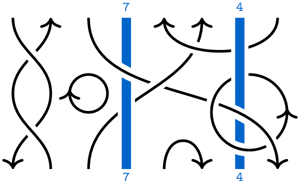
Link invariants and orbifolds

Or: What makes types ABCD special?

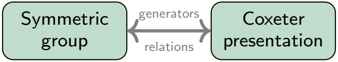
Daniel Tubbenhauer



Joint work in progress (take it with a grain of salt) with Catharina Stroppel and Arik Wilbert
(Based on an idea of Mikhail Khovanov)

April 2018

Symmetric group


Symmetric group
generators
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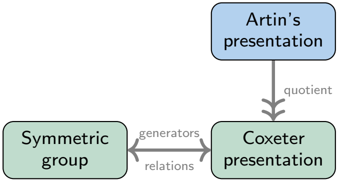
Coxeter presentation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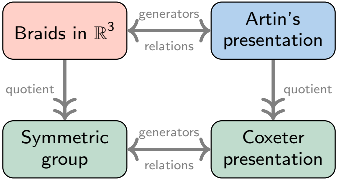
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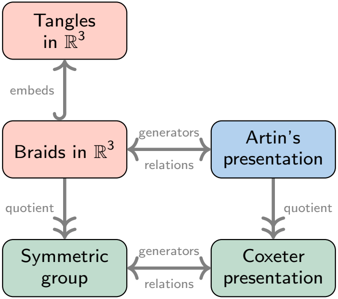
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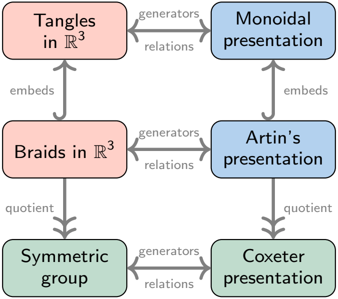
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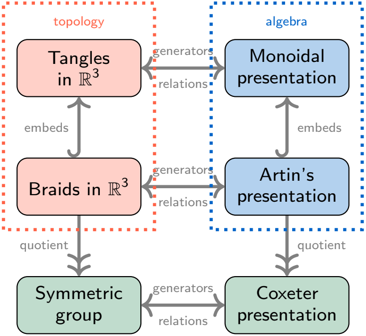
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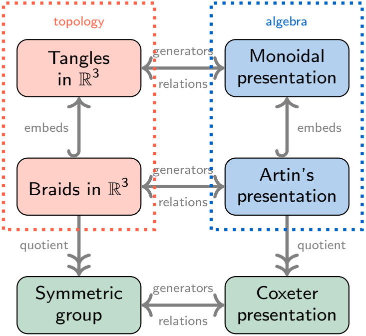
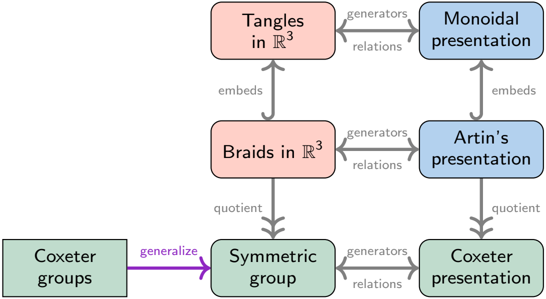


This is well-understood, neat and has many applications and connections.
So: How does this generalize?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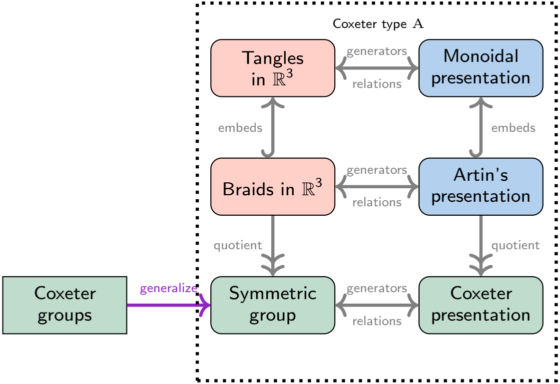
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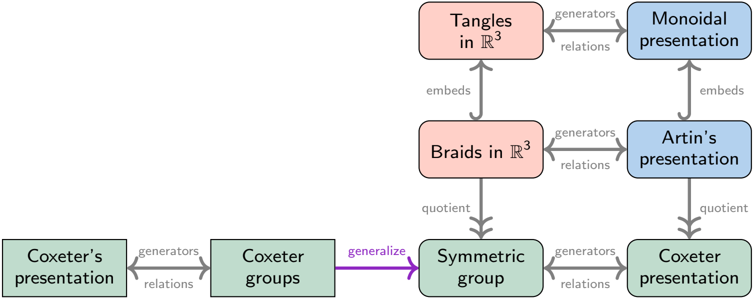

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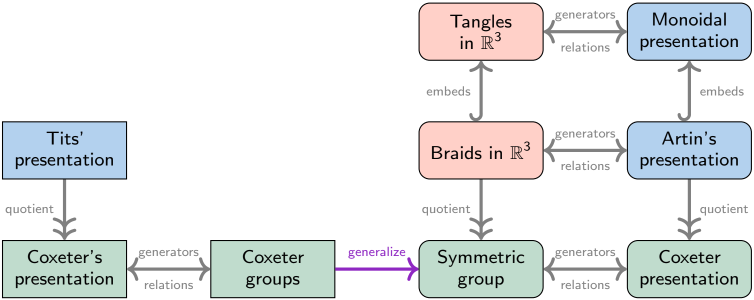
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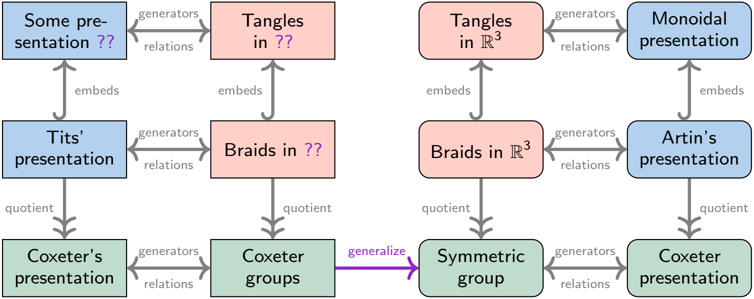
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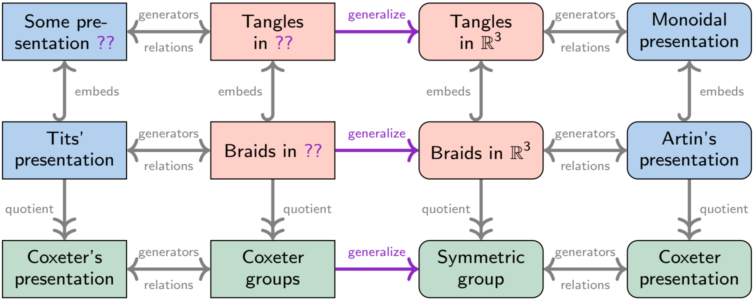
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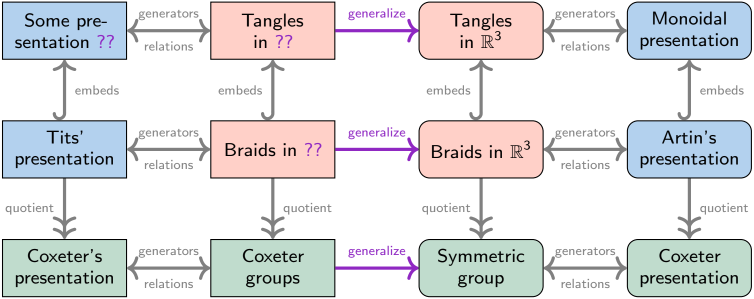
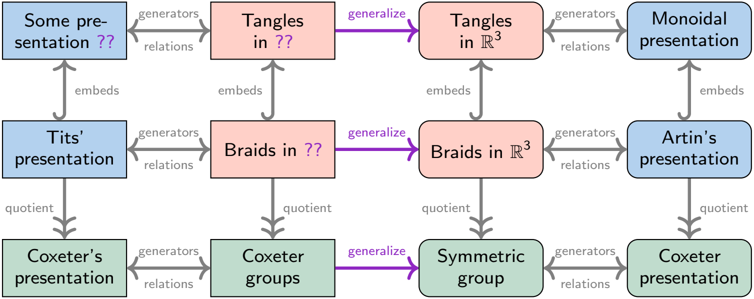
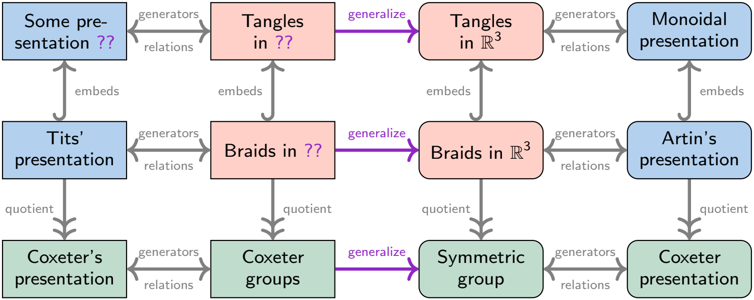


Question 1:
What fits into the questions marks?



Question 1:
What fits into the questions marks?

Question 2:
What is the analog of gadgets like Reshetikhin–Turaev or Khovanov theories?



Question 1:
What fits into the questions marks?

Question 2:
What is the analog of gadgets like Reshetikhin–Turaev or Khovanov theories?

Question 3:
Connections to other fields e.g. to representation theory?

1. **Tangle diagrams of orbifold tangles**
   - Diagrams
   - Tangles in orbifolds

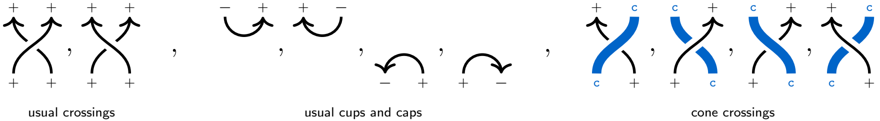
2. **Topology of Artin braid groups**
   - The Artin braid groups: algebra
   - Hyperplanes vs. configuration spaces

3. **Invariants**
   - Reshetikhin–Turaev-like theory for some coideals

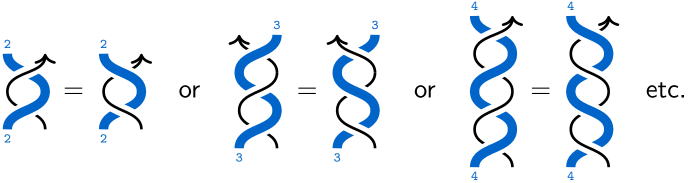
Tangle diagrams with cone strands

Let $\mathtt{c}\mathcal{T}an$ be the monoidal category ▸ defined as follows.

Generators. Object generators $\{+,-,\mathtt{c}\mid \mathtt{c}\in\mathbb{Z}_{\geq 2}\cup\{\infty\}\}$, morphism generators

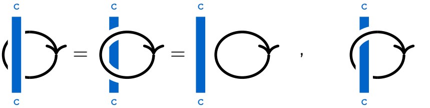


Relations. ▸ Reidemeister type relations, and the $\mathbb{Z}/\mathtt{c}\mathbb{Z}$-relations, e.g.



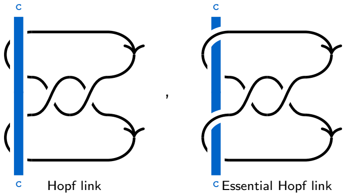
Let $\mathrm{c}\mathcal{T}an$ be the mon … f ll

Examples.



