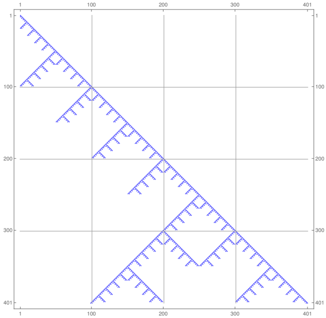
SL_2 and fractals

Or: Modular representation theory in a toy example

Daniel Tubbenhauer



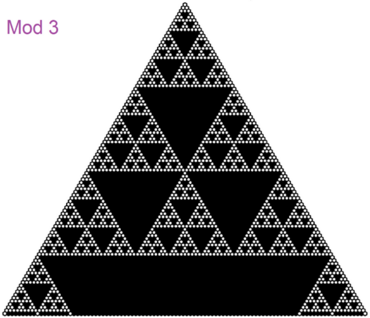
Joint with Lousie Sutton, Paul Wedrich, Jieru Zhu

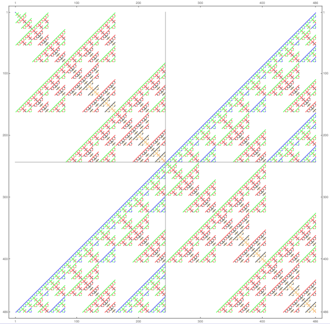
March 2021

Question. What can we say about finite-dimensional modules of SL_2...

- ...in the context of the representation theory of classical groups? $\rightsquigarrow$ The modules and their structure.
- ...in the context of the representation theory of Hopf algebras? $\rightsquigarrow$ Fusion rules *i.e.* tensor products rules.
- ...in the context of categories? $\rightsquigarrow$ Morphisms of representations and their structure.

The most amazing things happen if the characteristic of the underlying field $\mathbb{K}=\overline{\mathbb{K}}$ of $\mathrm{SL}_2=\mathrm{SL}_2(\mathbb{K})$ is finite, and we will see fractals, *e.g.*





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Spoiler: What will be the take away?

Well, in some sense modular (char $p < \infty$) representation theory so much harder than classical one (char ∞ a.k.a. char 0) because secretly we are doing fractal geometry.

In my toy example SL_2 we can do everything explicitly.

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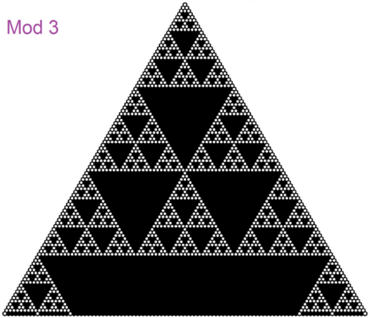

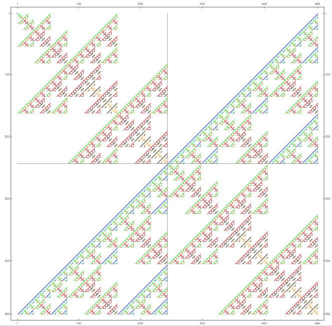

Weyl ~1923. The SL_2 (dual) Weyl modules $\Delta(v-1)$.

$\Delta(1-1)$	X^0Y^0
$\Delta(2-1)$	$X^1Y^0 \quad X^0Y^1$
$\Delta(3-1)$	$X^2Y^0 \quad X^1Y^1 \quad X^0Y^2$
$\Delta(4-1)$	$X^3Y^0 \quad X^2Y^1 \quad X^1Y^2 \quad X^0Y^3$
$\Delta(5-1)$	$X^4Y^0 \quad X^3Y^1 \quad X^2Y^2 \quad X^1Y^3 \quad X^0Y^4$
$\Delta(6-1)$	$X^5Y^0 \quad X^4Y^1 \quad X^3Y^2 \quad X^2Y^3 \quad X^1Y^4 \quad X^0Y^5$
$\Delta(7-1)$	$X^6Y^0 \quad X^5Y^1 \quad X^4Y^2 \quad X^3Y^3 \quad X^2Y^4 \quad X^1Y^5 \quad X^0Y^6$

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto$ matrix who's rows are expansions of $(aX+cY)^{v-i}(bX+dY)^{i-1}$.

▸ The simples

Example $\Delta(7-1) = \mathbb{K}X^6Y^0 \oplus \cdots \oplus \mathbb{K}X^0Y^6$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts as

