k)$. The $\mathbf{U}(\mathfrak{gl}_n)$-$\mathbf{U}(\mathfrak{gl}_k)$-bimodule decomposes as

$$\textstyle\bigoplus_{\lambda \in \mathfrak{P}} \mathrm{L}(\mathfrak{gl}_n, \lambda) \otimes \mathrm{L}(\mathfrak{gl}_k, \lambda^{\mathrm{T}}).$$

The λ’s are partitions with at most k columns and n rows.

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$1^1\!/_2$th statement

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$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowright \underbrace{{\textstyle\bigwedge}^{\bullet}\mathrm{V} \otimes \cdots \otimes {\textstyle\bigwedge}^{\bullet}\mathrm{V}}_{k \text{ times}} \circlearrowleft \overset{\text{use}}{\mathbf{U}(\mathfrak{gl}_k)}$$

Howe's first statement gives a functor

Dot version generated by weight space idempotents 1_λ, and E_i and F_i

$$\dot{\mathbf{U}}(\mathfrak{gl}_k) \xrightarrow{\Phi_{\mathbf{A}}^{\mathrm{ext}}} \mathcal{R}\mathbf{ep}(\mathfrak{gl}_n)$$

Again: The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowright \underbrace{{\textstyle\bigwedge}^{\bullet}\mathrm{V} \otimes \cdots \otimes {\textstyle\bigwedge}^{\bullet}\mathrm{V}}_{k \text{ times}} \circlearrowleft \overset{\text{use}}{\mathbf{U}(\mathfrak{gl}_k)}$$

Howe's second statement gives a full functor

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Howe's third statement gives a full functor

$$\dot{\mathbf{U}}(\mathfrak{gl}_k) \xrightarrow[\text{full}]{\Phi_{\mathbf{A}}^{\mathrm{ext}}} \mathcal{R}\mathrm{ep}(\mathfrak{gl}_n)$$

$$\dot{\mathbf{U}}(\mathfrak{gl}_k)/\text{“}\ker(\Phi_{\mathbf{A}}^{\mathrm{ext}})\text{”} \xrightarrow[\text{fully faithful}]{\Phi_{\mathbf{A}}^{\mathrm{ext}}} \mathcal{R}\mathrm{ep}(\mathfrak{gl}_n)$$

whose “kernel $\ker(\Phi_{\mathbf{A}}^{\mathrm{ext}})$” we can calculate.

Again: The diagrammatic presentation machine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{gl}_n)} \circlearrowright \underbrace{{\textstyle\bigwedge}^{\bullet}\mathrm{V} \otimes \cdots \otimes {\textstyle\bigwedge}^{\bullet}\mathrm{V}}_{k \text{ times}} \circlearrowleft \overset{\text{use}}{\mathbf{U}(\mathfrak{gl}_k)}$$

Howe's $1^1/_2$th statement ▸ defines a ▸ diagrammatic category $\mathcal{W}\mathbf{eb}^{\curlywedge}$ such that

$$\begin{tikzcd}
\dot{\mathbf{U}}(\mathfrak{gl}_k) \arrow[r, "\Phi^{\mathrm{ext}}_{\mathbf{A}}", "\text{full}"'] \arrow[dr] & \mathcal{R}\mathbf{ep}(\mathfrak{gl}_n) \\
\mathcal{S} \arrow[r, "\beta_{\mathbf{A}}", "\text{fully faithful}"'] & \mathcal{W}\mathbf{eb}^{\curlywedge} \arrow[u, "\text{full}", "\Gamma^{\mathrm{ext}}_{\mathbf{A}}"']
\end{tikzcd}$$

commutes. In particular, $\mathcal{W}\mathbf{eb}^{\curlywedge}$ is a ▸ thickened version of the symmetric group.

The presentation functor

Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{gl}_n)$-intertwiners

$$\curlywedge_{a,b}^{a+b}\colon {\textstyle\bigwedge}^a\mathrm{V}\otimes{\textstyle\bigwedge}^b\mathrm{V}\twoheadrightarrow{\textstyle\bigwedge}^{a+b}\mathrm{V},\qquad \curlyvee_{a+b}^{a,b}\colon{\textstyle\bigwedge}^{a+b}\mathrm{V}\hookrightarrow{\textstyle\bigwedge}^a\mathrm{V}\otimes{\textstyle\bigwedge}^b\mathrm{V}$$

given by projection and inclusion.

The presentation functor is

$$\Gamma_{\mathbf{A}}^{\mathrm{ext}}\colon \mathcal{W}\mathrm{eb}^{\curlywedge}\to\mathcal{R}\mathrm{ep}(\mathfrak{gl}_n),\qquad a\mapsto{\textstyle\bigwedge}^a\mathrm{V},$$

$\mapsto \curlywedge_{a,b}^{a+b}$,

$\mapsto \curlyvee_{a+b}^{a,b}$

The presentation functor

Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{gl}_n)$-intertwiners

$$\curlywedge^{a+b}_{a,b}\colon \textstyle\bigwedge^{a}\mathrm{V}\otimes\bigwedge^{b}\mathrm{V}\twoheadrightarrow\bigwedge^{a+b}\mathrm{V},\qquad \curlyvee^{a,b}_{a+b}\colon \bigwedge^{a+b}\mathrm{V}\hookrightarrow\bigwedge^{a}\mathrm{V}\otimes\bigwedge^{b}\mathrm{V}$$

given by projection a

The (co)associativity relations say that $\bigwedge^{\bullet}\mathrm{V}$ is a (co)algebra with (co)multiplication $\curlywedge^{a+b}_{a,b}$ ($\curlyvee^{a,b}_{a+b}$).

The presentation functor is

$$\Gamma^{\mathrm{ext}}_{\mathbf{A}}\colon \mathcal{W}\mathbf{eb}^{\curlywedge}\to\mathcal{R}\mathbf{ep}(\mathfrak{gl}_n),\qquad a\mapsto\textstyle\bigwedge^{a}\mathrm{V},$$

$\mapsto \curlywedge^{a+b}_{a,b},$

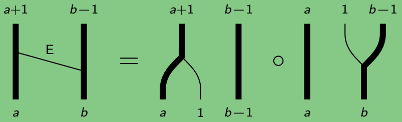
$\mapsto \curlyvee^{a,b}_{a+b}$

The presentation functor

Observe that there are (up to scalars) unique $\mathbf{U}(\mathfrak{gl}_n)$-intertwiners

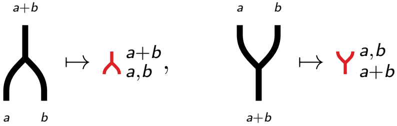
$$\curlywedge^{a+b}: \textstyle\bigwedge^{a}\mathrm{V} \otimes \bigwedge^{b}\mathrm{V} \twoheadrightarrow \bigwedge^{a+b}\mathrm{V}, \qquad \curlyvee^{a,b}: \bigwedge^{a+b}\mathrm{V} \hookrightarrow \bigwedge^{a}\mathrm{V} \otimes \bigwedge^{b}\mathrm{V}$$

We can play the game the other way around as well by defining Howe's action via:



etc.

$$\textstyle\bigwedge^{a}\mathrm{V} \otimes \bigwedge^{b}\mathrm{V} \to \bigwedge^{a}\mathrm{V} \otimes \mathrm{V} \otimes \bigwedge^{b-1}\mathrm{V} \to \bigwedge^{a+1}\mathrm{V} \otimes \bigwedge^{b-1}\mathrm{V}$$



$\mapsto \curlywedge^{a+b}_{a,b}, \qquad \mapsto \curlyvee^{a,b}_{a+b}$

Another pioneer of representation theory

▸ Brauer's remarkable relationship between $\mathfrak{g}_n = \mathfrak{so}_n, \mathfrak{sp}_n$ and the Brauer algebra Br_n^k:

Brauer ∼1937. Let $\mathrm{V} = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{g}_n) \circlearrowright \underbrace{\mathrm{V} \otimes \cdots \otimes \mathrm{V}}_{k \text{ times}} \circlearrowleft \mathrm{Br}_n^k$$

generating each other's centralizer. The $\mathbf{U}(\mathfrak{g}_n)$-$\mathrm{Br}_n^k$-bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} \mathrm{L}(\mathfrak{g}_n, \lambda) \otimes \mathrm{D}(\mathrm{Br}_n^k, \lambda^{\mathrm{T}}).$$

The λ's are partitions of $k, k-2, k-4, \ldots$ whose precise form depend on $\mathfrak{g}_n$.

Another pioneer of representation theory

Be careful: One needs to work with $\mathfrak{o}_n$ in type **D**.
Today, I silently stay with $\mathfrak{so}_n$, and thus, in type **B**.

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The λ's are partitions of $k, k-2, k-4, \ldots$ whose precise form depend on $\mathfrak{g}_n$.

