The diagrammatic beauty of $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_n))$: Part II

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The categorified story

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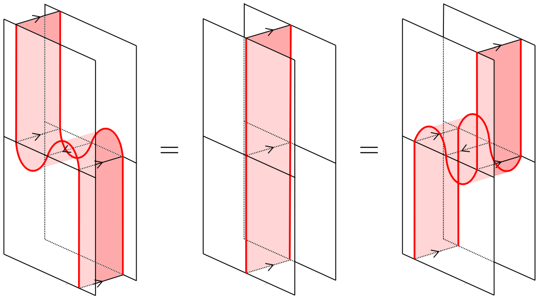


1. What is categorification?
    - From the viewpoint of "natural" constructions
    - From the viewpoint of topology
    - From the viewpoint of algebra
2. The $\mathfrak{sl}_2$-web algebra
    - The algebra
    - Straightening helps again!
    - Categorified q-skew Howe duality
3. Connection to the $\mathfrak{sl}_n$-link-homology
    - The Khovanov homology
    - Connection to the $\mathfrak{sl}_2$-web algebra
    - The KL-R algebra and Khovanov homology

What is categorification?

Forced to reduce this presentation to one sentence, the author would choose:

Interesting integers are shadows of richer structures in categories.

The basic idea can be seen as follows. Take a "set-based" structure S and try to find a "category-based" structure $\mathcal{C}$ such that S is just a shadow of $\mathcal{C}$.

Categorification, which can be seen as "remembering" or "inventing" information, comes with an "inverse" process called decategorification, which is more like "forgetting" or "identifying".

Note that decategorification should be easy.

The underlying basic example

Take $\mathcal{C} = \mathbf{FinVec}_K$ for a fixed field K, i.e. objects are finite dimensional K-vector spaces $V, V', \dots$ and morphisms are K-linear maps $f\colon V \to V'$ between them. $\mathcal{C}$ categorifies $\mathbb{N}$: We can go back by taking the dimension $\dim V \in \mathbb{N}$.

What is the upshot? Note the following:

- Much information is lost if we only consider $\mathbb{N}$, i.e. we can only say that two objects are isomorphic (aka equal) instead of how they are isomorphic. Thus,

$$n = n' \Leftrightarrow V \cong V'.$$

- A vector space can carry additional structure as for example inner products.
- We have the power of linear algebra between V and V', i.e. $\hom_K(V, W)$.

Never forget the original structure

The structure of $\mathbb{N}$ is reflected on a "higher" level!

- The product and coproduct $\oplus$ and the monoidal structure $\otimes_K$ categorify addition and multiplication, i.e. $\dim(V \oplus V') = \dim V + \dim V'$ and $\dim(V \otimes_K V') = \dim V \cdot \dim V'$.
- The zero object 0 and the identity of $\otimes_K$ categorify the identities, i.e $V \oplus 0 \simeq V$ and $V \otimes_K K \simeq V$.
- We have $V \hookrightarrow W$ iff $\dim V \leq \dim W$ and $V \twoheadrightarrow W$ iff $\dim V \geq \dim W$, i.e. injections and surjections categorify the order relation.

One can write down the categorified statements of other properties as "Addition and multiplication are associative and commutative", "Multiplication distributes over addition" or "Addition and multiplication preserve order".

Integer based invariants

A more topological flavoured example goes back to Riemann (1857), Betti (1871) and Poincaré (1895): The Betti numbers $b_k(X)$ and Euler characteristic $\chi(X)$ of a reasonable topological space X. Noether, Hopf and Alexandroff (1925) "categorified" these invariants as follows.

If we lift $m, n \in \mathbb{N}$ to the two K-vector spaces V, W with dimensions $\dim V = m, \dim W = n$, then the difference $m - n$ lifts to the complex

$$0 \longrightarrow V \xrightarrow{d} W \longrightarrow 0,$$

for any linear map d and V in even homology degree. As before, some of the basic properties of the integers $\mathbb{Z}$ can be lifted to the category $\mathbf{Kom}_b(\mathcal{C})$.