Unknot

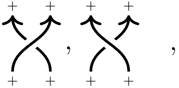
Essential unknot



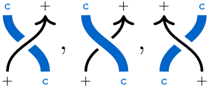
Hopf link

Essential Hopf link

Generators. Object … morphism generators

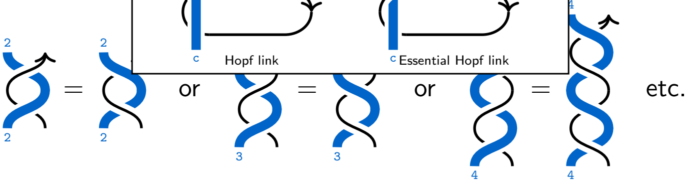


usual crossings



cone crossings

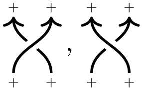
Relations. ▸ Reidemeister



Tangle diagrams with cone strands

Let $\mathrm{c}\mathcal{T}an$ be the monoidal category ▸ defined as follows.

Generators. Object generator … $\cup\{\infty\}\}$, morphism generators



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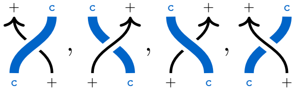

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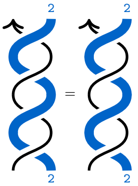
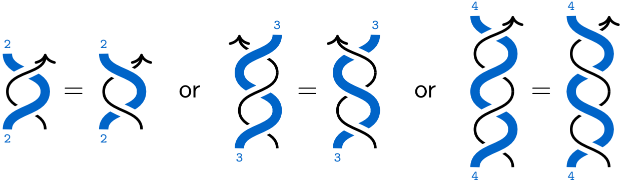


Relations. ▸ Reidemeister type relations, … , e.g.



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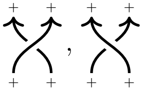

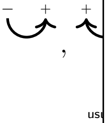

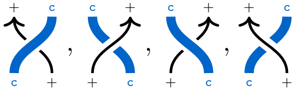

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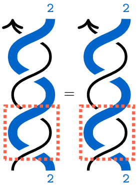

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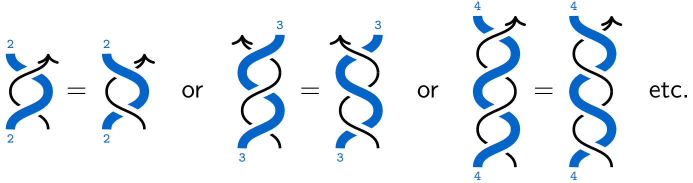

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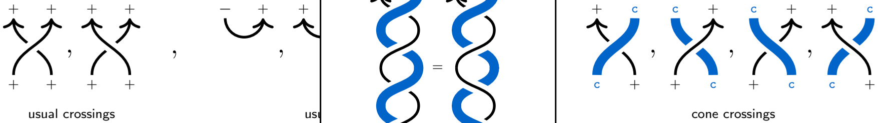

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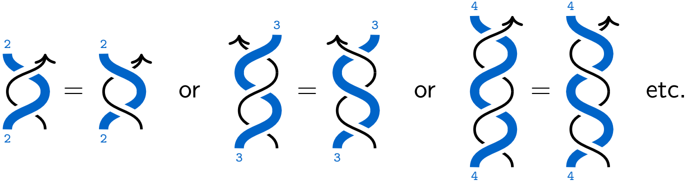

Tangle diagrams with cone strands

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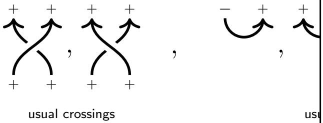
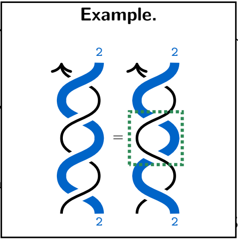
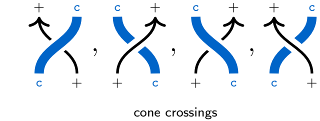
Generators. Object generator … $\cup\{\infty\}\}$, morphism generators







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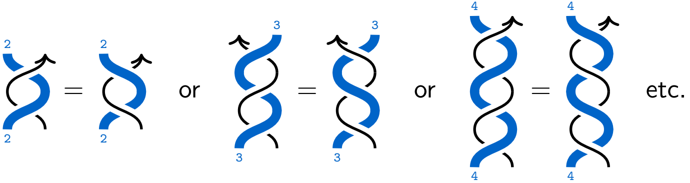

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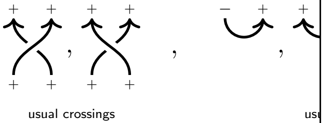
Generators. Object generator … $\cup\{\infty\}\}$, morphism generators



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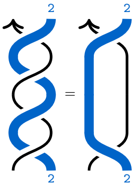

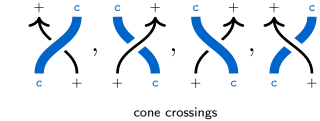

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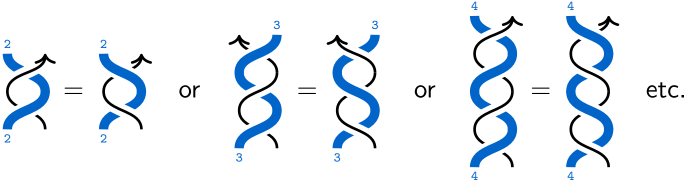

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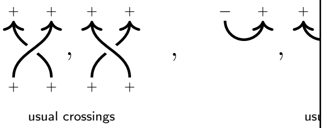
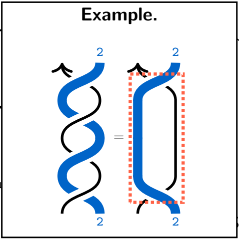
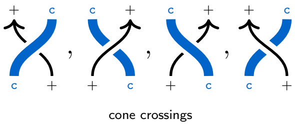
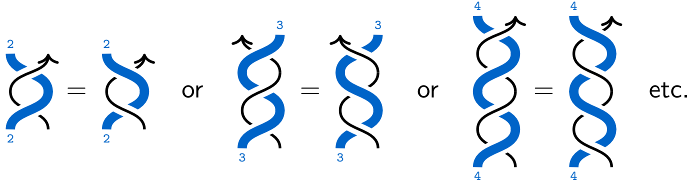
Generators. Object generator ... $\cup\{\infty\}\}$, morphism generators







Relations. ▸ Reidemeister type relations , ..., e.g.



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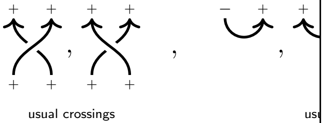

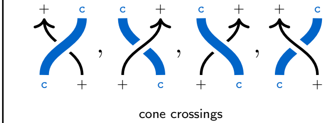

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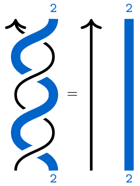
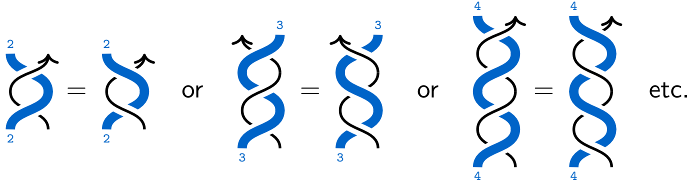


Relations. ▸ Reidemeister type relations, …, e.g.



Tangle diagrams with cone strands

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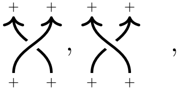
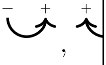
Generators. Object generator … $\cup\{\infty\}\}$, morphism generators



usual crossings



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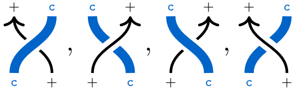
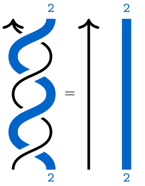


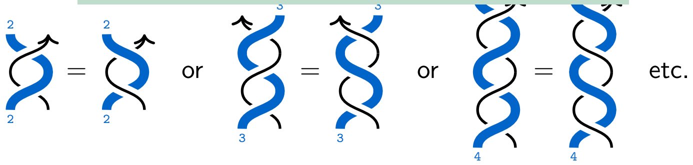
cone crossings

Example.



Relations. ▸ Reidemeister type relations , … , e.g.

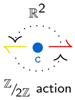
Exercise. The relations are actually equivalent.

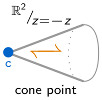


Two-dimensional orbifolds

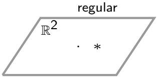
"Definition". An ▸ orbifold is locally modeled on the standard Euclidean space modulo an action of some finite group.

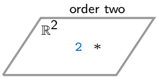
Main example. $\mathbb{Z}/_{c\mathbb{Z}}$ acts on $\mathbb{R}^2$ by rotation around a fixed point c, e.g.:

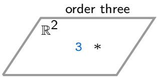
$\mathtt{O}rb = \mathbb{R}^2 \big/ \mathbb{Z}/_{2\mathbb{Z}}$ 

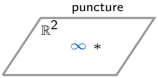
$\rightsquigarrow \mathtt{X}_{\mathtt{O}rb} \approx$ 

Philosophy. The c's are in between regular points and punctures:









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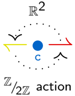

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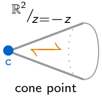

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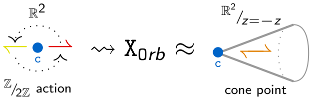

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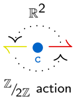

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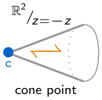

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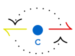

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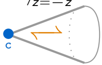

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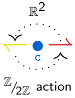

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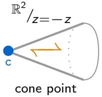

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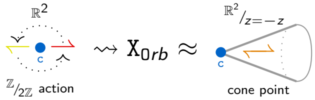

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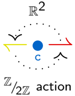

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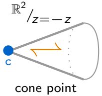

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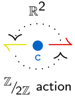

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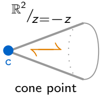

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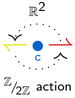

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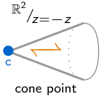

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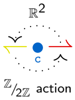

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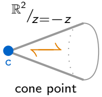

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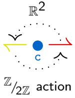

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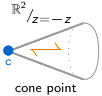

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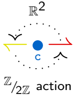

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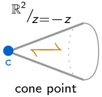

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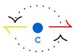
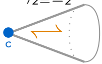
$\mathbb{R}^2$



$\mathbb{Z}/_{2\mathbb{Z}}$ action

$\rightsquigarrow X_{\mathtt{Orb}} \approx$

$\mathbb{R}^2/_{z=-z}$



cone point

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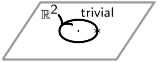
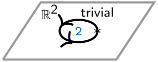
regular



$\pi_1^{\mathtt{Orb}} = 1$

order two



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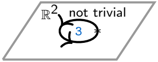
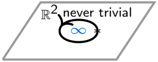


puncture



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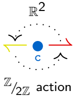

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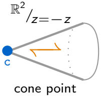

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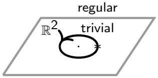

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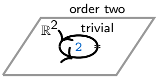

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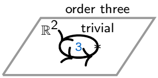

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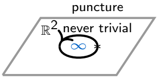

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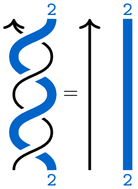
If we draw tangles in $2\mathtt{O}rb$, then:



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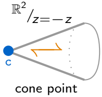

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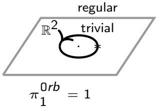

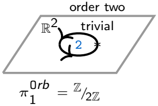

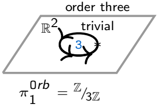

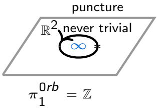

Pioneers of algebra

Let Γ be a ▸ Coxeter graph.

Artin ∼1925, Tits ∼1961++. The Artin braid groups and its Coxeter group quotients are given by generators-relations:

$$\mathcal{A}r_\Gamma = \langle b_i \mid \underbrace{\cdots b_i b_j b_i}_{m_{ij}\text{ factors}} = \underbrace{\cdots b_j b_i b_j}_{m_{ij}\text{ factors}} \rangle$$

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Artin braid groups generalize classical braid groups, Coxeter groups Weyl groups.

We want to understand these better.

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Only algebra:
No "interpretation" yet.