The diagrammatic presentation machine – it still works fine

$$\overset{\text{fix}}{\mathbf{U}(\mathfrak{g}_n)} \circlearrowright \underbrace{\mathrm{V} \otimes \cdots \otimes \mathrm{V}}_{k \text{ times}} \circlearrowleft \overset{\text{use}}{\mathrm{Br}_n^k}$$

As usual, Brauer's insights give a full functor

Categorical version of the Brauer algebra

$$\boldsymbol{\mathcal{B}}\mathbf{r}_n \xrightarrow[\text{full}]{\Phi} \boldsymbol{\mathcal{R}}\mathbf{ep}(\mathfrak{g}_n)$$

$$\boldsymbol{\mathcal{B}}\mathbf{r}_n/\text{“}\ker(\Phi)\text{”} \xrightarrow[\text{fully faithful}]{\Phi} \boldsymbol{\mathcal{R}}\mathbf{ep}(\mathfrak{g}_n)$$

whose “kernel ker(Φ)” can be calculated.

Hence, up to Spin's and additive/Karoubi closures, Brauer gave us a diagrammatic presentation of the representation category $\boldsymbol{\mathcal{R}}\mathbf{ep}(\mathfrak{g}_n)$ of $\mathfrak{g}_n$.

"Thickened" Schur–Weyl–Brauer duality

Another one of Howe's remarkable relationships:

Howe ∼1975. Let $\mathrm{V} = \mathbb{C}^n$. There are commuting actions

Today I stay with the $\mathfrak{so}$-$\mathfrak{so}$ story, but the other three work analogously.

$$\mathbf{U}(\mathfrak{so}_n) \circlearrowright \underbrace{{\bigwedge}^{\bullet}\mathrm{V} \otimes \cdots \otimes {\bigwedge}^{\bullet}\mathrm{V}}_{k \text{ times}} \circlearrowleft \mathbf{U}(\mathfrak{so}_{2k})$$

generating each other's centralizer, and ${\bigwedge}^{a_1}\mathrm{V} \otimes \cdots \otimes {\bigwedge}^{a_k}\mathrm{V}$ is the $(\overline{a}_1, \dots, \overline{a}_k)$th weight space of $\mathbf{U}(\mathfrak{so}_{2k})$. The $\mathbf{U}(\mathfrak{so}_n)$-$\mathbf{U}(\mathfrak{so}_{2k})$-bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} \mathrm{L}(\mathfrak{so}_n, \lambda) \otimes \mathrm{L}(\mathfrak{so}_{2k}, \textstyle\sum_{j=1}^{k}(\lambda_j^{\mathrm{T}} - {}^{n}\!/_{2})\varepsilon_j).$$

The λ's again satisfy certain explicit conditions and $\overline{a}_i = a_i + {}^{n}\!/_{2}$.

“Thickened” Schur–Weyl–Brauer duality

Another one of Howe’s remarkable relations

Note that the action of $\mathbf{U}(\mathfrak{so}_{2k})$ is not as clear as it was for $\mathbf{U}(\mathfrak{gl}_k)$!

Howe ∼1975. Let $\mathrm{V} = \mathbb{C}^n$. There are commuting actions

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The λ’s again satisfy certain explicit conditions and $\bar{a}_i = a_i + {}^{n}\!/\!{}_{2}$.

The restriction game

$$\begin{array}{ccccc} \text{fix} & & & & \text{use} \\ \mathbf{U}(\mathfrak{gl}_n) & \circlearrowright & \bigwedge^{\bullet}\mathrm{V}\otimes\cdots\otimes\bigwedge^{\bullet}\mathrm{V} & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_k) \\ \cup & & \| & & \cap \\ \mathbf{U}(\mathfrak{so}_n) & \circlearrowright & \underbrace{\bigwedge^{\bullet}\mathrm{V}\otimes\cdots\otimes\bigwedge^{\bullet}\mathrm{V}}_{k\text{ times}} & \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k}) \end{array}$$

Howe's $1^1/_2$th statement ▸ defines a ▸ diagrammatic category $\mathcal{W}\mathbf{eb}^{\cup}$ such that

$$\begin{tikzcd}
\dot{\mathbf{U}}(\mathfrak{so}_{2k}) \arrow[r, "\Phi^{\mathrm{ext}}_{\mathbf{BD}}", "\text{full}"'] \arrow[dr] & \mathcal{R}\mathbf{ep}(\mathfrak{so}_n) \\
\mathcal{B}\mathbf{r}_n \arrow[r, "\beta_{\cup}", "\text{fully faithful}"'] & \mathcal{W}\mathbf{eb}^{\cup} \arrow[u, "\text{full}", "\Gamma^{\mathrm{ext}}_{\mathbf{BD}}"']
\end{tikzcd}$$

commutes. In particular, $\mathcal{W}\mathbf{eb}^{\cup}$ is a ▸ thickened version of the Brauer algebra.

The restriction game

Restricting the action on one side

$$\begin{array}{ccccc} \text{fix} & & & & \text{use} \\ \mathbf{U}(\mathfrak{gl}_n) & \circlearrowright & {\textstyle\bigwedge}^{\bullet}\mathrm{V}\otimes\cdots\otimes{\textstyle\bigwedge}^{\bullet}\mathrm{V} & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_k) \\ \cup & & \parallel & & \cap \\ \mathbf{U}(\mathfrak{so}_n) & \circlearrowright & \underbrace{{\textstyle\bigwedge}^{\bullet}\mathrm{V}\otimes\cdots\otimes{\textstyle\bigwedge}^{\bullet}\mathrm{V}}_{k\text{ times}} & \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k}) \end{array}$$

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\cup & & \| & & \cap\\
\mathbf{U}(\mathfrak{so}_n) & \circlearrowright & \underbrace{\textstyle\bigwedge^{\bullet}\mathrm{V}\otimes\cdots\otimes\bigwedge^{\bullet}\mathrm{V}}_{k\text{ times}} & \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k})
\end{array}$$

Increases the centralizer on the other

Howe's $1^1/_2$th statement ▸ defines a ▸ diagrammatic category $\mathcal{W}\mathbf{eb}^{\cup}$ such that

$$\begin{tikzcd}
\dot{\mathbf{U}}(\mathfrak{so}_{2k}) \arrow[r, "\Phi^{\mathrm{ext}}_{\mathbf{BD}}", "\text{full}"'] \arrow[dr] & \mathcal{R}\mathbf{ep}(\mathfrak{so}_n) \\
\mathcal{B}\mathbf{r}_n \arrow[r, "\beta_{\cup}", "\text{fully faithful}"'] & \mathcal{W}\mathbf{eb}^{\cup} \arrow[u, "\text{full}", "\Gamma^{\mathrm{ext}}_{\mathbf{BD}}"']
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commutes. In particular, $\mathcal{W}\mathbf{eb}^{\cup}$ is a ▸ thickened version of the Brauer algebra.

The restriction game

Restricting the action on one side

fix

$\mathbf{U}(\mathfrak{gl}_n)$

$\cup$

$\mathbf{U}(\mathfrak{so}_n)$

Hence, we get
"old diagram generators"
and
"new diagram generators"

k times

use

$\mathbf{U}(\mathfrak{gl}_k)$

$\cap$

$\mathbf{U}(\mathfrak{so}_{2k})$

Increases the centralizer on the other

Howe's $1^1/_2$th statement ▸ defines a ▸ diagrammatic category $\mathcal{W}\mathbf{eb}^{\cup}$ such that

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Some delicate quantizations

$$\begin{array}{ccccc}
\mathbf{U}(\mathfrak{gl}_n) & \circlearrowright & \bigwedge^{\bullet}\mathrm{V} \otimes \cdots \otimes \bigwedge^{\bullet}\mathrm{V} & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_k) \\
\cup & & \| & & \cap \\
\mathbf{U}(\mathfrak{so}_n) & \circlearrowright & \underbrace{\bigwedge^{\bullet}\mathrm{V} \otimes \cdots \otimes \bigwedge^{\bullet}\mathrm{V}}_{k \text{ times}} & \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k})
\end{array}$$

Some delicate quantizations

$$\begin{array}{ccc} \mathbf{U}(\mathfrak{gl}_n) & {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q & \mathbf{U}(\mathfrak{gl}_k) \\ \cup & \| & \cap \\ \mathbf{U}(\mathfrak{so}_n) & \underbrace{{\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q}_{k \text{ times}} & \mathbf{U}(\mathfrak{so}_{2k}) \end{array}$$

Some delicate quantizations

$$\begin{tikzcd}[column sep=small, row sep=small]
\mathbf{U}_q(\mathfrak{gl}_n) \arrow[r, phantom, "\circlearrowright"] & \textstyle\bigwedge_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes \textstyle\bigwedge_q^{\bullet}\mathrm{V}_q \arrow[d, phantom, "||"] & \mathbf{U}_q(\mathfrak{gl}_k) \arrow[l, phantom, "\circlearrowleft"] \\
\mathbf{U}(\mathfrak{so}_n) & \underbrace{\textstyle\bigwedge_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes \textstyle\bigwedge_q^{\bullet}\mathrm{V}_q}_{k \text{ times}} & \mathbf{U}(\mathfrak{so}_{2k})
\end{tikzcd}$$

Quantum skew Howe duality:
Lehrer–Zhang–Zhang ∼2009.
(But its quite easy and not their main point.)