$$\begin{pmatrix}
a^6 & 6a^5c & 15a^4c^2 & 20a^3c^3 & 15a^2c^4 & 6ac^5 & c^6 \\
a^5b & 5a^4bc+a^5d & 10a^3bc^2+5a^4cd & 10a^2bc^3+10a^3c^2d & 5abc^4+10a^2c^3d & bc^5+5ac^4d & c^5d \\
a^4b^2 & 4a^3b^2c+2a^4bd & 6a^2b^2c^2+8a^3bcd+a^4d^2 & 4ab^2c^3+12a^2bc^2d+4a^3cd^2 & b^2c^4+8abc^3d+6a^2c^2d^2 & 2bc^4d+4ac^3d^2 & c^4d^2 \\
a^3b^3 & 3a^2b^3c+3a^3b^2d & 3ab^3c^2+9a^2b^2cd+3a^3bd^2 & b^3c^3+9ab^2c^2d+9a^2bcd^2+a^3d^3 & 3b^2c^3d+9abc^2d^2+3a^2cd^3 & 3bc^3d^2+3ac^2d^3 & c^3d^3 \\
a^2b^4 & 2ab^4c+4a^2b^3d & b^4c^2+8ab^3cd+6a^2b^2d^2 & 4b^3c^2d+12ab^2cd^2+4a^2bd^3 & 6b^2c^2d^2+8abcd^3+a^2d^4 & 4bc^2d^3+2acd^4 & c^2d^4 \\
ab^5 & b^5c+5ab^4d & 5b^4cd+10ab^3d^2 & 10b^3cd^2+10ab^2d^3 & 10b^2cd^3+5abd^4 & 5bcd^4+ad^5 & cd^5 \\
b^6 & 6b^5d & 15b^4d^2 & 20b^3d^3 & 15b^2d^4 & 6bd^5 & d^6
\end{pmatrix}$$

The rows are expansions of $(aX+cY)^{7-i}(bX+dY)^{i-1}$. Binomials!

$\Delta(3-1)$ X^2Y^0 X^1Y^1 X^0Y^2

$\Delta(4-1)$ X^3Y^0 X^2Y^1 X^1Y^2 X^0Y^3

$\Delta(5-1)$ X^4Y^0 X^3Y^1 X^2Y^2 X^1Y^3 X^0Y^4

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The rows are expansions of $(aX + cY)^{7-i}(bX + dY)^{i-1}$. Binomials!

$\Delta(3{-}1)$ X^2Y^0 X^1Y^1 X^0Y^2

Example $\Delta(7{-}1)$, characteristic 0**.**

No common eigensystem $\Rightarrow \Delta(7{-}1)$ simple.

Example $\Delta(7{-}1)$, characteristic 2**.**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ acts as } \begin{pmatrix}
a^6 & 0 & a^4c^2 & 0 & a^2c^4 & 0 & c^6 \\
a^5b & a^4bc+a^5d & a^4cd & 0 & abc^4 & bc^5+ac^4d & c^5d \\
a^4b^2 & 0 & a^4d^2 & 0 & b^2c^4 & 0 & c^4d^2 \\
a^3b^3 & a^2b^3c+a^3b^2d & ab^3c^2+a^2b^2cd+a^3bd^2 & b^3c^3+ab^2c^2d+a^2bcd^2+a^3d^3 & b^2c^3d+abc^2d^2+a^2cd^3 & bc^3d^2+ac^2d^3 & c^3d^3 \\
a^2b^4 & 0 & b^4c^2 & 0 & a^2d^4 & 0 & c^2d^4 \\
ab^5 & b^5c+ab^4d & b^4cd & 0 & abd^4 & bcd^4+ad^5 & cd^5 \\
b^6 & 0 & b^4d^2 & 0 & b^2d^4 & 0 & d^6
\end{pmatrix}$$

$(0, 0, 0, 1, 0, 0, 0)$ is a common eigenvector, so we found a submodule.

▸ The simples

Weyl $\sim$1923. The SL_2 (dual) Weyl modules $\Delta(v-1)$.

$\Delta(1-1)$

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$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$ matrix who's rows are expansions of $(aX+cY)$ $(bX+dY)^{i-1}$.

When is $\Delta(v-1)$ simple?

$\Delta(v-1)$ is simple

$\Leftrightarrow$

$\binom{v-1}{w-1} \neq 0$ for all $w \leq v$

$\Leftrightarrow$ (Lucas's theorem)

$v = [a_r, 0, \ldots, 0]_p$.

Lucas $\sim$1878.
"Binomials mod p are the product of binomials of the p-adic digits":
$\binom{a}{b} = \prod_{i=0}^{r} \binom{a_i}{b_i}$ mod p,
where $a = [a_r, \ldots, a_0]_p = \sum_{i=0}^{r} a_i p^i$ *etc.*