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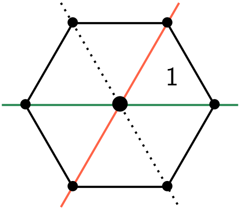
I follow hyperplanes

$\mathcal{W}_{A_2} = \langle s, t \rangle$ acts faithfully on $\mathbb{R}^2$ by reflecting in hyperplanes (for each reflection):



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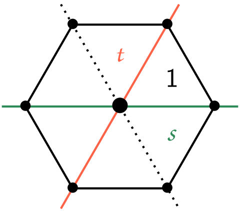

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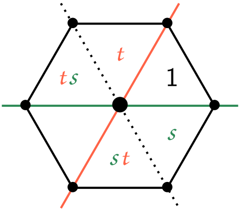

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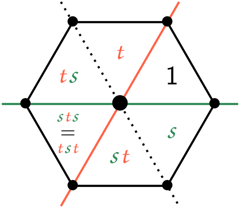

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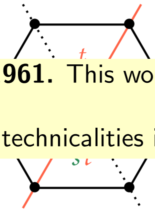

Coxeter ∼1934, Tits ∼1961. This works in ridiculous generality.

(Up to some minor technicalities in the infinite case.)

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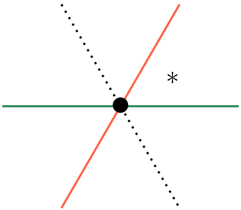

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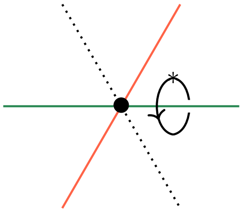

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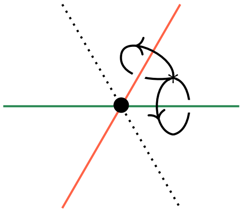

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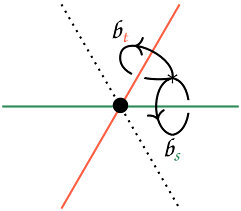

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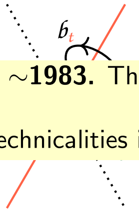

Brieskorn ∼1971, van der Lek ∼1983. This works in ridiculous generality.

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Configuration spaces

Artin $\sim$1925. There is a topological model of $\mathcal{A}r_{\mathrm{A}}$ via configuration spaces.

Example. Take $\mathtt{Conf}_{\mathrm{A}_2} = (\mathbb{R}^2)^3 \setminus \text{fat diagonal}/\mathrm{S}_3$. Then $\pi_1(\mathtt{Conf}_{\mathrm{A}_2}) \cong \mathcal{A}r_{\mathrm{A}_2}$.

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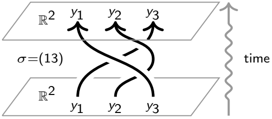

a usual braid

Crucial. Note that, by explicitly calculating the ▸ equations defining the hyperplanes, one can directly check that:

"Hyperplane picture equals configuration space picture."

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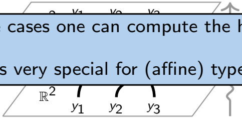
Example. Take Conf ... $(\mathbb{R}^2)^3 \setminus$ fat diagonal/... Then π(Conf...) $\cong \mathcal{A}r_{\mathrm{A}_2}$.

Lambropoulou ∼1993, tom Dieck ∼1998, Allcock ∼2002.

Type	A	$\widetilde{\mathrm{A}}$	B=C	$\widetilde{\mathrm{B}}$	$\widetilde{\mathrm{C}}$	D	$\widetilde{\mathrm{D}}$
Orbifold feature	none ("1")	∞	∞	$\infty, 2$	∞, ∞	2	2, 2

Additional inside: Works for tangles as well.

Philosop... diagrams:



In those cases one can compute the hyperplanes!

This is very special for (affine) types ABCD.

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Configuration spaces

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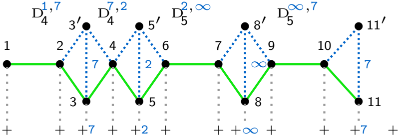
Exampl … A_2.

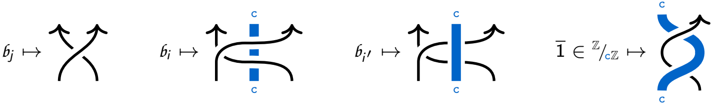
Philoso … iagrams:

Hope.

The same works for Coxeter diagrams Γ which are "locally ABCD-like graphs", e.g.:



$$\mathcal{A}r_\Gamma \rtimes (\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}) \xrightarrow{\cong} \text{orbifold braids}$$



But we can't compute the hyperplanes...

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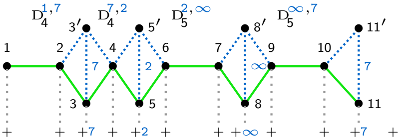

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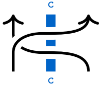
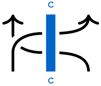
$b_j \mapsto$ 

$b_i \mapsto$ 

$b_{i'} \mapsto$ 

$\overline{1} \in \mathbb{Z}/c\mathbb{Z} \mapsto$ 

Crucial. ... ne can directly c...

In words: The $\mathbb{Z}/c\mathbb{Z}$-orbifolds provide the framework to study Artin braid groups of classical (affine) type and their "glued-generalizations".

"Hyperplane picture equals configuration space picture."

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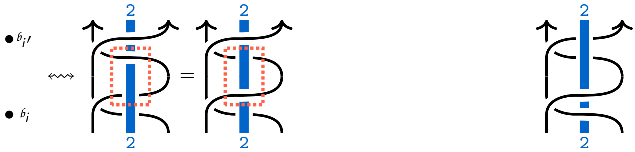

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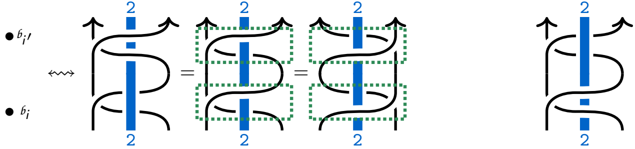

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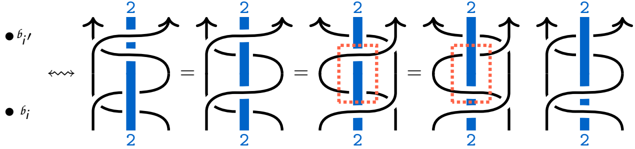

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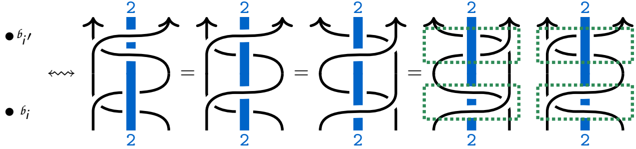

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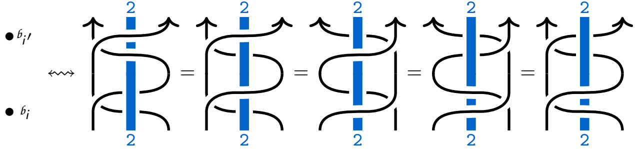

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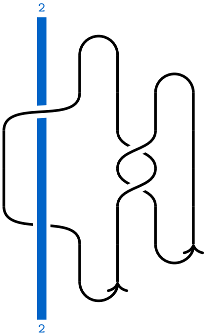
Reshetikhin–Turaev ∼1991. Construct link and tangle invariants as functors

$$\mathrm{u}\mathcal{RT}\colon \mathrm{u}\mathcal{T}an \to \text{well-behaved target category.}$$

Today: Target categories $= \mathcal{R}ep(\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2))$ and friends.

Question. What could the $\mathbb{Z}/2\mathbb{Z}$-analog be?



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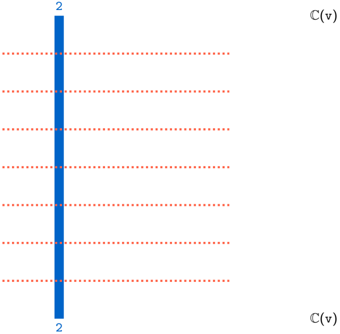

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$\mathbb{C}(\mathtt{v})$

$\mathbb{C}(\mathtt{v})$ = ground field,
$\mathbb{C}^2_{\mathtt{v}}$ = vector representation
of $\mathcal{U}_{\mathtt{v}} = \mathcal{U}_{\mathtt{v}}(\mathfrak{sl}_2)$.

$\mathbb{C}^2_{\mathtt{v}} \otimes \mathbb{C}^2_{\mathtt{v}}$
$\uparrow \mathrm{ev}^*$
$\mathbb{C}(\mathtt{v})$

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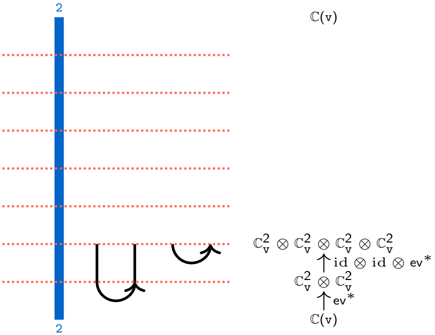

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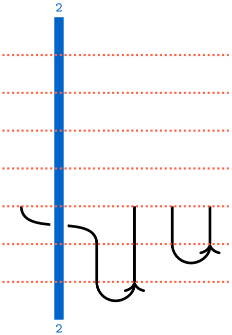
??



$\mathbb{C}(\mathrm{v})$

??: $\mathbb{C}_{\mathrm{v}}^2 \to \mathbb{C}_{\mathrm{v}}^2$ should be non-trivial.
But $\mathbb{C}_{\mathrm{v}}^2$ is irreducible for $\mathcal{U}_{\mathrm{v}}$...?

$$\mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2$$
$$\uparrow ?? \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id}$$
$$\mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2$$
$$\uparrow \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ev}^*$$
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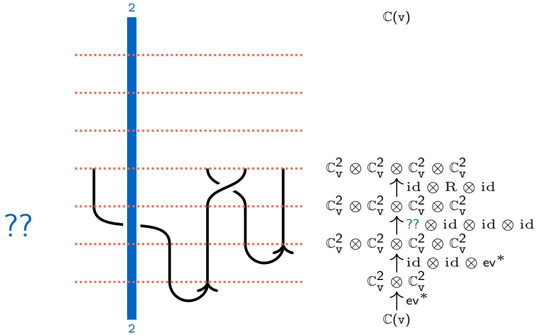

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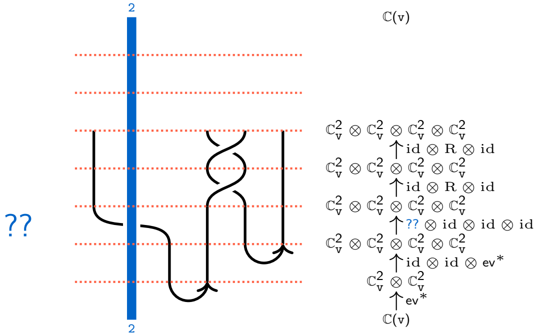

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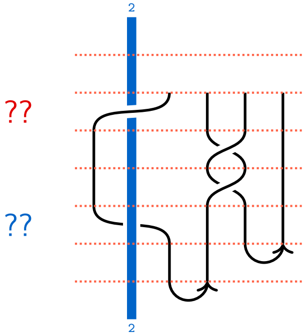


$\mathbb{C}(\mathrm{v})$

Same issue...

$$\begin{array}{c}
\mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \\
\uparrow ?? \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \\
\mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \\
\uparrow \mathrm{id} \otimes \mathrm{R} \otimes \mathrm{id} \\
\mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \\
\uparrow \mathrm{id} \otimes \mathrm{R} \otimes \mathrm{id} \\
\mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \\
\uparrow ?? \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \\
\mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \\
\uparrow \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ev}^* \\
\mathbb{C}_{\mathrm{v}}^2 \otimes \mathbb{C}_{\mathrm{v}}^2 \\
\uparrow \mathrm{ev}^* \\
\mathbb{C}(\mathrm{v})
\end{array}$$

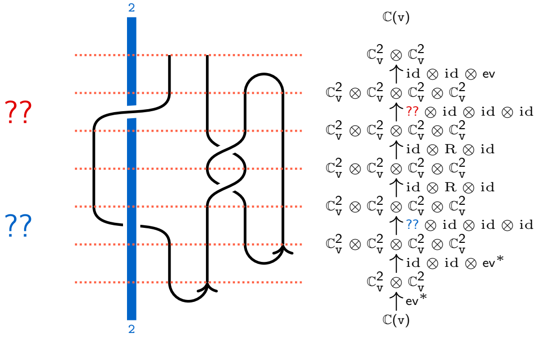
Reshetikhin–Turaev theory half-way in between

Reshetikhin–Turaev ∼1991. Construct link and tangle invariants as functors

$$\mathrm{u}\mathcal{RT}\colon \mathrm{u}\mathcal{T}an \to \text{well-behaved target category.}$$

Today: Target categories $= \mathcal{R}ep(\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2))$ and friends.