Some delicate quantizations

Does not quantize!

$$\begin{array}{ccccc}
\mathbf{U}_q(\mathfrak{gl}_n) & \circlearrowright & {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q & \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) \\
\not\cup & & \parallel & & \cap \\
\mathbf{U}_q(\mathfrak{so}_n) & & \underbrace{{\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q}_{k \text{ times}} & & \mathbf{U}_q(\mathfrak{so}_{2k})
\end{array}$$

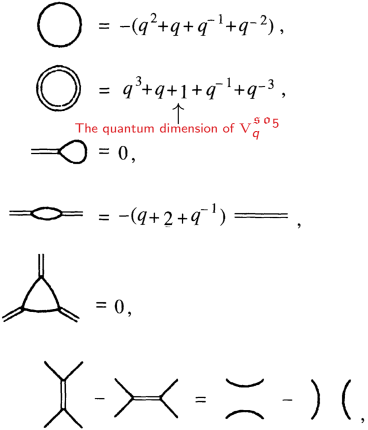
Quantizes easily

Some delicate quantizations

$$\begin{tikzcd}[column sep=small]
 & \mathbf{U}_q(\mathfrak{gl}_n) \arrow[d, phantom, "\not\cup"] & \circlearrowright & \bigwedge_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^{\bullet}\mathrm{V}_q \arrow[d, phantom, "||"] & \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) \arrow[d, phantom, "\cap"] & \\
\text{No action at all.} & \mathbf{U}_q(\mathfrak{so}_n) & \not\circlearrowright & \underbrace{\bigwedge_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^{\bullet}\mathrm{V}_q}_{k\text{ times}} & ??? & \mathbf{U}_q(\mathfrak{so}_{2k}) & \text{Action unclear.}
\end{tikzcd}$$

Some delicate

The quantum dimension of $\mathrm{V}_q^{\mathfrak{gl}}$ is $[n]$.
The quantum dimension of $\mathrm{V}_q^{\mathfrak{so}}$ is $[n-1]+1$.
Hence, $\mathrm{V}_q^{\mathfrak{so}}$ does not come from $\mathrm{V}_q^{\mathfrak{gl}}$!
This "flaw" propagates all the way through:
$\bigwedge_q^a \mathrm{V}_q^{\mathfrak{so}}$ have "weird" quantum dimensions.



The quantum dimension of $\mathrm{V}_q^{\mathfrak{so}_5}$

Above: Kuperberg's $\mathbf{B}_2$ web relations ∼1995.

Some delicate

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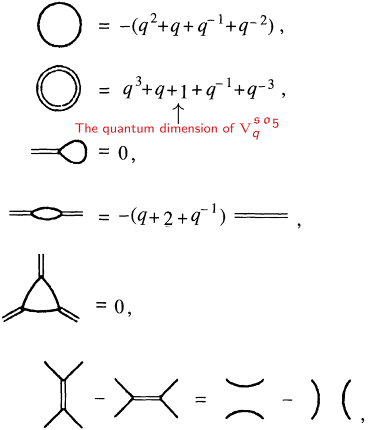

The quantum dimension of $\mathrm{V}_q^{\mathfrak{so}_5}$

Above: Kuperberg's $\mathbf{B}_2$ web relations $\sim$1995.

We wanted to generalize Kuperberg's results and his construction of link invariants.
We failed.

But let me explain what we can do.

Some delicate quantizations

Using a coideal subalgebra does the trick.

$$\begin{array}{ccccc} \mathbf{U}_q(\mathfrak{gl}_n) & \circlearrowright & {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q & \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) \\ \cup & & \| & & \cap \\ \mathbf{U}'_q(\mathfrak{so}_n) & \circlearrowright & \underbrace{{\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes \cdots \otimes {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q}_{k \text{ times}} & \circlearrowleft & \mathbf{U}_q(\mathfrak{so}_{2k}) \end{array}$$

The action is constructed using the unquantized diagrammatics.

Some delicate quantizations

$$\begin{array}{ccccc} \mathbf{U}_q(\mathfrak{gl}_n) & \circlearrowright & {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes\cdots\otimes {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q & \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) \\ \cup & & \| & & \cap \\ \mathbf{U}'_q(\mathfrak{so}_n) & \circlearrowright & \underbrace{{\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q \otimes\cdots\otimes {\textstyle\bigwedge}_q^{\bullet}\mathrm{V}_q}_{k\text{ times}} & \circlearrowleft & \mathbf{U}_q(\mathfrak{so}_{2k}) \end{array}$$

Using a q-monoidal ▸ diagrammatic category $\mathcal{W}\mathbf{eb}^{\cup}_{q,q^n}$ we can ▸ define a full Howe functor $\Phi^{\mathrm{ext}}_{\mathbf{BD}}$ such that we get a commuting diagram

$$\begin{tikzcd} \dot{\mathbf{U}}_q(\mathfrak{so}_{2k}) \arrow[r, "\Phi^{\mathrm{ext}}_{\mathbf{BD}}"{name=P}] \arrow[dr] & \mathcal{R}\mathbf{ep}'_q(\mathfrak{so}_n) \\ \mathcal{B}\mathbf{r}_{q,q^n} \arrow[r, "\beta_{\cup}", "\text{fully faithful}"'] & \mathcal{W}\mathbf{eb}^{\cup}_{q,q^n} \arrow[u, "\Gamma^{\mathrm{ext}}_{\mathbf{BD}}"'{name=G}] \arrow[from=G, to=P, dashed, bend left, "\text{define}"] \end{tikzcd}$$

Hereby, $\mathcal{R}\mathbf{ep}'_q(\mathfrak{so}_n)$ is the q-monoidal representation category of ▸ $\mathbf{U}'_q(\mathfrak{so}_n)$, and $\mathcal{B}\mathbf{r}_{q,q^n}$ is the q-Brauer category (Molev ∼2002). (▸ Time flies)

Maybe its a feature rather than a flaw

- Coideals appear naturally outside of type **A** and they give rise to some nice patterns, e.g. Howe's picture quantizes in two different ways:

$* = {}^{n-1}\!/_2$ for type **B**,
$* = {}^{n}\!/_2$ for type **D**.

$$\begin{array}{ccccc}
\mathbf{U}(\mathfrak{gl}_n) & \circlearrowright & \bigwedge^{\bullet}\mathbb{C}^n\otimes\cdots\otimes\bigwedge^{\bullet}\mathbb{C}^n & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_k)\\
\cup & & \| & & \cap\\
\mathbf{U}(\mathfrak{so}_n) & \circlearrowright & \bigwedge^{\bullet}\mathbb{C}^n\otimes\cdots\otimes\bigwedge^{\bullet}\mathbb{C}^n & \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k})\\
\cup & & \| & & \cap\\
\mathbf{U}(\mathfrak{gl}_*) & \circlearrowright & \bigwedge^{\bullet}\mathbb{C}^n\otimes\cdots\otimes\bigwedge^{\bullet}\mathbb{C}^n & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_{2k})
\end{array}$$

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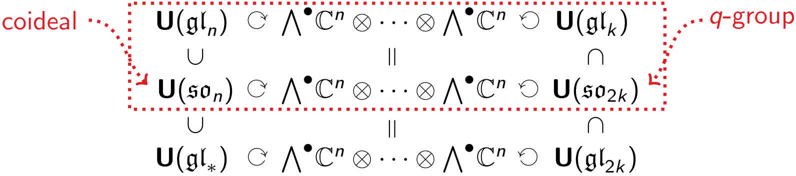

coideal

q-group

$$\begin{array}{ccccc}
\mathbf{U}(\mathfrak{gl}_n) & \circlearrowright & {\bigwedge}^{\bullet}\mathbb{C}^n \otimes \cdots \otimes {\bigwedge}^{\bullet}\mathbb{C}^n & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_k) \\
\cup & & \| & & \cap \\
\mathbf{U}(\mathfrak{so}_n) & \circlearrowright & {\bigwedge}^{\bullet}\mathbb{C}^n \otimes \cdots \otimes {\bigwedge}^{\bullet}\mathbb{C}^n & \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k}) \\
\cup & & \| & & \cap \\
\mathbf{U}(\mathfrak{gl}_*) & \circlearrowright & {\bigwedge}^{\bullet}\mathbb{C}^n \otimes \cdots \otimes {\bigwedge}^{\bullet}\mathbb{C}^n & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_{2k})
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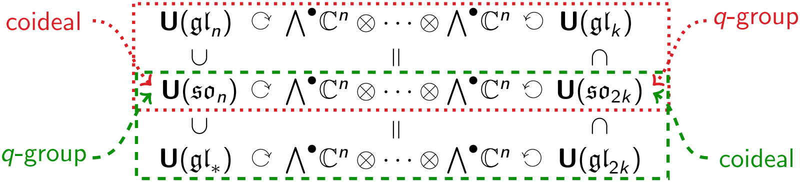

$$\begin{array}{ccccc}
\mathbf{U}(\mathfrak{gl}_n) & \circlearrowright & \bigwedge^{\bullet}\mathbb{C}^n \otimes \cdots \otimes \bigwedge^{\bullet}\mathbb{C}^n & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_k) \\
\cup & & \| & & \cap \\
\mathbf{U}(\mathfrak{so}_n) & \circlearrowright & \bigwedge^{\bullet}\mathbb{C}^n \otimes \cdots \otimes \bigwedge^{\bullet}\mathbb{C}^n & \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k}) \\
\cup & & \| & & \cap \\
\mathbf{U}(\mathfrak{gl}_*) & \circlearrowright & \bigwedge^{\bullet}\mathbb{C}^n \otimes \cdots \otimes \bigwedge^{\bullet}\mathbb{C}^n & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_{2k})
\end{array}$$

This should give the quantum group story,
but it is much trickier since e.g.

$$\mathrm{V}_q^{\mathfrak{so}} \cong \mathrm{V}_q^{\mathfrak{gl}} \oplus (\mathrm{V}_q^{\mathfrak{gl}})^* \oplus \mathbb{C}$$

as $\mathbf{U}_q(\mathfrak{gl}_*)$-modules in type **B**.
Thus, the above is not the usual $\mathbf{U}(\mathfrak{gl}_*)$-$\mathbf{U}(\mathfrak{gl}_{2k})$ duality,
but rather similar to work of Queffelec–Sartori (∼2015).