Conclusion (Noether): The homology groups $H_k(X, \mathbb{Z})$ categorify $b_k(X)$ and chain complexes $(C(X), c_*)$ categorify $\chi(X)$.

Well-known upshots

We note the following observations.

- The space $H_k(X, \mathbb{Z})$ is a graded abelian group and more information of the space X is encoded. Again, homomorphisms between the groups tell how some groups are related.
- Singular homology works for all topological spaces and the homological Euler characteristic can be defined for a huge class of spaces.
- The homology extends to a functor and provides information about continuous maps as well.
- More sophisticated constructions like multiplication in cohomology provide even more information.
- Although it is not the main point: The $H_k(X, \mathbb{Z})$ are better invariants.

Categorified symmetries

Another viewpoint comes from representation theory. Let A be some algebra, M be a A-module and $\mathcal{C}$ be a suitable category.

$$\text{“Usual”} \rightsquigarrow \text{“Higher”}$$

$$a \mapsto f_a \in \mathrm{End}(M) \rightsquigarrow a \mapsto \mathcal{F}_a \in \mathrm{End}(\mathcal{C})$$

$$(f_{a_1} \cdot f_{a_2})(m) = f_{a_1 a_2}(m) \rightsquigarrow (\mathcal{F}_{a_1} \circ \mathcal{F}_{a_2})\binom{X}{\varphi} \cong \mathcal{F}_{a_1 a_2}\binom{X}{\varphi}$$

A (weak) categorification of the A-module M should be though of a categorical action of A on a suitable category $\mathcal{C}$ with an isomorphism ψ such that

$$\begin{tikzcd} K_0(\mathcal{C}) \otimes A \arrow[r, "{[\mathcal{F}_a]}"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowleft"] & K_0(\mathcal{C}) \otimes A \arrow[d, "\psi"] \\ M \arrow[r, "\cdot a"] & M. \end{tikzcd}$$

There is no direct minus

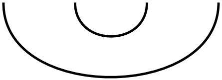
We have several upshots again.

- The natural transformations between functors give information invisible in "classical" representation theory. This gives a hint that we can go even "higher", e.g. actions of 2-categories on 2-categories.
- If $\mathcal{C}$ is suitable, e.g. module categories over an algebra, then its indecomposable objects X gives a basis $[X]$ of M with positivity properties.
- In particular, consider A as a A-module. Then $[X]$ gives a basis of A with positive structure coefficients c_k^{ij} via

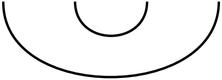
$$X_{a_i} \otimes X_{a_j} \cong \bigoplus_k X_{a_k}^{c_k^{ij}} \rightsquigarrow a_i a_j = \sum_k c_k^{ij} a_k,\ c_k^{ij} \in \mathbb{N}.$$

Natural transformations between $\mathfrak{sl}_2$-webs

Recall that we are interested in the intertwiner of $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$ or in pictures

$$u\colon \bar{\mathbb{Q}}(q) \to \bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}}^2 =$$ 

How can we describe higher structure between these intertwiners? That is, what can we say about

hom ( , )?

Note that the intertwiners "are" 1-dimensional. Thus, the natural transformations between them should be 2-dimensional.

Moreover, we can again "restrict" to $\mathbf{Inv}_{\mathbf{U}_q(\mathfrak{sl}_2)}(\bigotimes_{2n}\bar{\mathbb{Q}}^2)$. Recall that the invariant tensors form a $\bar{\mathbb{Q}}(q)$-vector space with basis $\mathrm{Arc}(n)$, that is all $\mathfrak{sl}_2$-arc diagrams with $2n$ boundary points.