Question. What could the $\mathbb{Z}/2\mathbb{Z}$-analog be?



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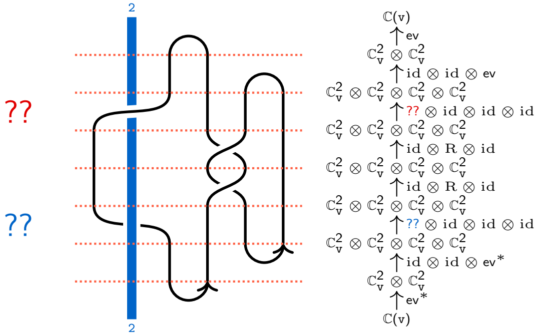

Reshetikhin–Turaev theory half-way in between

Reshetikhin–Turaev ∼1991. Construct link and tangle invariants as functors

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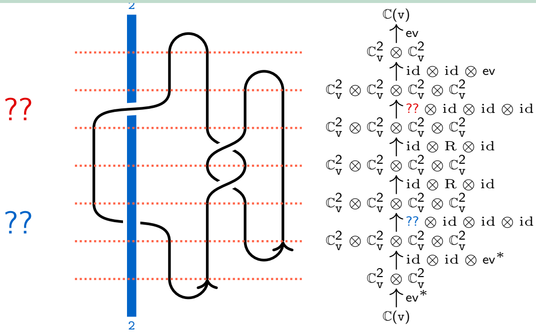
Today: Target categories $= \mathcal{R}ep(\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2))$ and friends.

Question. What could the $\mathbb{Z}/2\mathbb{Z}$-analog be?

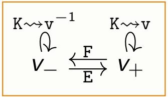
Orbifold-philosophy. We need something half-way in between $\mathbb{C}(\mathrm{v})$ and $\mathcal{U}_{\mathrm{v}}$.



Kulish–Reshetikhin ∼1981. $\mathcal{U}_{\mathrm{v}}$ is the associative, unital $\mathbb{C}(\mathrm{v})$-algebra generated by $\mathtt{E}, \mathtt{F}, \mathtt{K}^{\pm 1}$ subject to the usual relations.

Not really important...

$$\mathbb{C}^2_{\mathrm{v}}\colon \begin{array}{lll} \mathtt{E}v_+ = 0, & \mathtt{F}v_+ = v_-, & \mathtt{K}v_+ = \mathrm{v}v_+, \\ \mathtt{E}v_- = v_+, & \mathtt{F}v_- = 0, & \mathtt{K}v_- = \mathrm{v}^{-1}v_-. \end{array}$$



Define $\mathcal{U}_{\mathrm{v}}$-intertwiners:

$$\cup\colon \mathbb{C}(\mathrm{v}) \to \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}}, \quad 1 \mapsto v_- \otimes v_+ - \mathrm{v}^{-1} v_+ \otimes v_-,$$

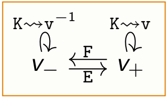
$$\cap\colon \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}} \to \mathbb{C}(\mathrm{v}), \quad \begin{cases} v_+ \otimes v_+ \mapsto 0, & v_+ \otimes v_- \mapsto 1, \\ v_- \otimes v_+ \mapsto -\mathrm{v}, & v_- \otimes v_- \mapsto 0, \end{cases}$$

$$\times\colon \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}} \to \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}}, \quad \times = \mathrm{v}\,|\,| + \mathrm{v}^2 \asymp.$$

Half-way in between trivial $\subset$?? $\subset \mathcal{U}_{\mathtt{v}}$ – part I

Kulish–Reshetikhin ∼1981. $\mathcal{U}_{\mathtt{v}}$ is the associative, unital $\mathbb{C}(\mathtt{v})$-algebra generated by $\mathtt{E}, \mathtt{F}, \mathtt{K}^{\pm 1}$ subject to the usual relations.

$$\mathbb{C}^2_{\mathtt{v}}\colon \begin{matrix} \mathtt{E}v_+ = 0, & \mathtt{F}v_+ = v_-, & \mathtt{K}v_+ = \mathtt{v}v_+, \\ \mathtt{E}v_- = v_+, & \mathtt{F}v_- = 0, & \mathtt{K}v_- = \mathtt{v}^{-1}v_-. \end{matrix}$$



Fact. $\mathcal{U}_{\mathtt{v}}$ is a Hopf algebra ⇒ We can tensor representations.

Define $\mathcal{U}_{\mathtt{v}}$-intertwiners:

$$\smile\colon \mathbb{C}(\mathtt{v}) \to \mathbb{C}^2_{\mathtt{v}} \otimes \mathbb{C}^2_{\mathtt{v}}, \quad 1 \mapsto v_- \otimes v_+ - \mathtt{v}^{-1} v_+ \otimes v_-,$$

$$\frown\colon \mathbb{C}^2_{\mathtt{v}} \otimes \mathbb{C}^2_{\mathtt{v}} \to \mathbb{C}(\mathtt{v}), \quad \begin{cases} v_+ \otimes v_+ \mapsto 0, & v_+ \otimes v_- \mapsto 1, \\ v_- \otimes v_+ \mapsto -\mathtt{v}, & v_- \otimes v_- \mapsto 0, \end{cases}$$

$$\times\colon \mathbb{C}^2_{\mathtt{v}} \otimes \mathbb{C}^2_{\mathtt{v}} \to \mathbb{C}^2_{\mathtt{v}} \otimes \mathbb{C}^2_{\mathtt{v}}, \quad \times = \mathtt{v}\,\mid\mid + \mathtt{v}^2 \asymp.$$

Half-way in between trivial $\subset$?? $\subset \mathcal{U}_{\mathrm{v}}$ – part I

Kulish–Reshetikhin ∼1981. $\mathcal{U}_{\mathrm{v}}$ is the associative, unital $\mathbb{C}(\mathrm{v})$-algebra generated by $\mathtt{E}, \mathtt{F}, \mathtt{K}^{\pm 1}$ subject to the usual relations.

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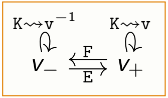


Define $\mathcal{U}_{\mathrm{v}}$-intertwiners:

Example. $(\cap \circ \cup)(1) = \cap(v_- \otimes v_+) - \mathrm{v}^{-1}\cap(v_+ \otimes v_-) = -\mathrm{v} - \mathrm{v}^{-1}$.

$$\cup\colon \mathbb{C}(\mathrm{v}) \to \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}}, \quad 1 \mapsto v_- \otimes v_+ - \mathrm{v}^{-1}v_+ \otimes v_-,$$

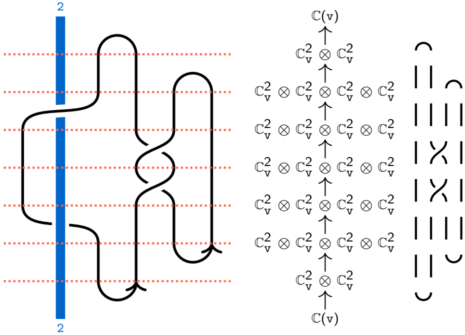
$$\cap\colon \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}} \to \mathbb{C}(\mathrm{v}), \quad \begin{cases} v_+ \otimes v_+ \mapsto 0, & v_+ \otimes v_- \mapsto 1, \\ v_- \otimes v_+ \mapsto -\mathrm{v}, & v_- \otimes v_- \mapsto 0, \end{cases}$$

$$\times\colon \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}} \to \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}}, \quad \times = \mathrm{v}\,|\,| + \mathrm{v}^2 \asymp.$$

Half-way in between trivial $\subset$?? $\subset \mathcal{U}_v$ – part I

Kulish–Reshetikhin $\sim$1981. $\mathcal{U}_v$ is the associative, unital $\mathbb{C}(v)$-algebra generated by $E, F, K^{\pm 1}$ s

Example. We can not see the cone strands.



$\mathbb{C}_v^2$:

$K \rightsquigarrow v$

v_+

Define $\mathcal{U}_v$-int

$\to 1,$

$\to 0,$

$: \mathbb{C}_v^2 \otimes \mathbb{C}_v^2 \to \mathbb{C}_v^2 \otimes \mathbb{C}_v^2, \quad = v \,||\, + v^2$.

Kulish–Reshetikhin ∼1981. $\mathcal{U}_{\mathrm{v}}$ is the associative, unital $\mathbb{C}(\mathrm{v})$-algebra generated by $\mathrm{E}, \mathrm{F}, \mathrm{K}^{\pm 1}$ s

$\mathbb{C}^2_{\mathrm{v}}$:

$\mathrm{K} \rightsquigarrow \mathrm{v}$

v_+

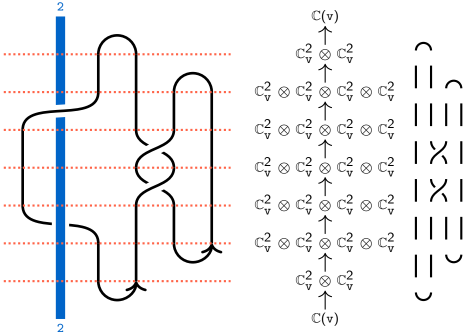
Define $\mathcal{U}_{\mathrm{v}}$-i

$\to 1,$

$\to 0,$

$: \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}} \to \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}}, \quad = \mathrm{v} \,||\, + \mathrm{v}^2$.

Example. We can not see the cone strands.



Up to scalars no choice for ||||

Let $\mathrm{c}\mathcal{U}_{\mathrm{v}}$ be the ▸ coideal subalgebra of $\mathcal{U}_{\mathrm{v}}$ generated by $\mathrm{B} = \mathrm{v}^{-1}\mathrm{EK}^{-1} + \mathrm{F}$.

$$\mathbb{C}^2_{\mathrm{v}}\colon\ \mathrm{B}v_+ = v_-, \quad \mathrm{B}v_- = v_+. \qquad v_- \underset{\mathrm{B}}{\overset{\mathrm{B}}{\rightleftarrows}} v_+$$

Define $\mathrm{c}\mathcal{U}_{\mathrm{v}}$-intertwiners:

 $: \mathbb{C}^2_{\mathrm{v}} \to \mathbb{C}^2_{\mathrm{v}}, \quad v_+ \mapsto v_-,\ v_- \mapsto v_+,$

 $: \mathbb{C}(\mathrm{v}) \to \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}}, \quad 1 \mapsto v_+ \otimes v_+ - \mathrm{v}^{-1} v_- \otimes v_-,$

 $: \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}} \to \mathbb{C}(\mathrm{v}), \quad \begin{cases} v_+ \otimes v_+ \mapsto -\mathrm{v}, & v_+ \otimes v_- \mapsto 0, \\ v_- \otimes v_+ \mapsto 0, & v_- \otimes v_- \mapsto 1, \end{cases}$

Aside. This drops out of a ▸ coideal version of Schur–Weyl duality.