Maybe its a feature rather than a flaw

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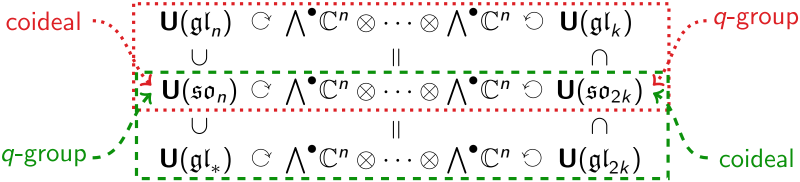


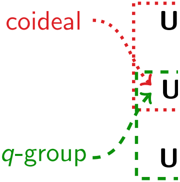
$$\begin{tikzcd}
\mathbf{U}(\mathfrak{gl}_n) \arrow[r, phantom, "\circlearrowright"] & \bigwedge^{\bullet}\mathbb{C}^n \otimes \cdots \otimes \bigwedge^{\bullet}\mathbb{C}^n & \mathbf{U}(\mathfrak{gl}_k) \arrow[l, phantom, "\circlearrowleft"] \\
\mathbf{U}(\mathfrak{so}_n) \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\circlearrowright"] & \bigwedge^{\bullet}\mathbb{C}^n \otimes \cdots \otimes \bigwedge^{\bullet}\mathbb{C}^n \arrow[u, phantom, "||"] & \mathbf{U}(\mathfrak{so}_{2k}) \arrow[u, phantom, "\cap"] \arrow[l, phantom, "\circlearrowleft"] \\
\mathbf{U}(\mathfrak{gl}_*) \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\circlearrowright"] & \bigwedge^{\bullet}\mathbb{C}^n \otimes \cdots \otimes \bigwedge^{\bullet}\mathbb{C}^n \arrow[u, phantom, "||"] & \mathbf{U}(\mathfrak{gl}_{2k}) \arrow[u, phantom, "\cap"] \arrow[l, phantom, "\circlearrowleft"]
\end{tikzcd}$$

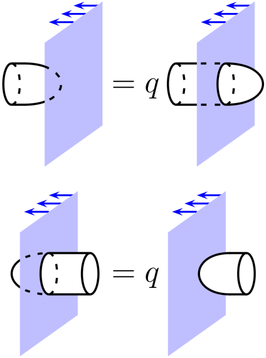
- Coideal subalgebras also arise in other work outside of type **A**, cf. Ehrig–Stroppel (∼2013) or Bao–Shan–Wang–Webster (∼2016).

We have $\mathbf{U}_q'(\mathfrak{so}_n) \subset \mathbf{U}_q'(\mathfrak{gl}_n)$, $\mathbf{U}_q'(\mathfrak{sp}_n) \subset \mathbf{U}_q'(\mathfrak{gl}_n)$,
they have $\mathbf{U}_q'(\mathfrak{gl}_n \times \mathfrak{gl}_n) \subset \mathbf{U}_q'(\mathfrak{gl}_{2n})$.

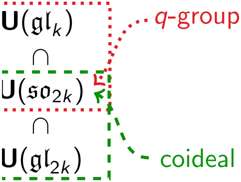
Maybe its a feature rather than a flaw

- Coideals appear na give rise to some nice patterns, e.g. How nt ways:





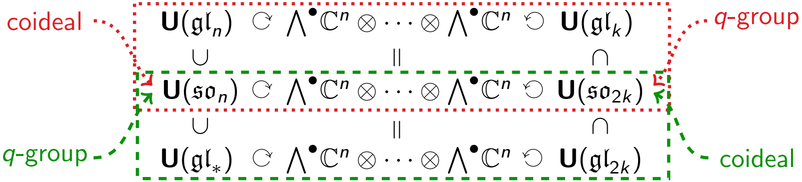
2-q-monoidal foams.
(Beliakova–Putyra–Wehrli's pictures, which I shamelessly stole, mean something different but "feel correct" to me.)



- Coideal subalgebra f type **A**, cf. Ehrig–Stroppel (∼ r (∼2016).
- Coideal subalgebras are amenable to categorification. Similarly, their representation categories should be amenable to categorification.

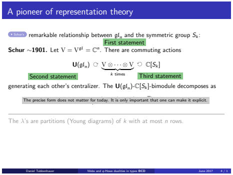
Maybe its a feature rather than a flaw

- Coideals appear naturally outside of type **A** and they give rise to some nice patterns, e.g. Howe's picture quantizes in two different ways:



$$\begin{array}{ccccc}
\mathbf{U}(\mathfrak{gl}_n) & \circlearrowright & {\bigwedge}^{\bullet}\mathbb{C}^n \otimes \cdots \otimes {\bigwedge}^{\bullet}\mathbb{C}^n & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_k) \\
\cup & & \| & & \cap \\
\mathbf{U}(\mathfrak{so}_n) & \circlearrowright & {\bigwedge}^{\bullet}\mathbb{C}^n \otimes \cdots \otimes {\bigwedge}^{\bullet}\mathbb{C}^n & \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k}) \\
\cup & & \| & & \cap \\
\mathbf{U}(\mathfrak{gl}_*) & \circlearrowright & {\bigwedge}^{\bullet}\mathbb{C}^n \otimes \cdots \otimes {\bigwedge}^{\bullet}\mathbb{C}^n & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_{2k})
\end{array}$$

- Coideal subalgebras also arise in other work outside of type **A**, cf. Ehrig–Stroppel (∼2013) or Bao–Shan–Wang–Webster (∼2016).
- Coideal subalgebras are amenable to categorification. Similarly, their representation categories should be amenable to categorification.
- (2)-q-monoidal categories are potentially useful to study representation categories of coideal subalgebras, and appear in other contexts e.g. Putyra (∼2013) or Brundan–Ellis (∼2017).



A pioneer of representation theory

Schur's remarkable relationship between $\mathfrak{gl}_n$ and the symmetric group S_k:

First statement

Schur ∼1901. Let $\mathrm{V} = \mathrm{V}^{\mathfrak{gl}} = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{gl}_n) \circlearrowright \underbrace{\mathrm{V}\otimes\cdots\otimes\mathrm{V}}_{k \text{ times}} \circlearrowleft \mathbb{C}[S_k]$$

Second statement   Third statement

generating each other's centralizer. The $\mathbf{U}(\mathfrak{gl}_n)$-$\mathbb{C}[S_k]$-bimodule decomposes as

The precise form does not matter for today. It is only important that one can make it explicit.

The λ's are partitions (Young diagrams) of k with at most n rows.

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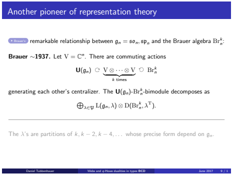

Another pioneer of representation theory

Brauer's remarkable relationship between $\mathfrak{g}_n = \mathfrak{so}_n, \mathfrak{sp}_n$ and the Brauer algebra Br_n^k:

Brauer ∼1937. Let $\mathrm{V} = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{g}_n) \circlearrowright \underbrace{\mathrm{V}\otimes\cdots\otimes\mathrm{V}}_{k \text{ times}} \circlearrowleft \mathrm{Br}_n^k$$

generating each other's centralizer. The $\mathbf{U}(\mathfrak{g}_n)$-$\mathrm{Br}_n^k$-bimodule decomposes as

$$\textstyle\bigoplus_{\lambda\in\mathfrak{P}} \mathrm{L}(\mathfrak{g}_n,\lambda)\otimes\mathrm{D}(\mathrm{Br}_n^k,\lambda^{\mathrm{T}}).$$

The λ's are partitions of $k, k-2, k-4, \ldots$ whose precise form depend on $\mathfrak{g}_n$.