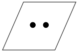
$\mathfrak{sl}_2$-foams

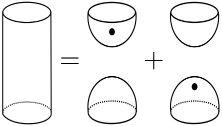
A $\mathfrak{sl}_2$-pre-foam is a cobordism between two $\mathfrak{sl}_2$-webs. Composition consists of placing one $\mathfrak{sl}_2$-pre-foam on top of the other. The following are called the saddle up and down respectively.

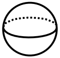
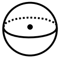
They have dots that can move freely about the facet on which they belong. Define the $q-$degree of a $\mathfrak{sl}_2$-foam F with d dots and b boundary components as

$$\mathrm{qdeg}(F) = -\chi(F) + 2d + \frac{b}{2}.$$

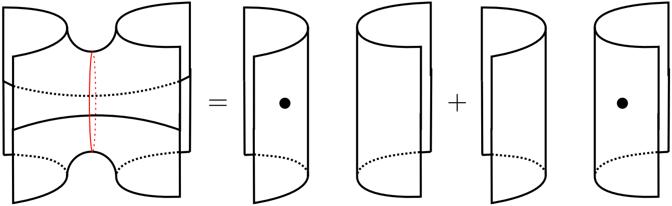
A $\mathfrak{sl}_2$-foam is a formal $\bar{\mathbb{Q}}$-linear combination of isotopy classes of $\mathfrak{sl}_2$-pre-foams modulo the following (degree preserving!) relations.

 $= 0$ (2D)

 (NC)

 $= 0$,  $= 1$ (S)

The relations $\ell = (2D, NC, S)$ suffice to evaluate $\mathfrak{sl}_2$-foam without boundary!



The $\mathfrak{sl}_2$-foam category

$\mathbf{Foam}_2$ is the $\mathbb{Z}$-graded, 2-category of $\mathfrak{sl}_2$-foams consisting of:

- The objects are sequences of points in the interval $[0,1]$.
- The 1-cells are formal direct sums of $\mathbb{Z}$-graded $\mathfrak{sl}_2$-webs with boundary corresponding to the sequences of points for the source and target.
- The 2-cells are formal matrices of $\bar{\mathbb{Q}}$-linear combinations of degree-zero dotted $\mathfrak{sl}_2$-foams modulo isotopy and $\mathfrak{sl}_2$-foam relations.
- Vertical composition $\circ_v$ is stacking on top of each other and horizontal composition $\circ_h$ is stacking next to each other. We write $\hom_{\mathbf{Foam}_2}(u,v) = \hom(u,v)$.

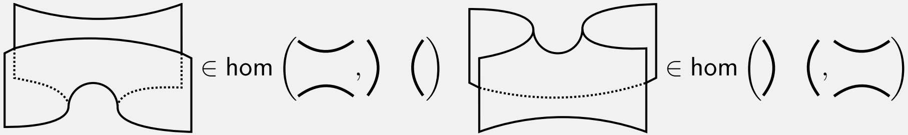
The $\mathfrak{sl}_2$-foam homology of a closed $\mathfrak{sl}_2$-web $w\colon \emptyset \to \emptyset$ is defined by

$$\mathcal{F}(w) = \hom_{\mathbf{Foam}_2}(\emptyset, w) = \hom(\emptyset, w).$$

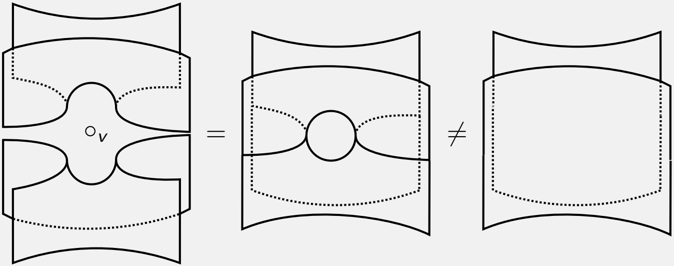
$\mathcal{F}(w)$ is a $\mathbb{Z}$-graded, $\bar{\mathbb{Q}}$-vector space.