Let $c\mathcal{U}_v$ be the ▸ coideal subalgebra of $\mathcal{U}_v$ generated by $B = v^{-1}EK^{-1} + F$.

$$\mathbb{C}_v^2 \colon \; Bv_+ = v_-, \quad Bv_- = v_+. \qquad v_- \underset{B}{\overset{B}{\rightleftarrows}} v_+$$

Define $c\mathcal{U}_v$-intertw

Observation. These are not $\mathcal{U}_v$-equivariant, but $\cup$ and $\cap$ are $c\mathcal{U}_v$-equivariant.

$$\cup \colon \mathbb{C}(v) \to \mathbb{C}_v^2 \otimes \mathbb{C}_v^2, \quad 1 \mapsto v_+ \otimes v_+ - v^{-1} v_- \otimes v_-,$$

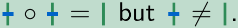
$$\cap \colon \mathbb{C}_v^2 \otimes \mathbb{C}_v^2 \to \mathbb{C}(v), \quad \begin{cases} v_+ \otimes v_+ \mapsto -v, & v_+ \otimes v_- \mapsto 0, \\ v_- \otimes v_+ \mapsto 0, & v_- \otimes v_- \mapsto 1, \end{cases}$$

Aside. This drops out of a ▸ coideal version of Schur–Weyl duality.

Let $\mathtt{c}\mathcal{U}_{\mathrm{v}}$ be the ‣ coideal subalgebra of $\mathcal{U}_{\mathrm{v}}$ generated by $\mathtt{B} = \mathrm{v}^{-1}\mathtt{E}\mathtt{K}^{-1} + \mathtt{F}$.

$$\mathbb{C}^2_{\mathrm{v}}\colon\ \mathtt{B}v_+ = v_-, \quad \mathtt{B}v_- = v_+. \qquad \boxed{v_- \underset{\mathtt{B}}{\overset{\mathtt{B}}{\rightleftarrows}} v_+}$$

Define $\mathtt{c}\mathcal{U}$ **Example.** $(\curlywedge \circ \smile)(1) = \curlywedge(v_- \otimes v_+) - \mathrm{v}^{-1}\curlywedge(v_+ \otimes v_-) = 0$



$: \mathbb{C}(\mathrm{v}) \to \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}}, \quad 1 \mapsto v_+ \otimes v_+ - \mathrm{v}^{-1} v_- \otimes v_-,$

$$\curlywedge : \mathbb{C}^2_{\mathrm{v}} \otimes \mathbb{C}^2_{\mathrm{v}} \to \mathbb{C}(\mathrm{v}), \quad \begin{cases} v_+ \otimes v_+ \mapsto -\mathrm{v}, & v_+ \otimes v_- \mapsto 0, \\ v_- \otimes v_+ \mapsto 0, & v_- \otimes v_- \mapsto 1, \end{cases}$$

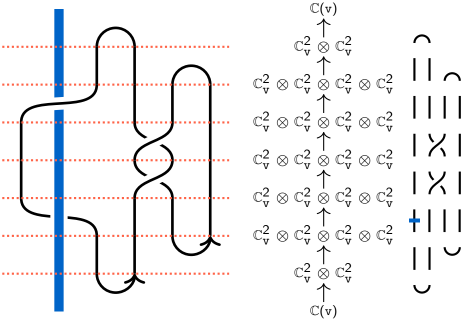
Aside. This drops out of a ‣ coideal version of Schur–Weyl duality.

Half-way in between trivial $\subset$?? $\subset$ $\mathcal{U}_v$ – part II

Let $c\mathcal{U}_v$ be the coideal subalgebra of $\mathcal{U}_v$ generated by $B = v^{-1}EK^{-1} + F$.

Define $c\mathcal{U}_v$-i

Example. We can see the cone strands.



0,

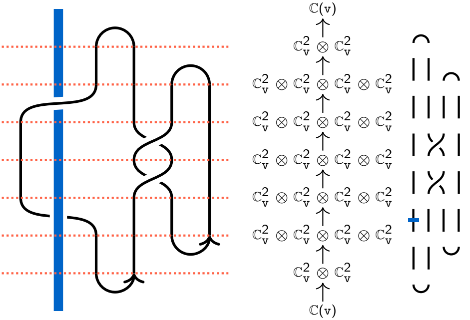
1,

Aside. This drops out of a coideal version of Schur–Weyl duality.

Half-way in between trivial ⊂ ?? ⊂ $\mathcal{U}_v$ – part II

Let $c\mathcal{U}_v$ be the ▸ coideal subalgebra of $\mathcal{U}_v$ generated by $B = v^{-1}EK^{-1} + F$.

Example. We can see the cone strands.



Define $c\mathcal{U}_v$-ir

We have now

0,

1,

Aside. This drops out of a ▸ coideal version of Schur–Weyl duality.

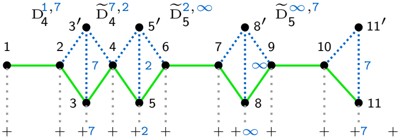
Half-way in between trivial $\subset$?? $\subset$ $\mathcal{U}_v$ – part II

Let $c\mathcal{U}_v$ be the ▸ coideal subalgebra of $\mathcal{U}_v$ generated by $B = v^{-1}EK^{-1} + F$.

Define $c\mathcal{U}_v$

Hope.

The same works for



But what is the replacement of $c\mathcal{U}_v$ outside of classical or affine classical type?

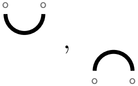
(Affine) ABCD are again very special.

Aside. This drops out of a ▸ coideal version of Schur–Weyl duality.

Back to diagrams

Let $\mathcal{TL}_{\mathbb{Z}[q^{\pm 1}]}$ be the monoidal category defined as follows.

Generators. Object generator $\{\circ\}$, morphism generators

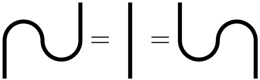


$\circ$ cups and caps

Relations. Temperley–Lieb relations, i.e.

A technicality: $q = -v$.

 $= q + q^{-1}$,



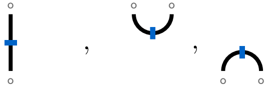
$\circ$ circle removal

isotopies

And to left-handed diagrams

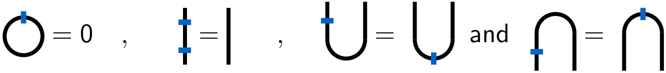
Let ◂$\mathtt{m}\mathcal{A}rc_{\mathbb{Z}[\mathtt{q}^{\pm 1}]}$ be the ▸ right $\mathcal{TL}_{\mathbb{Z}[\mathtt{q}^{\pm 1}]}$-category defined as follows.

Generators. No object generators, morphism generators



marked identity   marked cups and caps

Relations. Coideal relations, i.e.

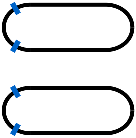


$\mathtt{m}$ circle removal   marker removal   marked isotopies

And to left-handed diagrams

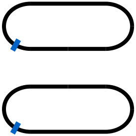
Let $\mathtt{m}\mathcal{A}rc_{\mathbb{Z}[q^{\pm 1}]}$ be the ▸ rig… …s follows.

Examples.



$= (q + q^{-1})^2$

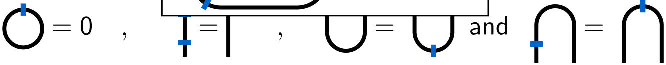
But in contrast:



$= 0$

Generators. No object ge… …s

Relations. Coideal relatio…



$\mathtt{m}$ circle removal — marker removal — marked isotopies

A polynomial invariant à la Jones & Kauffman

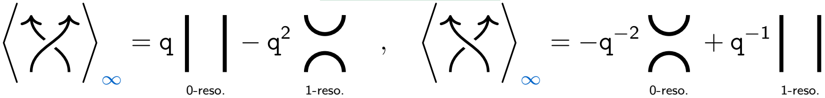
A left-handed version of $\mathtt{c}\mathcal{T}an$.

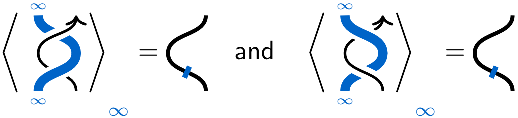
We define a functor $\langle\_\rangle_{\infty} : {}^{\blacktriangleleft}\infty f\mathcal{T}an \to {}^{\blacktriangleleft}\mathtt{m}\mathcal{A}rc_{\mathbb{Z}[\mathtt{q}^{\pm 1}]}$ intertwining the right actions as follows. On objects,

$$\langle + \rangle_{\infty} = \circ \quad , \quad \langle - \rangle_{\infty} = \circ \quad , \quad \langle \mathtt{c} \rangle_{\infty} = \varnothing$$

and on morphisms by

The skein relations.





A polynomial invariant à la Jones & Kauffman

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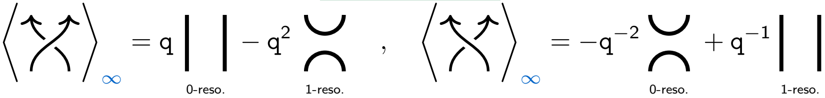
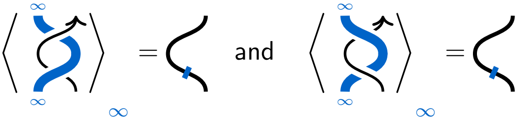


The $\mathbb{Z}/2\mathbb{Z}$-skein relations.



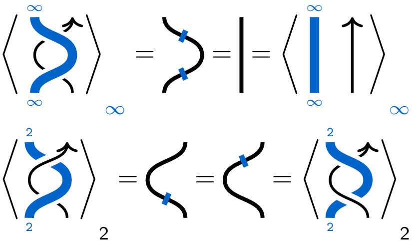
A polynomial invariant à la Jones & Kauffman

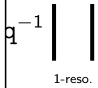
We define a functor $\langle \_ \rangle_{\infty} : \infty f\mathcal{T}an \to \mathrm{m}\mathcal{A}rc_{\mathbb{Z}[\mathrm{q}^{\pm 1}]}$ intertwining the right actions as follows. On objects

Theorem. Up to rescaling: This is a ∞-tangle invariant.
Up to framing: This is a $\mathbb{Z}/2\mathbb{Z}$-tangle invariant.

and on mor

Proof. Check relations, e.g.:

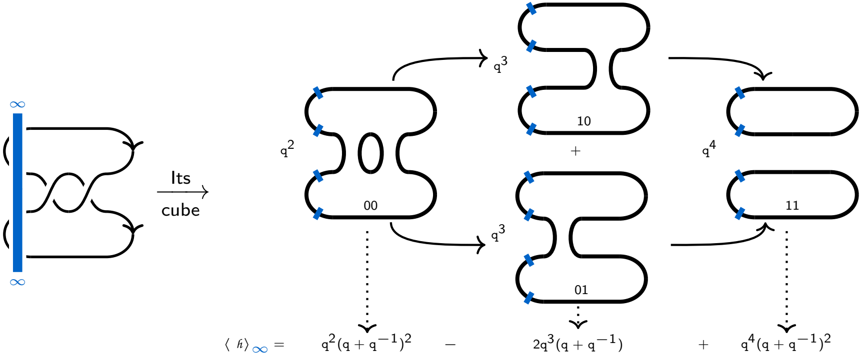




A polynomial invariant à la Jones & Kauffman

We define a functor $\langle \_ \rangle_{\infty} : \infty f\mathcal{T}an \to \mathtt{m}\mathcal{A}rc_{\mathbb{Z}[q^{\pm 1}]}$ intertwining the right actions as

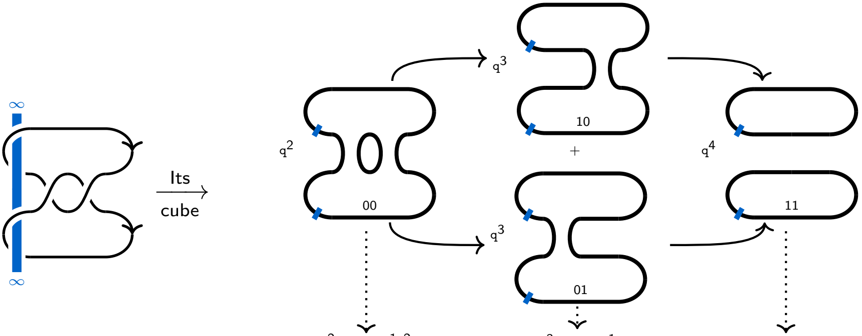
Example. Here the Hopf link.