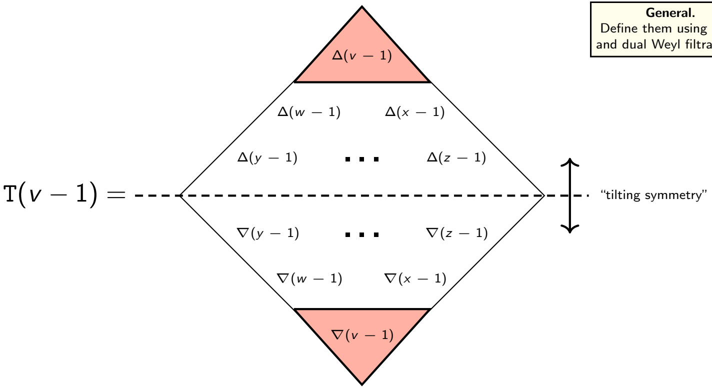
General.
Weyl $\Delta(\lambda)$ and dual Weyl $\nabla(\lambda)$
are easy a.k.a. standard;
are parameterized by dominant integral weights;
are highest weight modules;
are defined over $\mathbb{Z}$;
have the classical Weyl characters;
form a basis of the Grothendieck group unitriangular w.r.t. simples;
satisfy (a version of) Schur's lemma $\dim_{\mathbb{K}} \mathrm{Ext}^i(\Delta(\lambda), \Delta(\mu)) = \Delta_{i,0}\Delta_{\lambda,\mu}$;
are simple generically;
have a root-binomial-criterion to determine whether they are simple (Jantzen's thesis $\sim$1973).

The simples

Ringel, Donkin ∼1991. There is a class of modules $\mathtt{T}(v{-}1)$ indexed by $\mathbb{N}$. They are a bit tricky to define, but:

- They have Δ- and ∇ filtrations, which look the same if you tilt your head:

General.
Define them using Weyl and dual Weyl filtrations.



- Play the role of projective modules.
- $\mathtt{T}(v{-}1) \cong \mathtt{L}(v{-}1) \cong \Delta(v{-}1) \cong \nabla(v{-}1)$ over $\mathbb{C}$.
- They are more well-behaved than simples. Analogy

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$\mathtt{T}(v$

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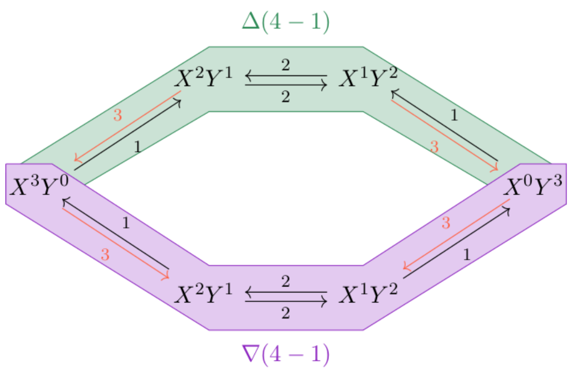
Example of $\mathtt{T}(4-1)$ for characteristic 3



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How many Weyl factors does $\mathtt{T}(v-1)$ have?

\# Weyl factors of $\mathtt{T}(v-1)$ is 2^k where

$k = \max\{\nu_p(\binom{v-1}{w-1}), w \leq v\}$. (Order of vanishing of $\binom{v-1}{w-1}$.)

determined by (Lucas's theorem)

non-zero non-leading digits of $v = [a_r, a_{r-1}, ..., a_0]_p$.

Example $\mathtt{T}(220540-1)$ for $p = 11$?

$v = 220540 = [1, 4, 0, 7, 7, 1]_{11}$;

Maximal vanishing for $w = 75594 = [0, 5, 1, 8, 8, 2]_{11}$;

$\binom{v-1}{w-1} = (\mathit{HUGE}) = [..., \neq 0, 0, 0, 0, 0]_{11}$.

$\Rightarrow \mathtt{T}(220540-1)$ has 2^4 Weyl factors.

- Play the ro[...]
- $\mathtt{T}(v-1) \cong$ [...]
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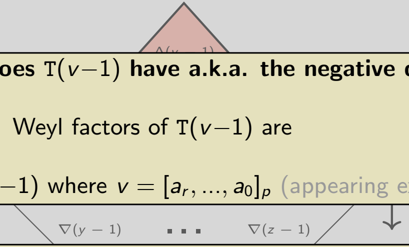

Which Weyl factors does $\mathtt{T}(v{-}1)$ have a.k.a. the negative digits game?

Weyl factors of $\mathtt{T}(v{-}1)$ are

$\Delta([a_r, \pm a_{r-1}, ..., \pm a_0]_p{-}1)$ where $v = [a_r, ..., a_0]_p$ (appearing exactly once).

Example $\mathtt{T}(220540{-}1)$ for $p = 11$?

$v = 220540 = [1, 4, 0, 7, 7, 1]_{11};$

has Weyl factors $[1, \pm 4, 0, \pm 7, \pm 7, \pm 1]_{11};$

e.g. $\Delta(218690 = [1, 4, 0, -7, -7, -1]_{11}{-}1)$ appears.

- Play the rol
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Ringel, Donkin ∼1991. There is a class of modules $\mathtt{T}(v{-}1)$ indexed by $\mathbb{N}$. They are a bit tricky to define, but:

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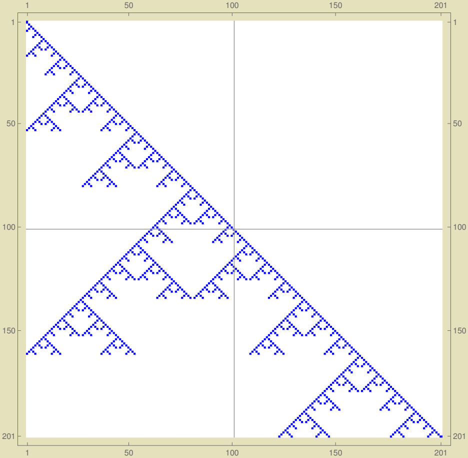

This is characteristic 3.