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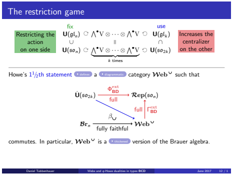

The restriction game

Restricting the action on one side

$$\begin{array}{ccccc} \text{fix} & & & & \text{use} \\ \mathbf{U}(\mathfrak{gl}_n) & \circlearrowright & {\textstyle\bigwedge^{\bullet}}\mathrm{V}\otimes\cdots\otimes{\textstyle\bigwedge^{\bullet}}\mathrm{V} & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_k) \\ \cup & & \| & & \cap \\ \mathbf{U}(\mathfrak{so}_n) & \circlearrowright & \underbrace{{\textstyle\bigwedge^{\bullet}}\mathrm{V}\otimes\cdots\otimes{\textstyle\bigwedge^{\bullet}}\mathrm{V}}_{k \text{ times}} & \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k}) \end{array}$$

Increases the centralizer on the other

Howe's $1^1/_2$th statement defines a diagrammatic category $\mathcal{W}\mathbf{eb}^{\cup}$ such that

$$\begin{tikzcd} \dot{\mathbf{U}}(\mathfrak{so}_{2k}) \arrow[r, "\Phi^{\mathrm{ext}}_{\mathbf{BD}}", "\text{full}"'] \arrow[dr] & \mathcal{R}\mathbf{ep}(\mathfrak{so}_n) \\ \mathcal{B}\mathbf{r}_n \arrow[r, "\beta_{\cup}", "\text{fully faithful}"'] & \mathcal{W}\mathbf{eb}^{\cup} \arrow[u, "\text{full}", "\Gamma^{\mathrm{ext}}_{\mathbf{BD}}"'] \end{tikzcd}$$

commutes. In particular, $\mathcal{W}\mathbf{eb}^{\cup}$ is a thickened version of the Brauer algebra.

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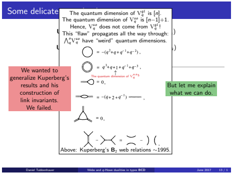

Some delicate

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Above: Kuperberg's $\mathbf{B}_2$ web relations ∼1995.

We wanted to generalize Kuperberg's results and his construction of link invariants. We failed.

But let me explain what we can do.

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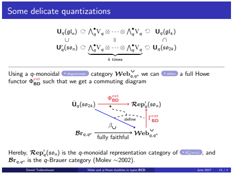

Some delicate quantizations

$$\begin{array}{ccccc} \mathbf{U}_q(\mathfrak{gl}_n) & \circlearrowright & {\textstyle\bigwedge_q^{\bullet}}\mathrm{V}_q\otimes\cdots\otimes{\textstyle\bigwedge_q^{\bullet}}\mathrm{V}_q & \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) \\ \cup & & \| & & \cap \\ \mathbf{U}'_q(\mathfrak{so}_n) & \circlearrowright & \underbrace{{\textstyle\bigwedge_q^{\bullet}}\mathrm{V}_q\otimes\cdots\otimes{\textstyle\bigwedge_q^{\bullet}}\mathrm{V}_q}_{k \text{ times}} & \circlearrowleft & \mathbf{U}_q(\mathfrak{so}_{2k}) \end{array}$$

Using a q-monoidal diagrammatic category $\mathcal{W}\mathbf{eb}^{\cup}_{q,q^n}$ we can define a full Howe functor $\Phi^{\mathrm{ext}}_{\mathbf{BD}}$ such that we get a commuting diagram

$$\begin{tikzcd} \dot{\mathbf{U}}_q(\mathfrak{so}_{2k}) \arrow[r, "\Phi^{\mathrm{ext}}_{\mathbf{BD}}"] \arrow[dr] & \mathcal{R}\mathbf{ep}'_q(\mathfrak{so}_n) \\ \mathcal{B}\mathbf{r}_{q,q^n} \arrow[r, "\beta_{\cup}", "\text{fully faithful}"'] & \mathcal{W}\mathbf{eb}^{\cup}_{q,q^n} \arrow[u, "\Gamma^{\mathrm{ext}}_{\mathbf{BD}}"'] \arrow[ul, dashed, "\text{define}"'] \end{tikzcd}$$

Hereby, $\mathcal{R}\mathbf{ep}'_q(\mathfrak{so}_n)$ is the q-monoidal representation category of $\mathbf{U}'_q(\mathfrak{so}_n)$, and $\mathcal{B}\mathbf{r}_{q,q^n}$ is the q-Brauer category (Molev ∼2002).

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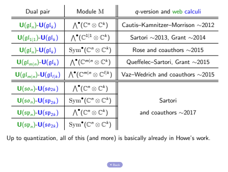
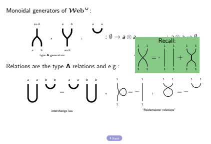


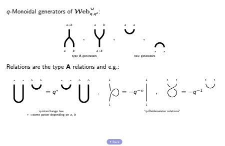
Dual pair	Module M	q-version and web calculi
$\mathbf{U}(\mathfrak{gl}_n)$-$\mathbf{U}(\mathfrak{gl}_k)$	${\textstyle\bigwedge^{\bullet}}(\mathbb{C}^n\otimes\mathbb{C}^k)$	Cautis–Kamnitzer–Morrison ∼2012
$\mathbf{U}(\mathfrak{gl}_{1\mid1})$-$\mathbf{U}(\mathfrak{gl}_k)$	${\textstyle\bigwedge^{\bullet}}(\mathbb{C}^{1\mid1}\otimes\mathbb{C}^k)$	Sartori ∼2013, Grant ∼2014
$\mathbf{U}(\mathfrak{gl}_n)$-$\mathbf{U}(\mathfrak{gl}_k)$	$\mathrm{Sym}^{\bullet}(\mathbb{C}^n\otimes\mathbb{C}^k)$	Rose and coauthors ∼2015
$\mathbf{U}(\mathfrak{gl}_{m\mid n})$-$\mathbf{U}(\mathfrak{gl}_k)$	${\textstyle\bigwedge^{\bullet}}(\mathbb{C}^{m\mid n}\otimes\mathbb{C}^k)$	Queffelec–Sartori, Grant ∼2015
$\mathbf{U}(\mathfrak{gl}_{m\mid n})$-$\mathbf{U}(\mathfrak{gl}_{l\mid k})$	${\textstyle\bigwedge^{\bullet}}(\mathbb{C}^{m\mid n}\otimes\mathbb{C}^{l\mid k})$	Vaz–Wedrich and coauthors ∼2015
$\mathbf{U}(\mathfrak{so}_n)$-$\mathbf{U}(\mathfrak{so}_{2k})$	${\textstyle\bigwedge^{\bullet}}(\mathbb{C}^n\otimes\mathbb{C}^k)$	Sartori and coauthors ∼2017
$\mathbf{U}(\mathfrak{so}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$	$\mathrm{Sym}^{\bullet}(\mathbb{C}^n\otimes\mathbb{C}^k)$	
$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$	${\textstyle\bigwedge^{\bullet}}(\mathbb{C}^n\otimes\mathbb{C}^k)$	
$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{so}_{2k})$	$\mathrm{Sym}^{\bullet}(\mathbb{C}^n\otimes\mathbb{C}^k)$	

Up to quantization, all of this (and more) is basically already in Howe's work.



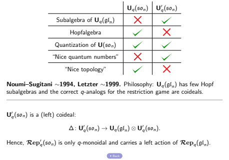
Monoidal generators of $\mathcal{W}\mathbf{eb}^{\cup}$:

Relations are the type **A** relations and e.g.:



q-Monoidal generators of $\mathcal{W}\mathbf{eb}^{\cup}_{q,q^n}$:

Relations are the type **A** relations and e.g.:



	$\mathbf{U}_q(\mathfrak{so}_n)$	$\mathbf{U}'_q(\mathfrak{so}_n)$
Subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$	✗	✓
Hopfalgebra	✓	✗
Quantization of $\mathbf{U}(\mathfrak{so}_n)$	✓	✓
"Nice quantum numbers"	✗	✓
"Nice topology"	✓	✗

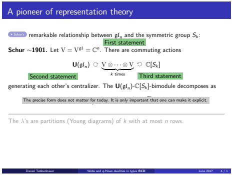
Noumi–Sugitani ∼1994, Letzter ∼1999. Philosophy: $\mathbf{U}_q(\mathfrak{gl}_n)$ has few Hopf subalgebras and the correct q-analogs for the restriction game are coideals.

$\mathbf{U}'_q(\mathfrak{so}_n)$ is a (left) coideal:

$$\Delta\colon \mathbf{U}'_q(\mathfrak{so}_n) \to \mathbf{U}_q(\mathfrak{gl}_n)\otimes\mathbf{U}'_q(\mathfrak{so}_n).$$

Hence, $\mathcal{R}\mathbf{ep}'_q(\mathfrak{so}_n)$ is only q-monoidal and carries a left action of $\mathcal{R}\mathbf{ep}_q(\mathfrak{gl}_n)$.

There is still much to do...



A pioneer of representation theory

Schur's remarkable relationship between $\mathfrak{gl}_n$ and the symmetric group S_k:

First statement

Schur ∼1901. Let $V = V^{\mathfrak{gl}} = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{gl}_n) \circlearrowright \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \circlearrowleft \mathbb{C}[S_k]$$

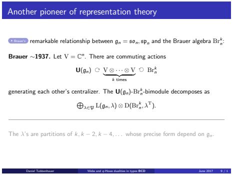
Second statement

Third statement

generating each other's centralizer. The $\mathbf{U}(\mathfrak{gl}_n)$-$\mathbb{C}[S_k]$-bimodule decomposes as

The precise form does not matter for today. It is only important that one can make it explicit.

The λ's are partitions (Young diagrams) of k with at most n rows.