A polynomial invariant à la Jones & Kauffman

We define a functor $\langle \_ \rangle_{\infty} : \infty f\mathcal{T}an \to \mathtt{m}\mathcal{A}rc_{\mathbb{Z}[\mathtt{q}^{\pm 1}]}$ intertwining the right actions as

Example. Here the essential Hopf link.

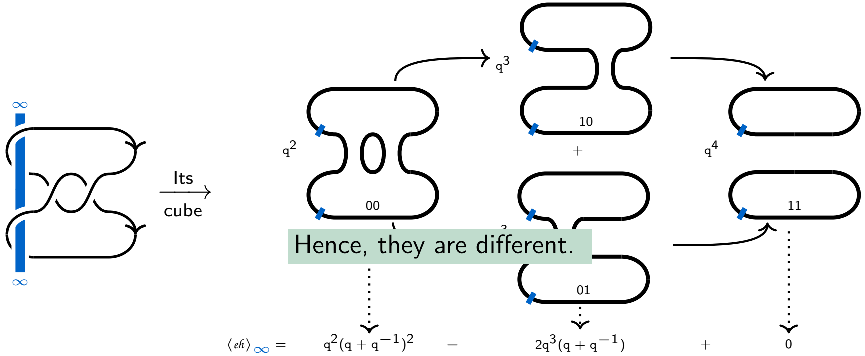


$\langle eh \rangle_{\infty} = \mathtt{q}^2(\mathtt{q}+\mathtt{q}^{-1})^2 - 2\mathtt{q}^3(\mathtt{q}+\mathtt{q}^{-1}) + 0$

A polynomial invariant à la Jones & Kauffman

We define a functor $\langle \_ \rangle_{\infty} : \infty f\mathcal{T}an \to \mathtt{m}\mathcal{A}rc_{\mathbb{Z}[q^{\pm 1}]}$ intertwining the right actions as

Example. Here the essential Hopf link.



Hence, they are different.

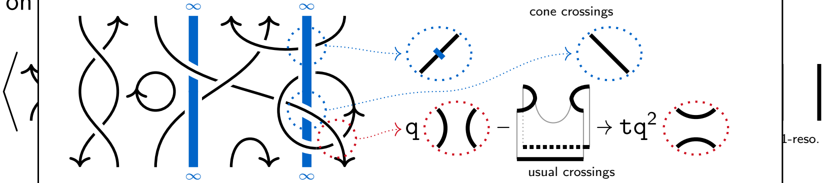
A polynomial invariant à la Jones & Kauffman

We defi… …ctions as follows.

and on

A homological invariant à la Khovanov & Bar-Natan.

Works mutatis mutandis. Here is the picture:



$$\mathtt{m}\mathcal{Z}\left(\bigcirc_{m}\right)=\begin{cases}\mathbb{Z}[X]/(X^2), & \text{if } m \text{ is even},\\ 0, & \text{if } m \text{ is odd}.\end{cases}$$

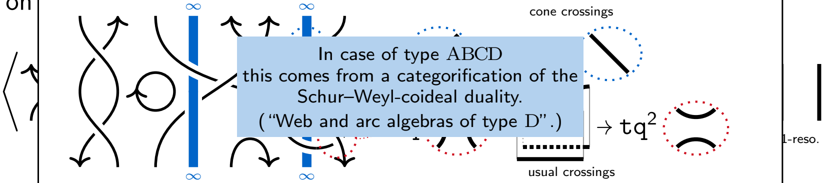
A polynomial invariant à la Jones & Kauffman

We defi... ...ctions as follows.

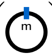
and on

A homological invariant à la Khovanov & Bar-Natan.

Works mutatis mutandis. Here is the picture:



In case of type ABCD
this comes from a categorification of the
Schur–Weyl-coideal duality.
(“Web and arc algebras of type D”.)

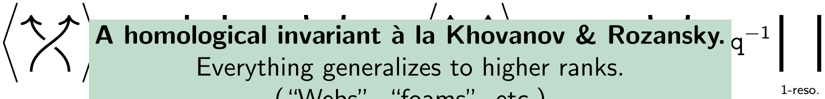
$\mathtt{m}\mathcal{Z}\Big($  $\Big) = \begin{cases} \mathbb{Z}[X]/(X^2), & \text{if } m \text{ is even,} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$

A polynomial invariant à la Jones & Kauffman

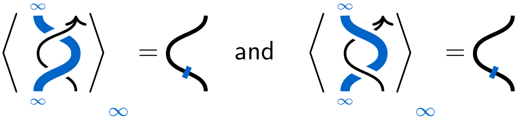
We define a functor $\langle \_ \rangle_{\infty} : \infty f\mathcal{T}an \to \mathtt{m}\mathcal{A}rc_{\mathbb{Z}[\mathtt{q}^{\pm 1}]}$ intertwining the right actions as follows. On objects,

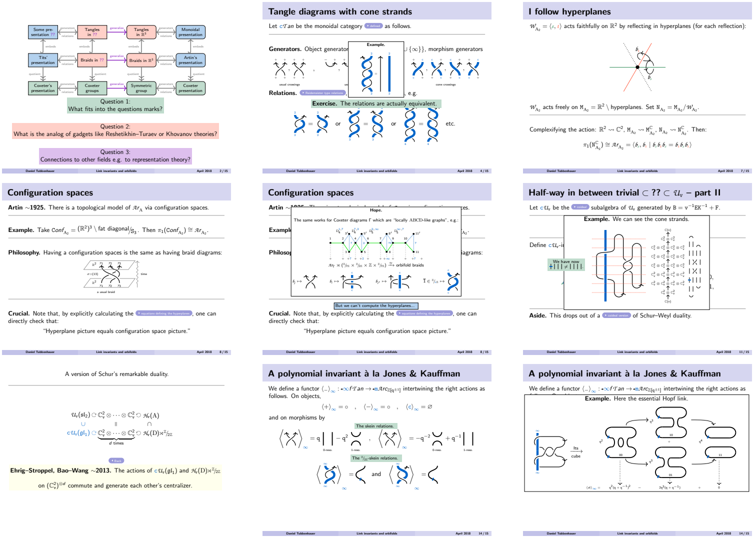
$$\langle + \rangle_{\infty} = \circ \quad , \quad \langle - \rangle_{\infty} = \circ \quad , \quad \langle \mathtt{c} \rangle_{\infty} = \varnothing$$

and on morphisms by

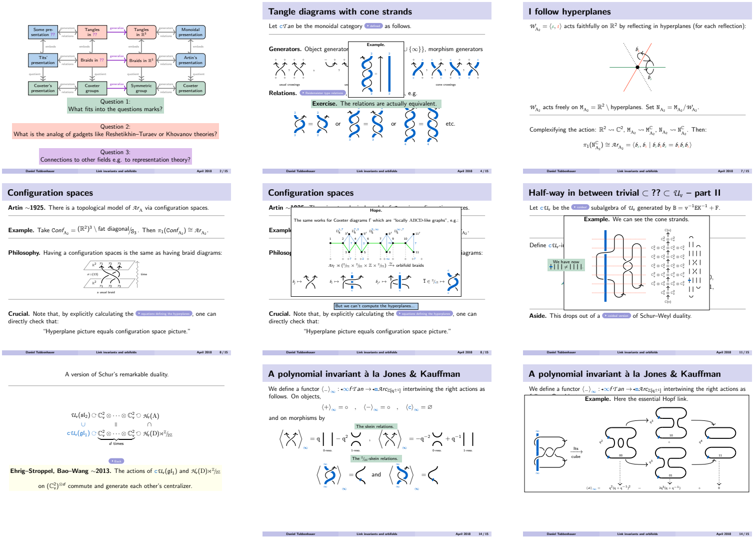


A homological invariant à la Khovanov & Rozansky.
Everything generalizes to higher ranks.
("Webs", "foams", etc.)





There is still much to do...



Thanks for your attention!

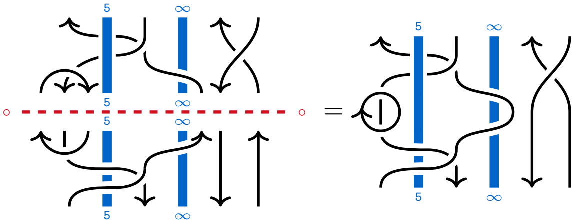
Slogan. (Monoidally) generated = building with Lego pieces.

$$\text{Lego} \otimes \text{Lego} = \text{new Lego},$$

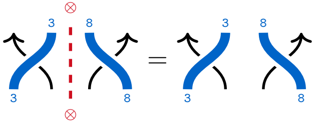
$$\text{e.g.} \quad --++ \otimes \mathtt{c}- = --++\mathtt{c}-$$

$$\text{Lego} \circ \text{Lego} = \text{new Lego} \quad \text{or} \quad \text{Lego} \otimes \text{Lego} = \text{new Lego}$$

e.g.

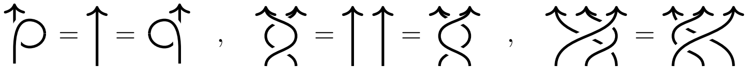


e.g.

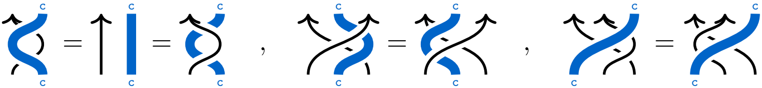


Back

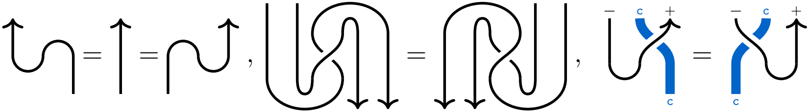
Examples of usual relations.



Examples of mixed relations.



Examples of planar isotopies.



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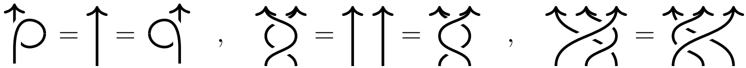
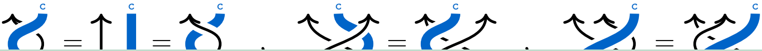
Examples of usual relations.

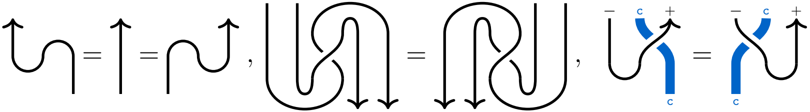


Examples of mixed relations.



In the spirit of Turaev ∼1989. Generators & relations in the monoidal setting.

Examples of planar isotopies.



Back

Satake ∼1956 ("V-manifold"), Thurston ∼1978, Haefliger ∼1990 ("orbihedron"), etc. A triple $\mathtt{O}rb = (\mathtt{X}_{\mathtt{O}rb}, \cup_i \mathtt{U}_i, \mathtt{G}_i)$ of a Hausdorff space $\mathtt{X}_{\mathtt{O}rb}$, a covering $\cup_i \mathtt{U}_i$ of it (closed under finite intersections) and a collection of finite groups $\mathtt{G}_i$ is called an orbifold (of dimension m) if for each $\mathtt{U}_i$ there exists a open subset $\mathtt{V}_i \subset \mathbb{R}^m$ carrying an action of $\mathtt{G}_i$, and some compatibility conditions.

Fact. A two-dimensional ("smooth") orbifold is locally modeled on:

- ▷ Cone points ⇜⇝ rotation action of $\mathbb{Z}/c\mathbb{Z}$.
- ▷ Reflector corners ⇜⇝ reflection action of the dihedral group.
- ▷ Mirror points ⇜⇝ reflection action of $\mathbb{Z}/2\mathbb{Z}$.