- Play the
- $\mathtt{T}(v{-}1) \cong$
- They are

General.
These facts hold in general, and
tilting modules form the "nicest possible" monoidal subcategory.

Tilting modules form a braided monoidal category $\mathcal{T}\mathrm{ilt}$.
Simple$\otimes$simple$\neq$simple, Weyl$\otimes$Weyl$\neq$Weyl, but tilting$\otimes$tilting$=$tilting.

The Grothendieck algebra $[\mathcal{T}\mathrm{ilt}]$ of $\mathcal{T}\mathrm{ilt}$ is a commutative algebra with basis $[\mathtt{T}(v-1)]$. So what I would like to answer on the object level, *i.e.* for $[\mathcal{T}\mathrm{ilt}]$:

- What are the fusion rules? ▸ Answer
- Find the $N_{v,w}^{x} \in \mathbb{N}[0]$ in $\mathtt{T}(v-1) \otimes \mathtt{T}(v-1) \cong \bigoplus_{x} N_{v,w}^{x}\mathtt{T}(x-1)$.
  - ▷ For $[\mathcal{T}\mathrm{ilt}]$ this means finding the structure constants.
- What are the thick $\otimes$-ideals? ▸ Answer
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The morphism. There exists a $\mathbb{K}$-algebra Z_p defined as a (very explicit) quotient of the path algebra of an infinite, fractal-like quiver. Let $\mathrm{p}\mathcal{M}\mathrm{od}\text{-}\mathrm{Z}_p$ denote the category of finitely-generated, projective (right-)modules for Z_p. There is an equivalence of additive, $\mathbb{K}$-linear categories

$$\mathcal{F}\colon \mathcal{T}\mathrm{ilt} \xrightarrow{\cong} \mathrm{p}\mathcal{M}\mathrm{od}\text{-}\mathrm{Z}_p,$$

sending indecomposable tilting modules to indecomposable projectives.

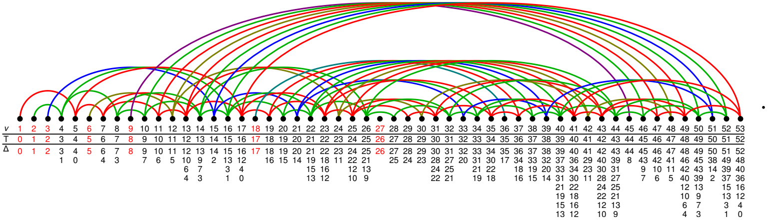

Figure: My favorite rainbow: The full subquiver containing the first 53 vertices of the quiver underlying Z_3.

▸ Proof? ▸ Time's up

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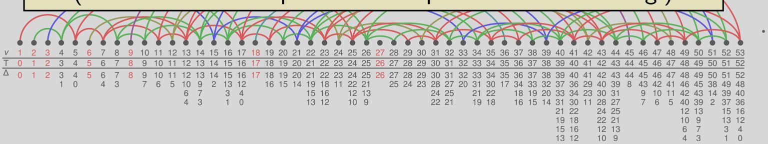
sending i

Example, generation 0, i.e. up to p.

In this case the quiver has no edges.

Continuing this periodically gives a quiver for $\mathcal{T}\mathrm{ilt}$ for char $p = \infty$.

(This is the semisimple case: the quiver has to be boring.)



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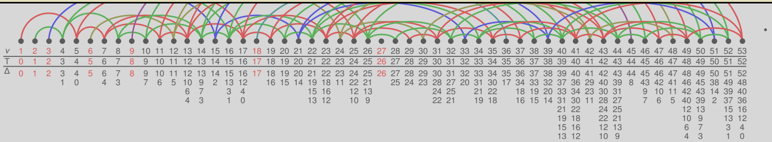
Example, generation 1, i.e. up to p^2.

In this case the quiver is a bunch of type A graphs. The algebra is a zigzag algebra, with arrows acting on the 0th digit.