Another pioneer of representation theory

Brauer's remarkable relationship between $\mathfrak{g}_n = \mathfrak{so}_n, \mathfrak{sp}_n$ and the Brauer algebra Br_n^k:

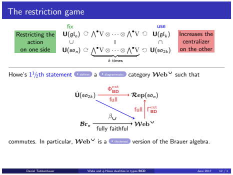
Brauer ∼1937. Let $V = \mathbb{C}^n$. There are commuting actions

$$\mathbf{U}(\mathfrak{g}_n) \circlearrowright \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \circlearrowleft \mathrm{Br}_n^k$$

generating each other's centralizer. The $\mathbf{U}(\mathfrak{g}_n)$-$\mathrm{Br}_n^k$-bimodule decomposes as

$$\bigoplus_{\lambda \in \mathfrak{P}} \mathrm{L}(\mathfrak{g}_n, \lambda) \otimes \mathrm{D}(\mathrm{Br}_n^k, \lambda^{\mathrm{T}}).$$

The λ's are partitions of $k, k-2, k-4, \dots$ whose precise form depend on $\mathfrak{g}_n$.



The restriction game

Restricting the action on one side

fix

use

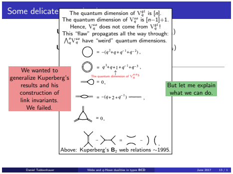
$$\begin{array}{ccccc} \mathbf{U}(\mathfrak{gl}_n) & \circlearrowright & \bigwedge^\bullet V \otimes \cdots \otimes \bigwedge^\bullet V & \circlearrowleft & \mathbf{U}(\mathfrak{gl}_k) \\ \cup & & \| & & \cap \\ \mathbf{U}(\mathfrak{so}_n) & \circlearrowright & \underbrace{\bigwedge^\bullet V \otimes \cdots \otimes \bigwedge^\bullet V}_{k \text{ times}} & \circlearrowleft & \mathbf{U}(\mathfrak{so}_{2k}) \end{array}$$

Increases the centralizer on the other

Howe's $1^1/_2$th statement ▸ defines a ▸ diagrammatic category $\mathcal{W}\mathbf{eb}^{\cup}$ such that

$$\begin{tikzcd} \dot{\mathbf{U}}(\mathfrak{so}_{2k}) \arrow[r, "\Phi_{\mathbf{BD}}^{\mathrm{ext}}", "\text{full}"'] \arrow[dr] & \mathcal{R}\mathbf{ep}(\mathfrak{so}_n) \\ \mathcal{B}\mathbf{r}_n \arrow[r, "\beta_{\cup}", "\text{fully faithful}"'] & \mathcal{W}\mathbf{eb}^{\cup} \arrow[u, "\text{full}", "\Gamma_{\mathbf{BD}}^{\mathrm{ext}}"'] \end{tikzcd}$$

commutes. In particular, $\mathcal{W}\mathbf{eb}^{\cup}$ is a ▸ thickened version of the Brauer algebra.



Some delicate

The quantum dimension of $V_q^{\mathfrak{gl}}$ is $[n]$.
The quantum dimension of $V_q^{\mathfrak{so}}$ is $[n-1]+1$.
Hence, $V_q^{\mathfrak{so}}$ does not come from $V_q^{\mathfrak{gl}}$!
This "flaw" propagates all the way through:
$\bigwedge_q^a V_q^{\mathfrak{so}}$ have "weird" quantum dimensions.

$= -(q^2+q+q^{-1}+q^{-2})$,

$= q^3+q+1+q^{-1}+q^{-3}$,

The quantum dimension of $V_q^{\mathfrak{so}_5}$

$= 0$,

$= -(q+2+q^{-1})$,

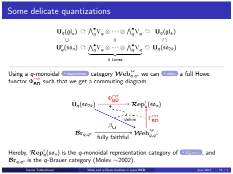
$= 0$,

$- \quad = \quad - \;)\,($,

Above: Kuperberg's $\mathbf{B}_2$ web relations ∼1995.

We wanted to generalize Kuperberg's results and his construction of link invariants. We failed.

But let me explain what we can do.



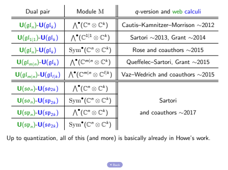
Some delicate quantizations

$$\begin{array}{ccccc} \mathbf{U}_q(\mathfrak{gl}_n) & \circlearrowright & \bigwedge_q^\bullet V_q \otimes \cdots \otimes \bigwedge_q^\bullet V_q & \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) \\ \cup & & \| & & \cap \\ \mathbf{U}_q'(\mathfrak{so}_n) & \circlearrowright & \underbrace{\bigwedge_q^\bullet V_q \otimes \cdots \otimes \bigwedge_q^\bullet V_q}_{k \text{ times}} & \circlearrowleft & \mathbf{U}_q(\mathfrak{so}_{2k}) \end{array}$$

Using a q-monoidal ▸ diagrammatic category $\mathcal{W}\mathbf{eb}_{q,q^n}^{\cup}$ we can ▸ define a full Howe functor $\Phi_{\mathbf{BD}}^{\mathrm{ext}}$ such that we get a commuting diagram

$$\begin{tikzcd} \dot{\mathbf{U}}_q(\mathfrak{so}_{2k}) \arrow[r, "\Phi_{\mathbf{BD}}^{\mathrm{ext}}"] \arrow[dr] & \mathcal{R}\mathbf{ep}_q'(\mathfrak{so}_n) \\ \mathcal{B}\mathbf{r}_{q,q^n} \arrow[r, "\beta_{\cup}", "\text{fully faithful}"'] & \mathcal{W}\mathbf{eb}_{q,q^n}^{\cup} \arrow[u, "\Gamma_{\mathbf{BD}}^{\mathrm{ext}}"'] \arrow[ul, dashed, bend left, "\text{define}"] \end{tikzcd}$$

Hereby, $\mathcal{R}\mathbf{ep}_q'(\mathfrak{so}_n)$ is the q-monoidal representation category of ▸ $\mathbf{U}_q'(\mathfrak{so}_n)$, and $\mathcal{B}\mathbf{r}_{q,q^n}$ is the q-Brauer category (Molev ∼2002).



Dual pair	Module M	q-version and web calculi
$\mathbf{U}(\mathfrak{gl}_n)$-$\mathbf{U}(\mathfrak{gl}_k)$	$\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	Cautis–Kamnitzer–Morrison ∼2012
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$\mathbf{U}(\mathfrak{gl}_{m\|n})$-$\mathbf{U}(\mathfrak{gl}_k)$	$\bigwedge^\bullet(\mathbb{C}^{m\|n} \otimes \mathbb{C}^k)$	Queffelec–Sartori, Grant ∼2015
$\mathbf{U}(\mathfrak{gl}_{m\|n})$-$\mathbf{U}(\mathfrak{gl}_{l\|k})$	$\bigwedge^\bullet(\mathbb{C}^{m\|n} \otimes \mathbb{C}^{l\|k})$	Vaz–Wedrich and coauthors ∼2015
$\mathbf{U}(\mathfrak{so}_n)$-$\mathbf{U}(\mathfrak{so}_{2k})$	$\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	Sartori and coauthors ∼2017
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$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$	$\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	
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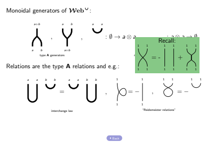
Up to quantization, all of this (and more) is basically already in Howe's work.

◂ Back



Monoidal generators of $\mathcal{W}\mathbf{eb}^{\cup}$:

$: \emptyset \to a \otimes a$

type **A** generators

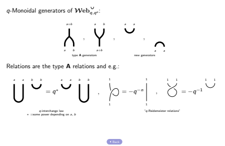
Recall:

Relations are the type **A** relations and e.g.:

interchange law

"Reidemeister relations"

◂ Back



q-Monoidal generators of $\mathcal{W}\mathbf{eb}_{q,q^n}^{\cup}$:

type **A** generators

new generators

Relations are the type **A** relations and e.g.:

$= q^*$

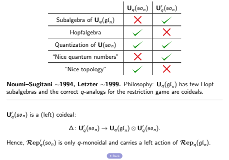
$= -q^{-n}$

$= -q^{-1}$

q-interchange law
$*$:=some power depending on a, b

"q-Reidemeister relations"

◂ Back



◂ Back

	$\mathbf{U}_q(\mathfrak{so}_n)$	$\mathbf{U}_q'(\mathfrak{so}_n)$
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Noumi–Sugitani ∼1994, Letzter ∼1999. Philosophy: $\mathbf{U}_q(\mathfrak{gl}_n)$ has few Hopf subalgebras and the correct q-analogs for the restriction game are coideals.

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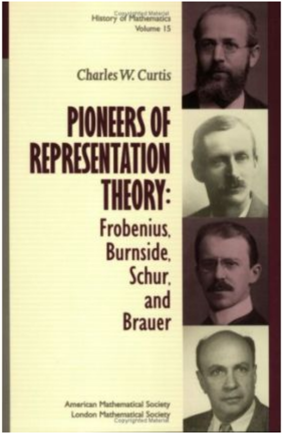
Hence, $\mathcal{R}\mathbf{ep}_q'(\mathfrak{so}_n)$ is only q-monoidal and carries a left action of $\mathcal{R}\mathbf{ep}_q(\mathfrak{gl}_n)$.

◂ Back

Thanks for your attention!