Back

Satake $\sim$1956 (“V-manifold”), Thurston $\sim$1978, Haefliger $\sim$1990 (“or

Not super important. Only one thing to stress:
Topologically an orbifold is sometimes the same as its underlying space.
So all notions concerning orbifolds have to take this into account.

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Fact. A two-dimensional (“smooth”) orbifold is locally modeled on:

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Quote by Thurston about the name orbifold:

“This terminology should not be blamed on me. It was obtained by a democratic process in my course of 1976-77. An orbifold is something with many folds; unfortunately, the word ‘manifold’ already has a different definition. I tried ‘foldamani’, which was quickly displaced by the suggestion of ‘manifolded’. After two months of patiently saying ‘no, not a manifold, a manifol**dead**,’ we held a vote, and ‘orbifold’ won.”

Back

Satake ∼1956 ("V-manifold"), Thurston ∼1978, Haefliger ∼1990 ("orbihedron"), etc. A triple $\mathtt{O}rb = (\mathtt{X}_{\mathtt{O}rb}, \cup_i \mathtt{U}_i, \mathtt{G}_i)$ of a Hausdorff space $\mathtt{X}_{\mathtt{O}rb}$, a covering $\cup_i \mathtt{U}_i$ of it (closed under finite intersections) and a collection of finite groups $\mathtt{G}_i$ is called an orbifold (of dimension m) if for each $\mathtt{U}_i$ there exists a open subset $\mathtt{V}_i \subset \mathbb{R}^m$ carrying an action of $\mathtt{G}_i$, and some compatibility conditions.

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Back

Satake ∼1956 (“ … **r ∼1990 (“orbihedron”), e** … lorff space X_{Orb}, a covering $\cup_i U_i$ of it … ection of finite groups G_i is called … here exists a open subset $V_i \subset \mathbb{R}^m$ ca … y conditions.

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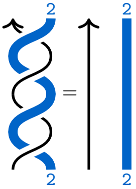
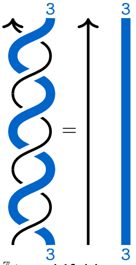
Examples.



A $\mathbb{Z}/2\mathbb{Z}$-orbifold tangle



A $\mathbb{Z}/3\mathbb{Z}$-orbifold tangle

etc.

“Puncture $= \lim_{c\to\infty}$ c-cone point”.

Back

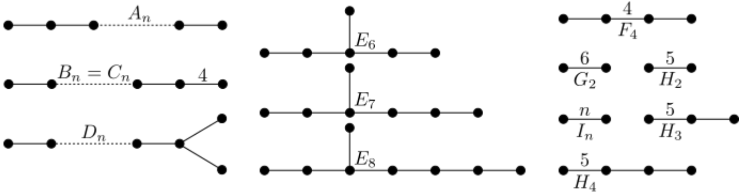
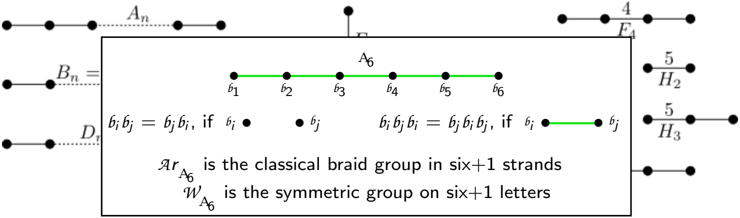


Figure: The Coxeter graphs of finite type.

Example. The type A family is given by the symmetric groups using the simple transpositions as generators.

(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Back



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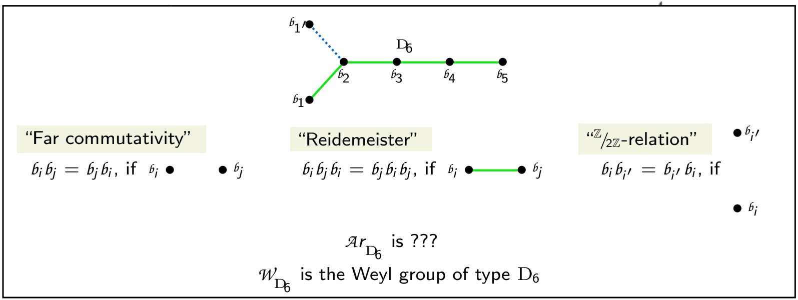

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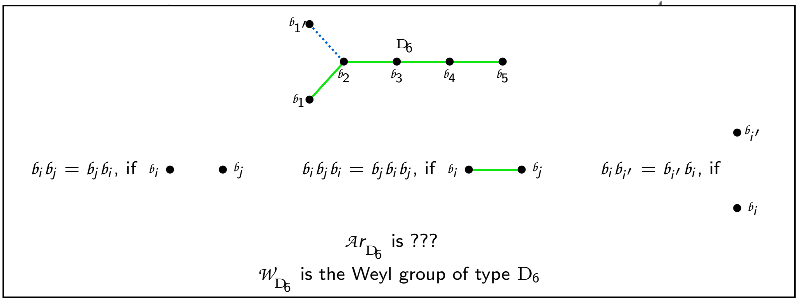


Example. The type A family is given by the symmetric groups using the simple transpositions a I want to answer ??? in this case, and partially in general.

(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Back

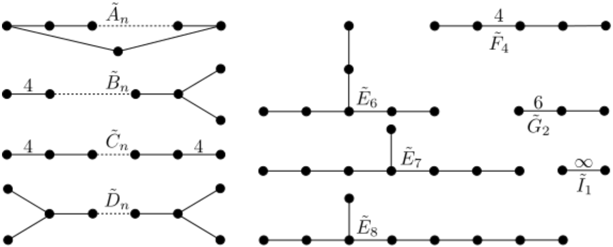
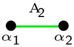


Figure: The Coxeter graphs of affine type.

Example. The type $\widetilde{\mathrm{A}}_n$ corresponds to the affine Weyl group for $\mathfrak{sl}_n$.

(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

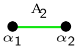
Back



positive root	$\alpha_1 = (1, -1, 0)$	$\alpha_2 = (0, 1, -1)$	$\alpha_1 + \alpha_2 = (1, 0, -1)$
reflection action	$x_1 \leftrightarrow x_2$	$x_2 \leftrightarrow x_3$	$x_1 \leftrightarrow x_3$
$\perp$-hyperplane	$\{(x, x, 0)\}$	$\{(0, y, y)\}$	$\{(z, 0, z)\}$

Hyperplane equations: $\{(x, y, z) \in (\mathbb{R}^2)^3 \mid x = y \text{ or } y = z \text{ or } x = z\}$

This is $\mathfrak{gl}$-notation.



positive root	$\alpha_1 = (1, -1, 0)$	$\alpha_2 = (0, 1, -1)$	$\alpha_1 + \alpha_2 = (1, 0, -1)$
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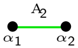
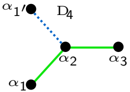
Observe that this matches the diagonal of the configuration space picture.

Back



positive root	$\alpha_1 = (1,-1,0)$	$\alpha_2 = (0,1,-1)$	$\alpha_1+\alpha_2 = (1,0,-1)$
reflection action	$x_1 \leftrightarrow x_2$	$x_2 \leftrightarrow x_3$	$x_1 \leftrightarrow x_3$
$\perp$-hyperplane	$\{(x,x,0)\}$	$\{(0,y,y)\}$	$\{(z,0,z)\}$

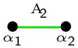
Hyperplane equations: $\{(x,y,z) \in (\mathbb{R}^2)^3 \mid x = y \text{ or } y = z \text{ or } x = z\}$



positive root	$\alpha_{1'} = (1,1,0)$	$\alpha_1 = (1,-1,0)$	more "type A-like"
reflection action	$x_{1'}, x_1 \leftrightarrow -x_{1'}, -x_1$	$x_1 \leftrightarrow x_2$	more "type A-like"
$\perp$-hyperplane	$\{(x,-x,0,0)\}$	$\{(x,x,0,0)\}$	more "type A-like"

Hyperplane equations: $\{(x,y,z,w) \in \mathbb{C}^4 \mid x = \pm y \text{ etc.}\}$

Back



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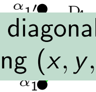
Hyperplane equations: $\{(x,y,z) \in (\mathbb{R}^2)^3 \mid x = y \text{ or } y = z \text{ or } x = z\}$



Observe that this matches the diagonal of the configuration space picture up to a 2-fold covering $(x,y,z,w) \mapsto (x^2,y^2,z^2,w^2)$.

positive root	$\alpha_{1'} = (1,1,0)$	$\alpha_1 = (1,-1,0)$	more "type A-like"
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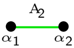

positive root	$\alpha_1 = (1, -1, 0)$	$\alpha_2 = (0, 1, -1)$	$\alpha_1 + \alpha_2 = (1, 0, -1)$
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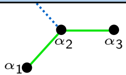
Hyperplane equations: $\{(x, y, z) \in (\mathbb{R}^2)^3 \mid x = y \text{ or } y = z \text{ or } x = z\}$

Similarly in (affine) types ABCD.



positive root	$\alpha_{1'} = (1, 1, 0)$	$\alpha_1 = (1, -1, 0)$	more "type A-like"
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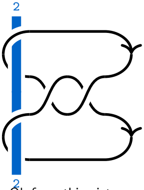
Back

Noumi–Sugitani ∼1994, Letzter ∼1999. Quantum groups have few Hopf subalgebras, but plenty of coideal subalgebras.

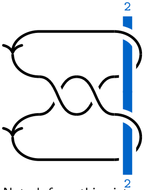
$\mathtt{c}\mathcal{U}_{\mathtt{v}}$ is not a Hopf algebra, but rather a right coideal (subalgebra) of $\mathcal{U}_{\mathtt{v}}$:

$$\Delta(\mathtt{B}) = \mathtt{B} \otimes \underbrace{\mathtt{K}^{-1}}_{\notin \mathtt{c}\mathcal{U}_{\mathtt{v}}} + 1 \otimes \mathtt{B} \in \mathtt{c}\mathcal{U}_{\mathtt{v}} \otimes \mathcal{U}_{\mathtt{v}},$$

which gives $\mathcal{R}ep(\mathtt{c}\mathcal{U}_{\mathtt{v}})$ the structure of a right $\mathcal{R}ep(\mathcal{U}_{\mathtt{v}})$-category ⇒ right handedness of diagrams, e.g.:



Ok from this picture



Not ok from this picture

◂ Back

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$^c\mathcal{U}_v$ is not a Hopf algebra, but rather a right coideal (subalgebra) of $\mathcal{U}_v$:

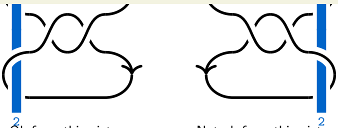
Example. The vector representations of $\mathfrak{gl}_n$, $\mathfrak{so}_n$ and $\mathfrak{sp}_n$ all agree, and indeed
$\mathfrak{so}_n \hookrightarrow \mathfrak{gl}_n$ and $\mathfrak{sp}_n \hookrightarrow \mathfrak{gl}_n$.

But the quantum vector representations do not agree, i.e.
$\mathcal{U}_v(\mathfrak{so}_n) \not\hookrightarrow \mathcal{U}_v(\mathfrak{gl}_n)$ and $\mathcal{U}_v(\mathfrak{sp}_n) \not\hookrightarrow \mathcal{U}_v(\mathfrak{gl}_n)$.

This is bad. Idea: Invent new quantizations such that
$\mathcal{U}'_v(\mathfrak{so}_n) \hookrightarrow \mathcal{U}_v(\mathfrak{gl}_n)$ and $\mathcal{U}'_v(\mathfrak{sp}_n) \hookrightarrow \mathcal{U}_v(\mathfrak{gl}_n)$.

w

h



Ok from this picture   Not ok from this picture

Back

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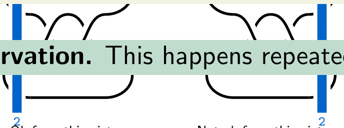
w

h



Observation. This happens repeatedly.