Continuing this periodically gives a quiver for $\mathcal{T}\mathrm{ilt}$ for the quantum group at a complex root of unity (due to Andersen ∼2014).

$$(v_0-1)\underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}}(v_1-1)\underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}}(v_2-1)\underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}}(v_3-1)\underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}}\dots,$$

$$\mathrm{D}_{\{0\}}\mathrm{D}_{\{0\}}\mathrm{e}_{v-1}=0,\mathrm{U}_{\{0\}}\mathrm{U}_{\{0\}}\mathrm{e}_{v-1}=0,\ \mathrm{D}_{\{0\}}\mathrm{U}_{\{0\}}\mathrm{e}_{v-1}=\mathrm{U}_{\{0\}}\mathrm{D}_{\{0\}}\mathrm{e}_{v-1}\text{ for }v\neq1,\ \mathrm{D}_{\{0\}}\mathrm{U}_{\{0\}}\mathrm{e}_0=0.$$



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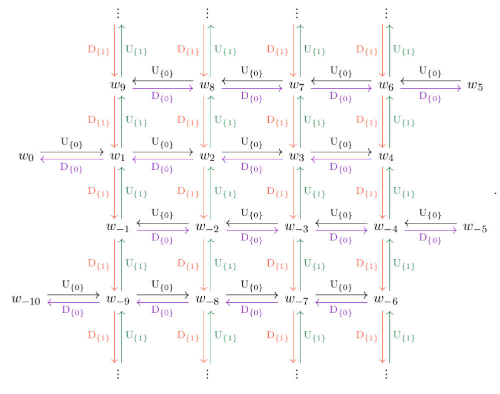
Proof? Time's up

Example, generation 2, i.e. up to p^3.

In this case every connected component
of the quiver is a bunch of type A graphs glued together in a matrix-grid.
Each row and column is a zigzag algebra, with arrows acting on the 0th digit or 1digit,
and there are “squares commute” relations.

Continuing this periodically gives a quiver for projective G_2T-modules
(due to Andersen ∼2019).



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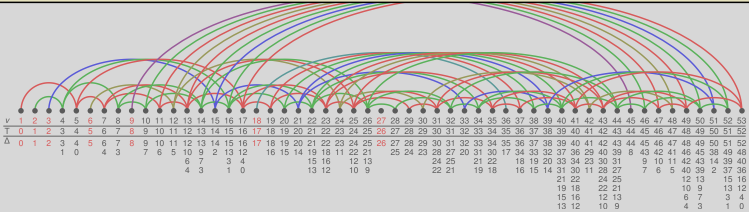
$$\mathcal{F}\colon \mathcal{T}\mathrm{ilt} \xrightarrow{\cong} \mathrm{p}\mathcal{M}\mathrm{od}\text{-}\mathrm{Z}_p,$$

sendin

In general, Z_p is basically a bunch of zigzag algebras
(there are scalars and a lower-order-error term, but never mind)
glued together in a fractal-way, according to the digits of $v = [a_r, \dots, a_0]_p$.



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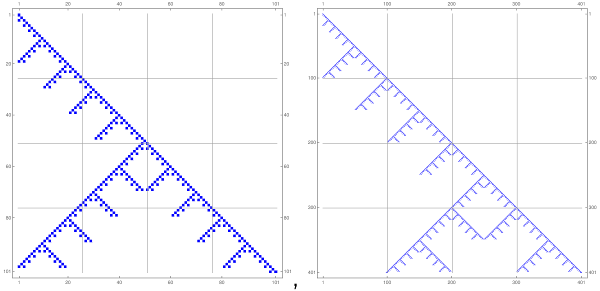
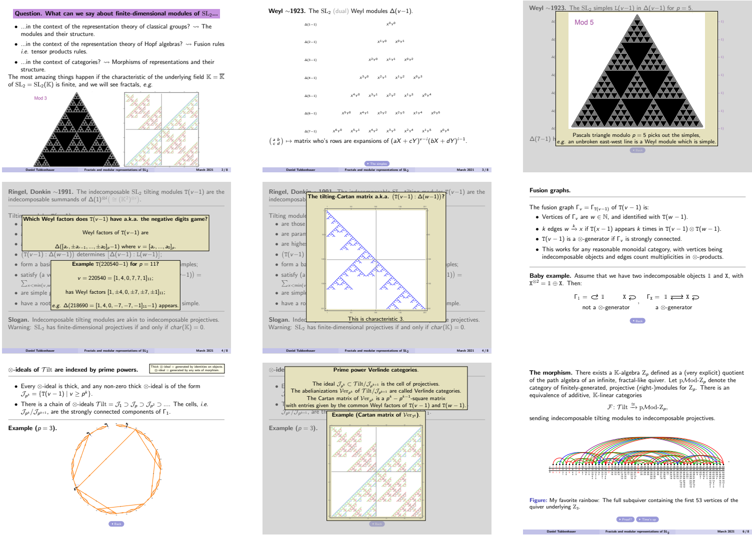
Proof? Time's up

The whole story generalizes to Lusztig's quantum group over $\mathbb{K}$ with $q \in \mathbb{K}$ via:

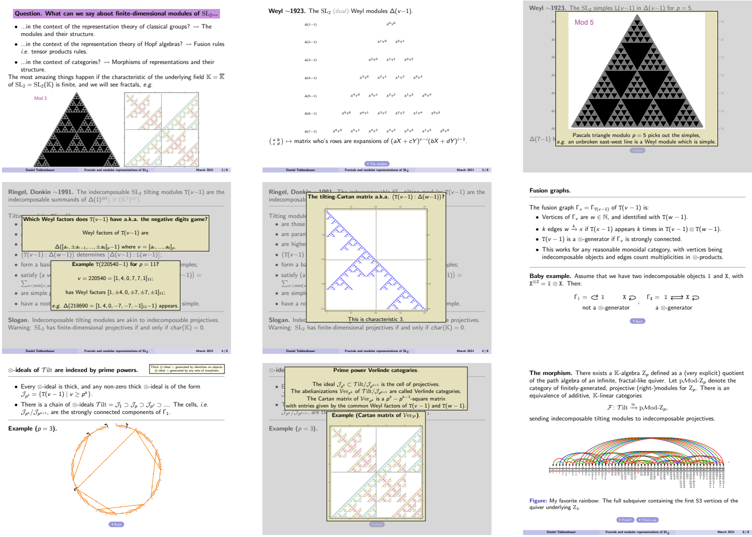
- We need p, the characteristic of $\mathbb{K}$, and l, the order of q^2.
- The p-l-adic expansion of $v = [a_r, \dots, a_0]_{p,l}$ is $v = \sum_{i=0}^{r} a_i p^{(r)}$ with $p^{(0)} = 1$ and $p^{(k)} = p^{k-1} l$. Here $0 \leq a_0 < l - 1$ and $0 \leq a_i < p - 1$.
  - ▷ Example. For $\mathbb{K} = \overline{\mathbb{F}_7}$ and $q = 2 \in \mathbb{F}_7$, we have $p = 7$ and $l = 3$.
  - ▷ Example. $68 = [68]_{p,\infty} = [66, 2]_{\infty,3} = [1, 2, 5]_{7,7} = [3, 1, 2]_{7,3}$
- Repeat everything I told you for these expansions.

Here is the tilting-Cartan matrix in mixed characteristic $p = 5$ and $l = 2$:





There is still much to do...



Thanks for your attention!

Weyl ~1923. The SL_2 simples $\mathtt{L}(v-1)$ in $\Delta(v-1)$ for $p = 5$.

$\Delta(1-1)$				X^0Y^0				$L(1-1)$
$\Delta(2-1)$				X^1Y^0 X^0Y^1				$L(2-1)$
$\Delta(3-1)$			X^2Y^0	X^1Y^1	X^0Y^2			$L(3-1)$
$\Delta(4-1)$			X^3Y^0 X^2Y^1		X^1Y^2 X^0Y^3			$L(4-1)$
$\Delta(5-1)$		X^4Y^0	X^3Y^1	X^2Y^2	X^1Y^3	X^0Y^4		$L(5-1)$
$\Delta(6-1)$		X^5Y^0 X^4Y^1	X^3Y^2		X^2Y^3	X^1Y^4 X^0Y^5		$L(6-1)$
$\Delta(7-1)$	X^6Y^0	X^5Y^1	X^4Y^2	X^3Y^3	X^2Y^4	X^1Y^5	X^0Y^6	$L(7-1)$

$\Delta(7-1)$ has (its head) $\mathtt{L}(7-1)$ and $\mathtt{L}(3-1)$ as factors.

Back

Weyl ∼1923. The SL_2 simples $\mathrm{L}(v-1)$ in $\Delta(v-1)$ for $p = 5$.

$\Delta($ ... $-1)$

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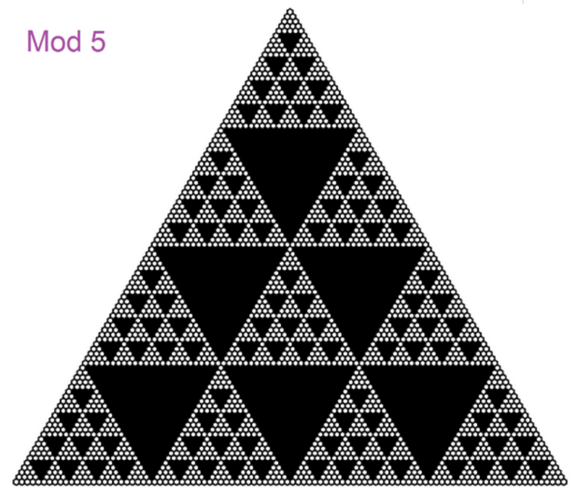

Pascals triangle modulo $p = 5$ picks out the simples,
e.g. an unbroken east-west line is a Weyl module which is simple.

$\Delta(7-1)$ h

Back

Two notions of "elements"

No substructure	Does not decompose
Simples	Indecomposables
$(*) \mathtt{V} \subset \mathtt{L} \Rightarrow \mathtt{V} \cong 0$ or $\mathtt{V} \cong \mathtt{L}$	$\mathtt{T} \cong \mathtt{V} \oplus \mathtt{W} \Rightarrow \mathtt{V} \cong 0$ or $\mathtt{V} \cong \mathtt{T}$

Both are legit elements of which one would like a periodic table.

G finite group, $\mathbb{K}[\mathrm{G}]$ the regular module (G acting on itself).