History of Mathematics
Volume 15

Charles W. Curtis

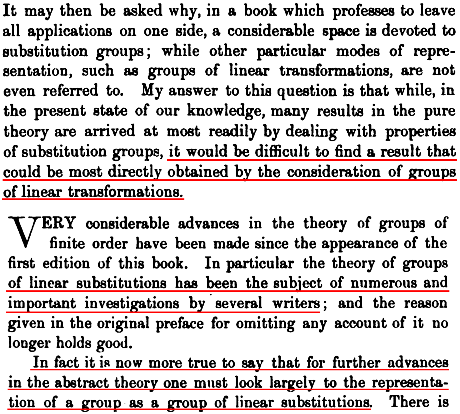
PIONEERS OF REPRESENTATION THEORY:
Frobenius,
Burnside,
Schur,
and
Brauer

American Mathematical Society
London Mathematical Society

Figure: Two of the main players for today: Schur and Brauer.

Curtis, C.W. *Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer*.

◂ Back



It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

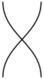
VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

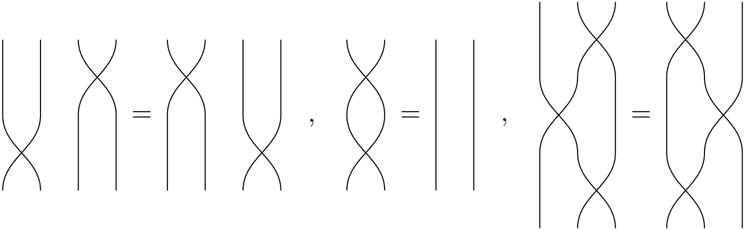
Back

Monoidal generator of $\boldsymbol{\mathcal{S}}$:



$: 2 \to 2.$

Relations e.g.:



interchange law

“Reidemeister relations”

◂ Back

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$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$	$\bigwedge^{\bullet}(\mathbb{C}^n \otimes \mathbb{C}^k)$	
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Up to quantization, all of this (and more) is basically already in Howe's work.

◀ Back

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$\mathbf{U}(\mathfrak{gl}_n)$-$\mathbf{U}(\mathfrak{gl}_k)$		authors ∼2015
$\mathbf{U}(\mathfrak{gl}_{m\vert n})$-$\mathbf{U}(\mathfrak{gl}_k)$	${\bigwedge}^\bullet(\mathbb{C}^{m\vert n} \otimes \mathbb{C}^k)$	Queffelec–Sartori, Grant ∼2015
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$\mathbf{U}(\mathfrak{so}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$	$\mathrm{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	
$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$	${\bigwedge}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	
$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{so}_{2k})$	$\mathrm{Sym}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$	

In type **A** we have
applications to link invariants
(going back to work of many people).

Up to quantization, all of this (and more) is basically already in Howe's work.

◂ Back

Dual pair	Module M	q-version and web calculi
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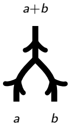
$\mathbf{U}(\mathfrak{so}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$	$\mathrm{Sym}^{\bullet}(\mathbb{C}^n \otimes \mathbb{C}^k)$	
$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{sp}_{2k})$		
$\mathbf{U}(\mathfrak{sp}_n)$-$\mathbf{U}(\mathfrak{so}_{2k})$	$\mathrm{Sym}^{\bullet}(\mathbb{C}^n \otimes \mathbb{C}^k)$	

Types **BCD** are not really understood.
But we firmly hope(d) for a similar story.

Up to quantization, all of this (and more) is basically already in Howe's work.

Back

Monoidal generators of $\mathcal{W}\mathrm{eb}^{\curlywedge}$:

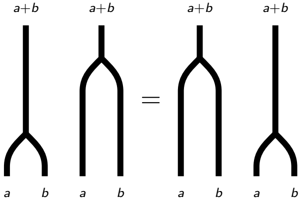


$: a \otimes b \to a + b$ and

$: a + b \to a \otimes b.$

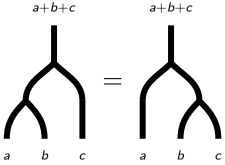
One needs orientations in type **A**, but I am going to ignore them.

Relations e.g.:



interchange law

,



Associativity

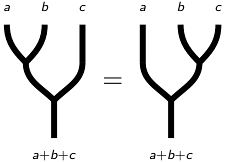
◂ Back

Monoidal generators of $\mathcal{W}\mathrm{eb}^{\curlywedge}$:

: $a \otimes b \to a + b$ and

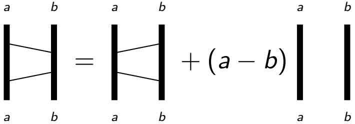
: $a + b \to a \otimes b$.

Relations e.g.:



Coassociativity

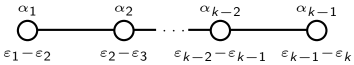
,



square switch

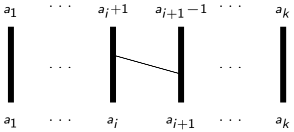
Back

Root conventions is type **A**:



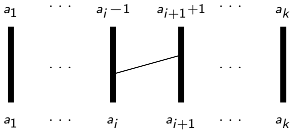
Thus, because of statement $1\frac{1}{2}$, we should set

$\mathrm{E}_i 1_\lambda \longmapsto$



, for all $i = 1, \ldots, k-1$,

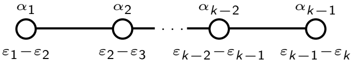
$\mathrm{F}_i 1_\lambda \longmapsto$



, for all $i = 1, \ldots, k-1$.

◂ Back

Root conventions is type **A**:

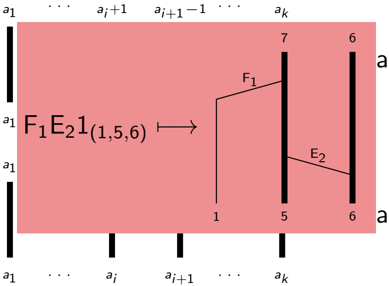


Thus, because of statement $1\frac{1}{2}$, we should set

$E_i1_\lambda \longmapsto$ all $i=1,\dots,k-1,$

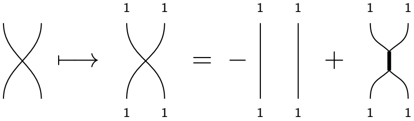
$F_1E_21_{(1,5,6)} \longmapsto$

$F_i1_\lambda \longmapsto$ all $i=1,\dots,k-1.$



◂ Back

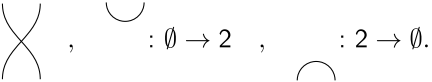
$$\beta_{\mathsf{A}} \colon \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{W}}\mathbf{eb}^{\curlywedge}$$



$$\mathbb{C}[S_k] \xrightarrow{\cong} \mathsf{End}_{\boldsymbol{\mathcal{W}}\mathbf{eb}^{\curlywedge}}(1^{\otimes k})$$

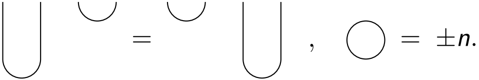
◂ Back

Monoidal generators of $\mathcal{B}\mathbf{r}_n$:



$$, \quad : \emptyset \to 2 \quad , \quad : 2 \to \emptyset.$$

Relations e.g.:



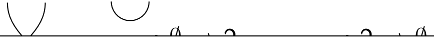
$$= \quad , \quad = \pm n.$$

interchange law

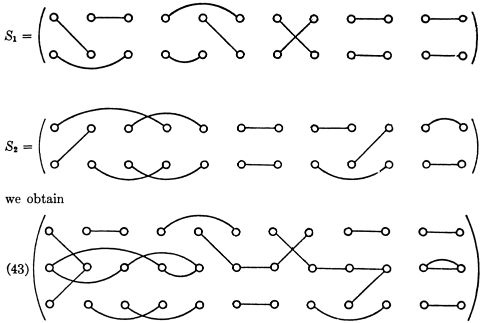
circle removal

Back

Monoidal generators of $\mathcal{B}\mathbf{r}_n$:



Relati



$S_1 =$

$S_2 =$

we obtain

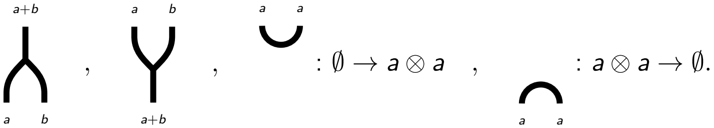
(43)

From “Brauer, R. *On algebras which are connected with the semisimple continuous groups*. Ann. of Math. (2) 38 (1937), no. 4, 857–872.”

Back

Monoidal generators of $\mathcal{W}\mathbf{eb}^{\cup}$:

No orientations needed in types **BCD**.