Ok from this picture

Not ok from this picture

◂ Back

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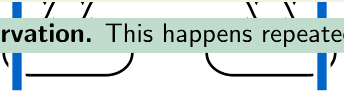
which gives ... right
handedness ...

This happens really often. In our case we have basically

$$\mathfrak{gl}_1 \hookrightarrow \mathfrak{sl}_2, (t) \mapsto \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$$

which does not quantize properly...



Observation. This happens repeatedly.

2 Ok from this picture

Not ok from this picture 2

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A version of Schur's remarkable duality.

Plain old $\mathfrak{sl}_2$:
Acts by matrices.

The symmetric group:
Acts by permutation.

$$\mathcal{U}_1(\mathfrak{sl}_2) \circlearrowright \underbrace{\mathbb{C}_1^2 \otimes \cdots \otimes \mathbb{C}_1^2}_{d \text{ times}} \circlearrowleft \mathcal{H}_1(\mathrm{A})$$

Schur ∼1901. The natural actions of $\mathcal{U}_1(\mathfrak{sl}_2)$ and $\mathcal{H}_1(\mathrm{A})$ on $(\mathbb{C}_1^2)^{\otimes d} = (\mathbb{C}^2)^{\otimes d}$ commute and generate each other's centralizer.

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A version of Schur's remarkable duality.

$$\mathcal{U}_1(\mathfrak{sl}_2) \circlearrowright \mathbb{C}_1^2 \otimes \cdots \otimes \mathbb{C}_1^2 \circlearrowleft \mathcal{H}_1(\mathrm{A})$$

$$\|$$

$$\underbrace{\mathbb{C}_1^2 \otimes \cdots \otimes \mathbb{C}_1^2}_{d \text{ times}}$$

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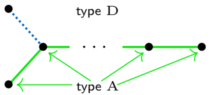
A version of Schur's remarkable duality.

Weyl group of type D:



$$\mathcal{U}_1(\mathfrak{sl}_2) \circlearrowright \mathbb{C}_1^2 \otimes \cdots \otimes \mathbb{C}_1^2 \circlearrowleft \mathcal{H}_1(\mathrm{A})$$

$$\begin{array}{ccc} & \| & \cap \\ & \underbrace{\mathbb{C}_1^2 \otimes \cdots \otimes \mathbb{C}_1^2}_{d \text{ times}} & \mathcal{H}_1(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

Ignore the component group $\mathbb{Z}/2\mathbb{Z}$.

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A version of Schur's remarkable duality.

$$\begin{array}{ccc} \mathcal{U}_1(\mathfrak{sl}_2) \circlearrowright & \mathbb{C}_1^2 \otimes \cdots \otimes \mathbb{C}_1^2 & \circlearrowleft \mathcal{H}_1(\mathrm{A}) \\ & \| & \cap \\ & \underbrace{\mathbb{C}_1^2 \otimes \cdots \otimes \mathbb{C}_1^2}_{d \text{ times}} & \circlearrowleft \mathcal{H}_1(\mathrm{D}) \rtimes {}^{\mathbb{Z}}\!/_{2\mathbb{Z}} \end{array}$$

Acts by signed permutations.

Back

A version of Schur's remarkable duality.

$$\begin{array}{ccc} \mathcal{U}_1(\mathfrak{sl}_2) \circlearrowright & \mathbb{C}_1^2 \otimes \cdots \otimes \mathbb{C}_1^2 & \circlearrowleft \mathcal{H}_1(\mathrm{A}) \\ \cup & \| & \cap \\ ?? \circlearrowright & \underbrace{\mathbb{C}_1^2 \otimes \cdots \otimes \mathbb{C}_1^2}_{d \text{ times}} & \circlearrowleft \mathcal{H}_1(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

Back

A version of Schur's remarkable duality.

The antidiagonal embedding:

$\mathfrak{gl}_1 \hookrightarrow \mathfrak{sl}_2,\ (t) \mapsto \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$

$$\begin{array}{ccccc} \mathcal{U}_1(\mathfrak{sl}_2) & \circlearrowright & \mathbb{C}_1^2 \otimes \cdots \otimes \mathbb{C}_1^2 & \circlearrowleft & \mathcal{H}_1(\mathrm{A}) \\ \cup & & \| & & \cap \\ \mathcal{U}_1(\mathfrak{gl}_1) & \circlearrowright & \underbrace{\mathbb{C}_1^2 \otimes \cdots \otimes \mathbb{C}_1^2}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_1(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

Acts by restriction.

Regev ∼1983. The actions of $\mathcal{U}_1(\mathfrak{gl}_1)$ and $\mathcal{H}_1(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}$ on $(\mathbb{C}_1^2)^{\otimes d}$ commute and generate each other's centralizer.

A version of Schur's remarkable duality.

$$\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) \circlearrowright \mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2 \circlearrowleft \mathcal{H}_{\mathrm{v}}(\mathrm{A})$$

Jimbo ∼1985. The natural actions of $\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2)$ and $\mathcal{H}_{\mathrm{v}}(\mathrm{A})$

on $(\mathbb{C}_{\mathrm{v}}^2)^{\otimes d} = (\mathbb{C}(\mathrm{v})^2)^{\otimes d}$ commute and generate each other's centralizer.

Back

A version of Schur's remarkable duality.

$$\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) \circlearrowright \mathbb{C}^2_{\mathrm{v}} \otimes \cdots \otimes \mathbb{C}^2_{\mathrm{v}} \circlearrowleft \mathcal{H}_{\mathrm{v}}(\mathrm{A})$$

$$\|$$

$$\underbrace{\mathbb{C}^2_{\mathrm{v}} \otimes \cdots \otimes \mathbb{C}^2_{\mathrm{v}}}_{d \text{ times}}$$

Back

A version of Schur's remarkable duality.

$$
\begin{array}{ccccc}
\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) & \circlearrowright & \mathbb{C}^2_{\mathrm{v}} \otimes \cdots \otimes \mathbb{C}^2_{\mathrm{v}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{A}) \\
 & & \| & & \cap \\
 & & \underbrace{\mathbb{C}^2_{\mathrm{v}} \otimes \cdots \otimes \mathbb{C}^2_{\mathrm{v}}}_{d \text{ times}} & & \mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}
\end{array}
$$

Back

A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) \circlearrowright & \mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2 & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{A}) \\ & \| & & \cap \\ & \underbrace{\mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

Quantizes nicely.

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A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) & \circlearrowright & \mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2 & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{A}) \\ \cup & & \| & & \cap \\ ?? & \circlearrowright & \underbrace{\mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

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A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) & \circlearrowright & \mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2 & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{A}) \\ \cup & & \| & & \cap \\ \mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_1) & \circlearrowright & \underbrace{\mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

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A version of Schur's remarkable duality.

$$
\begin{array}{ccccc}
\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) & \circlearrowright & \mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2 & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{A}) \\
\not\cup & & \| & & \cap \\
\mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_1) & \circlearrowright & \underbrace{\mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}
\end{array}
$$

Does not embed.

◂ Back

A version of Schur's remarkable duality.

$$\mathcal{U}_v(\mathfrak{sl}_2) \circlearrowright \mathbb{C}_v^2 \otimes \cdots \otimes \mathbb{C}_v^2 \circlearrowleft \mathcal{H}_v(\mathrm{A})$$

$$\not\cup \qquad \qquad || \qquad \qquad \cap$$

$$\mathcal{U}_v(\mathfrak{gl}_1) \not\circlearrowright \underbrace{\mathbb{C}_v^2 \otimes \cdots \otimes \mathbb{C}_v^2}_{d \text{ times}} \circlearrowleft \mathcal{H}_v(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}$$

No commuting action.

◂ Back

A version of Schur's remarkable duality.

$$\mathcal{U}_v(\mathfrak{sl}_2) \circlearrowright \mathbb{C}_v^2 \otimes \cdots \otimes \mathbb{C}_v^2 \circlearrowleft \mathcal{H}_v(\mathrm{A})$$

$$\not\cup \qquad \qquad \| \qquad \qquad \cap$$

$$\mathcal{U}_v(\mathfrak{gl}_1) \not\circlearrowright \underbrace{\mathbb{C}_v^2 \otimes \cdots \otimes \mathbb{C}_v^2}_{d \text{ times}} \circlearrowleft \mathcal{H}_v(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}$$

Back

A version of Schur's remarkable duality.

$$
\begin{array}{ccccc}
\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) & \circlearrowright & \mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2 & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{A}) \\
 & & \| & & \cap \\
\mathrm{c}\,\mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_1) & & \underbrace{\mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}
\end{array}
$$

◀ Back

A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) \circlearrowright & \mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2 & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{A}) \\ \cup & \| & & \cap \\ c\,\mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_1) & \underbrace{\mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

Is a subalgebra.

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A version of Schur's remarkable duality.

$$
\begin{array}{ccccc}
\mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) & \circlearrowright & \mathbb{C}^2_{\mathrm{v}} \otimes \cdots \otimes \mathbb{C}^2_{\mathrm{v}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{A}) \\
\cup & & \| & & \cap \\
\mathrm{c}\,\mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_1) & \circlearrowright & \underbrace{\mathbb{C}^2_{\mathrm{v}} \otimes \cdots \otimes \mathbb{C}^2_{\mathrm{v}}}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}
\end{array}
$$

Act by restriction.

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A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) & \circlearrowright & \mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2 & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{A}) \\ \cup & & \| & & \cap \\ \mathrm{c}\,\mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_1) & \circlearrowright & \underbrace{\mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

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Ehrig–Stroppel, Bao–Wang ∼2013. The actions of $\mathrm{c}\,\mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_1)$ and $\mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z}$ on $(\mathbb{C}_{\mathrm{v}}^2)^{\otimes d}$ commute and generate each other's centralizer.

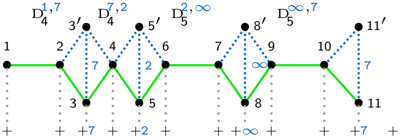
A version of Schur's remarkable duality.

$$\mathcal{U}_v(\mathfrak{sl}_2) \circlearrowright \mathbb{C}_v^2 \otimes \cdots \otimes \mathbb{C}_v^2 \circlearrowleft \mathcal{H}_v(\mathrm{A})$$

$\cup \qquad || \qquad \cap$

Hope.

The same works for the Coxeter diagrams



But, again, only in the special case of type ABCD this is known.

A version of Schur's remarkable duality.

$$\begin{array}{ccccc} \mathcal{U}_{\mathrm{v}}(\mathfrak{sl}_2) & \circlearrowright & \mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2 & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{A}) \\ \cup & & \| & & \cap \\ \mathrm{c}\,\mathcal{U}_{\mathrm{v}}(\mathfrak{gl}_1) & \circlearrowright & \underbrace{\mathbb{C}_{\mathrm{v}}^2 \otimes \cdots \otimes \mathbb{C}_{\mathrm{v}}^2}_{d \text{ times}} & \circlearrowleft & \mathcal{H}_{\mathrm{v}}(\mathrm{D}) \rtimes \mathbb{Z}/2\mathbb{Z} \end{array}$$

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Message to take away. Coideal naturally appear in Schur–Weyl-like games.

And these pull the strings from the background for tangle and link invariants.

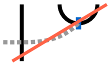
Slogan. Right generated = building with left- and right-Lego pieces.

$$\text{left-Lego} \otimes \text{right-Lego} = \text{new left-Lego},$$

$$\text{e.g.} \quad \emptyset \otimes \text{o} = \text{o}$$

$$\text{any-Lego} \circ \text{any-Lego} = \text{new any-Lego} \quad \text{but}$$

$$\text{left-Lego} \otimes \text{right-Lego} = \text{new left-Lego}$$

e.g.  $\otimes$ =  but 

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