No substructure	Does not decompose
Simples	Projective indecomposables
$(*)$	$\oplus$-summands of $\mathbb{K}[\mathrm{G}]$

SL_2, $\Delta(1)$ the regular module (matrices acting by matrices).

No substructure	Does not decompose
Simples	Tilting modules
$(*)$	$\oplus$-summands of $\Delta(1)^{\otimes k}$

Back

Two notions of "elements"

No substructure	Does not decompose
Simples	Indecomposables
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Both are legit elements of wh… riodic table.

In good cases:
Simple=indecomposable
but not always.

G finite group, $\mathbb{K}[\mathrm{G}]$ the regu… n itself).

No substructure	Does not decompose
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Back

Fusion graphs.

The fusion graph $\Gamma_v = \Gamma_{\mathtt{T}(v-1)}$ of $\mathtt{T}(v-1)$ is:

- Vertices of Γ_v are $w \in \mathbb{N}$, and identified with $\mathtt{T}(w-1)$.
- k edges $w \xrightarrow{k} x$ if $\mathtt{T}(x-1)$ appears k times in $\mathtt{T}(v-1) \otimes \mathtt{T}(w-1)$.
- $\mathtt{T}(v-1)$ is a $\otimes$-generator if Γ_v is strongly connected.
- This works for any reasonable monoidal category, with vertices being indecomposable objects and edges count multiplicities in $\otimes$-products.

Baby example. Assume that we have two indecomposable objects $\mathbb{1}$ and $\mathtt{X}$, with $\mathtt{X}^{\otimes 2} = \mathbb{1} \oplus \mathtt{X}$. Then:

$$\underset{\text{not a } \otimes\text{-generator}}{\Gamma_{\mathbb{1}} = \circlearrowright \mathbb{1} \qquad \mathtt{X} \circlearrowleft}\ , \qquad \underset{\text{a } \otimes\text{-generator}}{\Gamma_{\mathtt{X}} = \mathbb{1} \rightleftarrows \mathtt{X} \circlearrowleft}$$

Back

Fusion graphs.

The fusion graph Γ

- Vertices of Γ_v .
- k edges $w \xrightarrow{k}$) $\otimes \mathtt{T}(w-1)$.
- $\mathtt{T}(v-1)$ is a $\otimes$
- This works for vertices being indecomposabl in $\otimes$-products.

Baby example. As objects $\mathbb{1}$ and $\mathtt{X}$, with $\mathtt{X}^{\otimes 2} = \mathbb{1} \oplus \mathtt{X}$. Then

$\Gamma_{\mathbb{1}}$ $\mathtt{X} \circlearrowleft$

tor

The fusion graph of $\mathtt{T}(1) \cong \mathbb{K}^2$ for $p = \infty$:

The fusion graph of $\mathtt{T}(1) \cong \mathbb{K}^2$ for $p = 2$:

Fusion graphs.

The fusion graph Γ

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tor

The fusion graph of $\mathtt{T}(1) \cong \mathbb{K}^2$ for $p = \infty$:

The fusion graph of $\mathtt{T}(1) \cong \mathbb{K}^2$ for $p = 2$:

In general, there is are cycles of length p with edges jumping $1 = p^0, p^1, p^2, \ldots$, units, reaping every $1 = p^0, p^1, p^2, \ldots$, steps.

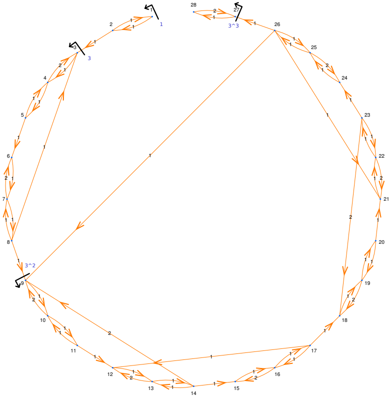
Back

$\otimes$-ideals of $\mathcal{T}\mathrm{ilt}$ are indexed by prime powers.

Thick $\otimes$-ideal $=$ generated by identities on objects.
$\otimes$-ideal $=$ generated by any sets of morphism.

- Every $\otimes$-ideal is thick, and any non-zero thick $\otimes$-ideal is of the form $\mathcal{J}_{p^k} = \{\mathtt{T}(v-1) \mid v \geq p^k\}$.
- There is a chain of $\otimes$-ideals $\mathcal{T}\mathrm{ilt} = \mathcal{J}_1 \supset \mathcal{J}_p \supset \mathcal{J}_{p^2} \supset \dots$. The cells, *i.e.* $\mathcal{J}_{p^k}/\mathcal{J}_{p^{k+1}}$, are the strongly connected components of Γ_1.

Example ($p = 3$).