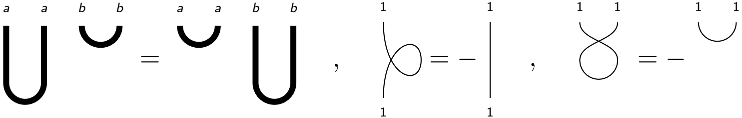


: $\emptyset \rightarrow a \otimes a$, : $a \otimes a \rightarrow \emptyset$.

type **A** generators

new generators

Relations are the type **A** relations and e.g.:

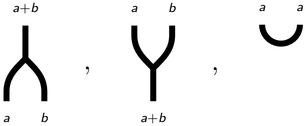


interchange law

“Reidemeister relations”

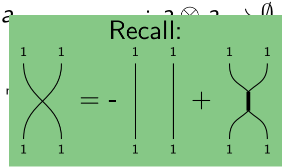
Back

Monoidal generators of $\mathcal{W}\mathbf{eb}^{\cup}$:

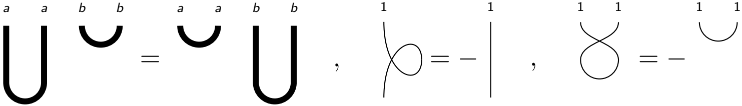


$: \emptyset \to a \otimes a$, $: a \otimes a \to \emptyset$

type **A** generators


Recall:

Relations are the type **A** relations and e.g.:

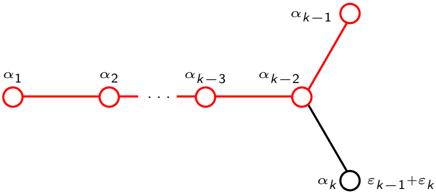


interchange law

“Reidemeister relations”

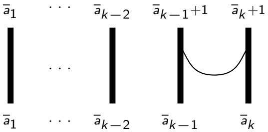
Back

Root conventions is type **D**:

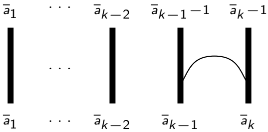


Thus, because of statement $1^1/_2$, we should set

$$\mathsf{E}_k 1_\lambda \longmapsto$$

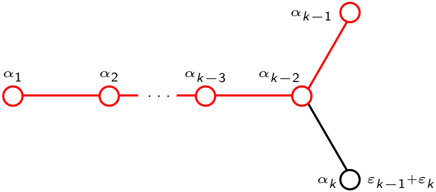


$$\mathsf{F}_k 1_\lambda \longmapsto$$



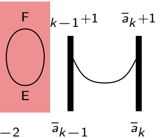
Back

Root conventions is type **D**:

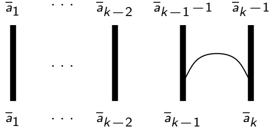


Thus, because of statement $1^1/_2$, we should set

E_k $FE1_{(-n/2,-n/2)} \longmapsto$

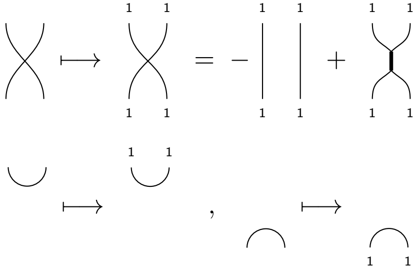


$F_k 1_\lambda \longmapsto$



◂ Back

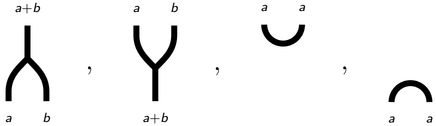
$$\beta_{\cup}\colon \boldsymbol{\mathcal{B}}\mathbf{r}_n \to \boldsymbol{\mathcal{W}}\mathbf{eb}^{\cup}$$



$$\mathrm{Br}_n^k \xrightarrow{\cong} \mathsf{End}_{\boldsymbol{\mathcal{W}}\mathbf{eb}^{\cup}}(1^{\otimes k})$$

◂ Back

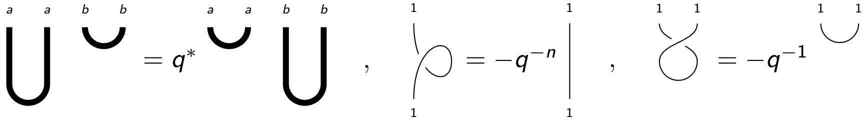
q-Monoidal generators of $\mathcal{W}\mathbf{eb}^{\cup}_{q,q^n}$:



type **A** generators

new generators

Relations are the type **A** relations and e.g.:

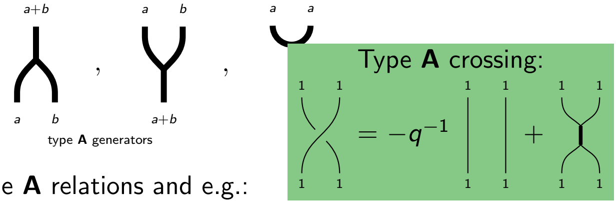


q-interchange law
$* =$some power depending on a, b

“q-Reidemeister relations”

Back

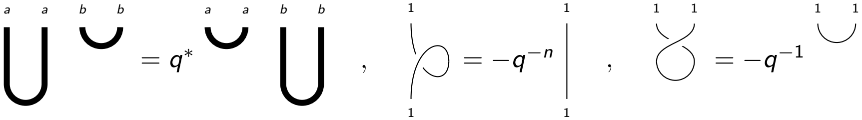
q-Monoidal generators of $\mathcal{W}\mathbf{eb}^{\cup}_{q,q^n}$:



Type **A** crossing:

type **A** generators

Relations are the type **A** relations and e.g.:

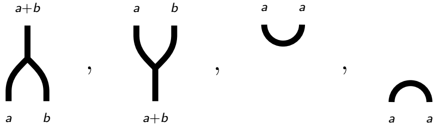


q-interchange law
$*$ =some power depending on a, b

“q-Reidemeister relations”

Back

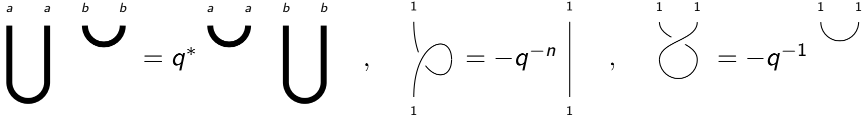
q-Monoidal generators of $\mathcal{W}\mathbf{eb}^{\cup}_{q,q^n}$:



type **A** generators

new generators

These relations say:
"One should be very careful when trying to define link invariants outside of type **A**".



q-interchange law
$*$ =some power depending on a, b

"q-Reidemeister relations"

Back

Via restriction, we see that the $\mathbf{U}_q(\mathfrak{gl}_n)$-intertwiners $\curlywedge^{a+b}_{a,b}$ and $\curlyvee^{a,b}_{a+b}$ are $\mathbf{U}'_q(\mathfrak{so}_n)$-equivariant as well.

Note that $\mathrm{V} \otimes \mathrm{V}$ contains a copy of the trivial $\mathbf{U}(\mathfrak{so}_n)$-module. One shows that the same holds with q's and one gets inclusions and projections

$$\cup : \mathbb{C}_q \to \mathrm{V}_q \otimes \mathrm{V}_q, \qquad \cap : \mathrm{V}_q \otimes \mathrm{V}_q \to \mathbb{C}_q.$$

As before, use these to quantize Howe's duality.

◂ Back

	$\mathbf{U}_q(\mathfrak{so}_n)$	$\mathbf{U}'_q(\mathfrak{so}_n)$
Subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$	✗	✓
Hopfalgebra	✓	✗
Quantization of $\mathbf{U}(\mathfrak{so}_n)$	✓	✓
"Nice quantum numbers"	✗	✓
"Nice topology"	✓	✗

Noumi–Sugitani ∼1994, Letzter ∼1999. Philosophy: $\mathbf{U}_q(\mathfrak{gl}_n)$ has few Hopf subalgebras and the correct q-analogs for the restriction game are coideals.

	$\mathbf{U}_q(\mathfrak{so}_n)$	$\mathbf{U}'_q(\mathfrak{so}_n)$
Subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$	✗	✓
Hopfalgebra	✓	✗
Quantization of $\mathbf{U}(\mathfrak{so}_n)$	✓	✓
"Nice quantum numbers"	✗	✓
"Nice topology"	✓	✗

Noumi–Sugitani ∼1994, Letzter ∼1999. Philosophy: $\mathbf{U}_q(\mathfrak{gl}_n)$ has few Hopf subalgebras and the correct q-analogs for the restriction game are coideals.

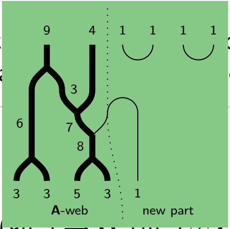
$\mathbf{U}'_q(\mathfrak{so}_n)$ is a (left) coideal:

$$\Delta\colon \mathbf{U}'_q(\mathfrak{so}_n) \to \mathbf{U}_q(\mathfrak{gl}_n) \otimes \mathbf{U}'_q(\mathfrak{so}_n).$$

Hence, $\mathcal{R}\mathbf{ep}'_q(\mathfrak{so}_n)$ is only q-monoidal and carries a left action of $\mathcal{R}\mathbf{ep}_q(\mathfrak{gl}_n)$.

◂ Back

	$\mathbf{U}_q(\mathfrak{so}_n)$	$\mathbf{U}'_q(\mathfrak{so}_n)$
Subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$	✗	✓
Hopfalgebra	✓	✗
Quantization of $\mathbf{U}(\mathfrak{so}_n)$	✓	✓
"Nice quantum numbers"	✗	✓
"Nice topology"	✓	✗



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◂ Back