◂ Back

$\otimes$-ide

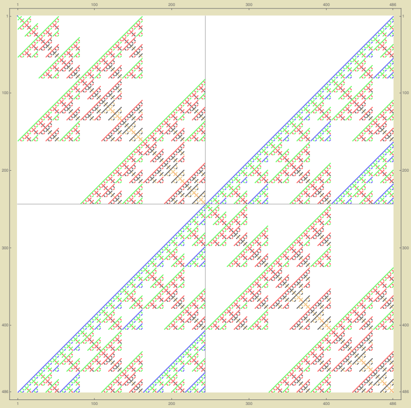
- E
  $\mathcal{J}$
- T .
  $\mathcal{J}_{p^k}/\mathcal{J}_{p^{k+1}}$, are th 1.

Example ($p = 3$).

Prime power Verlinde categories.

The ideal $\mathcal{J}_{p^k} \subset \mathcal{T}\mathrm{ilt}/\mathcal{J}_{p^{k+1}}$ is the cell of projectives.
The abelianizations $\mathcal{V}\mathrm{er}_{p^k}$ of $\mathcal{T}\mathrm{ilt}/\mathcal{J}_{p^{k+1}}$ are called Verlinde categories.
The Cartan matrix of $\mathcal{V}\mathrm{er}_{p^k}$ is a $p^k - p^{k-1}$-square matrix
with entries given by the common Weyl factors of $\mathtt{T}(v-1)$ and $\mathtt{T}(w-1)$.

Example (Cartan matrix of $\mathcal{V}\mathrm{er}_{3^4}$).



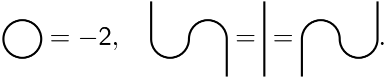
Back

Rumer–Teller–Weyl ∼1932, Temperley–Lieb ∼1971, Kauffman ∼1987.

The category $\mathcal{TL}$ is the monoidal $\mathbb{Z}$-linear category monoidally generated by

object generators : $\bullet$, morphism generators : $\cap : \mathbb{1} \to \bullet^{\otimes 2}$, $\cup : \bullet^{\otimes 2} \to \mathbb{1}$,

relations :



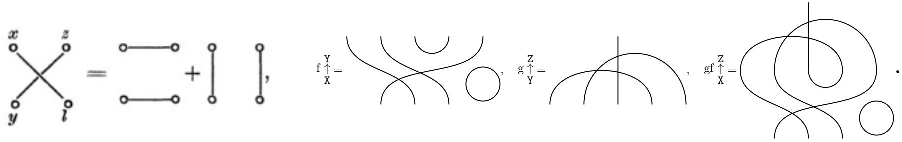


Figure: Conventions and examples. The crossing is from "G. Rumer, E. Teller, H. Weyl. Eine für die Valenztheorie geeignete Basis der binären Vektorinvarianten. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Volume: 1932, pages 499–504.".

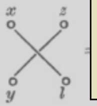
General–diagrammatics for $\mathcal{T}\mathrm{ilt}$.
For type A we have webs
à la Kuperberg ∼1997, Cautis–Kamnitzer–Morrison ∼2012.
For types BCD there are some partial results,
e.g. Brauer ∼1937, Kuperberg ∼1997,
Sartori ∼2017, Rose–Tatham ∼2020.

Rumer–Teller–Weyl ∼1932, Temperley–Lieb ∼1971, Kauffman ∼1987.

The category $\mathcal{TL}$ is the monoidal $\mathbb{Z}$-linear category monoidally generated by

object generators : $\bullet$, morphism generators : $\frown : \mathbb{1} \to \bullet^{\otimes 2}, \smile : \bullet^{\otimes 2} \to \mathbb{1}$,

relations : $\bigcirc = -2$, 



Theorem (folklore).

$\mathcal{TL}$ is an integral model of $\mathcal{T}\mathrm{ilt}$, *i.e.* fixing $\mathbb{K}$,

$\mathcal{TL} \to \mathcal{T}\mathrm{ilt}, \quad \bullet \mapsto \mathtt{T}(1)$

induces an equivalence upon additive, idempotent completion.

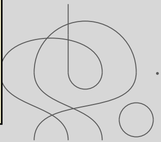
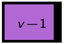


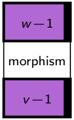
Figure: Conventions and examples. The crossing is from "G. Rumer, E. Teller, H. Weyl. Eine für die Valenztheorie geeignete Basis der binären Vektorinvarianten. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (1932), Volume: 1932, pages 499–504.".

By $\mathcal{TL} \to \mathcal{T}\mathrm{ilt}$, there are diagrammatic projectors

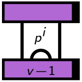
$$e_{v-1} =$$


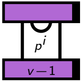
$$\in \mathrm{End}_{\mathcal{TL}}(\bullet^{\otimes(v-1)})$$

and the algebra we are looking for is

$$\mathrm{Z}_p = \bigoplus_{v,w} \mathrm{Hom}_{\mathcal{TL}}\, e_{w-1}(\bullet^{\otimes(v-1)}, \bullet^{\otimes(w-1)})e_{v-1} \rightsquigarrow$$


The generating morphisms are basically

$$D_i =$$


$$, \quad U_i =$$


Then calculate relations.

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