Example

A saddles are 2-morphisms



Vertical composition gives a non-trivial "natural transformation" in hom($\asymp$, $\asymp$)!



The $\mathfrak{sl}_2$-web algebra

Definition(Khovanov 2002)

The $\mathfrak{sl}_2$-web algebra $H_2(n)$ is defined by

$$H_2(n) = \bigoplus_{u,v \in \mathrm{Arc}(n)} {}_uH_v,$$

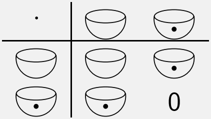
with

$${}_uH_v = \mathcal{F}(u^*v)\{n\}, \text{ i.e. all } \mathfrak{sl}_2\text{-foams: } \emptyset \to u^*v.$$

Multiplication is defined by composition $\mathcal{F}(u^*v) \cong \mathrm{hom}(u,v)$.

Example

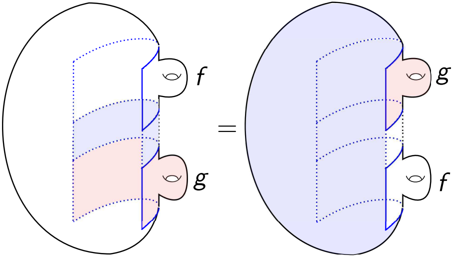
Since * = , we have $H_2(1) = \langle$, $\rangle_{\bar{\mathbb{Q}}} \cong \bar{\mathbb{Q}}[X]/X^2$ with multiplication



It's Frobenius!

There is a trace form $\mathrm{tr}\colon H_2(n) \to \bar{\mathbb{Q}}$ given by closing a $\mathfrak{sl}_2$-foam f_u with $\mathbf{1}_u$.

The trace is non-degenerated and symmetric, i.e. $\mathrm{tr}(fg) = \mathrm{tr}(gf)$:



Theorem(Khovanov 2002)

The algebra $H_2(n)$ is a graded, finite dimensional, symmetric Frobenius algebra.

Higher representation theory

Moreover, we define

$$W_{(2^\ell)} = \bigoplus_{\vec{k}\in\Lambda(2\ell,2\ell)_3} W_2(\vec{k}) \cong \bigoplus_{\vec{k}\in\Lambda(2\ell,2\ell)_3} \mathbf{Inv}_{\mathbf{U}_q(\mathfrak{sl}_2)}(\vec{k})$$

on the level of $\mathfrak{sl}_2$-webs and on the level of $\mathfrak{sl}_2$-foams we define (below the technical definition, but think: Take the module category over H_n)

$$\mathcal{W}^{(p)}_{(2^\ell)} = \bigoplus_{\vec{k}\in\Lambda(2\ell,2\ell)_2} H_2(\vec{k})\text{-}(\mathrm{p})\mathbf{Mod}_{gr}.$$

With this constructions we obtain the categorification result.

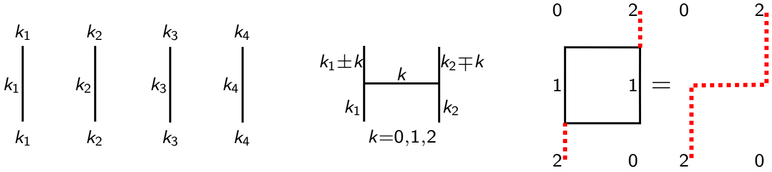
Theorem(Khovanov 2002, Brundan-Stroppel 2008)

$$K_0(\mathcal{W}_{(2^\ell)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \bar{\mathbb{Q}}(q) \cong W^*_{(2^\ell)} \text{ and } K_0^{\oplus}(\mathcal{W}^p_{(2^\ell)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \bar{\mathbb{Q}}(q) \cong W_{(2^\ell)}.$$

This is nice, but how to generalize to $n > 2$?

Recall: Rigid $\mathfrak{sl}_2$-spider

Recall that the rigid version of $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))$ consists of



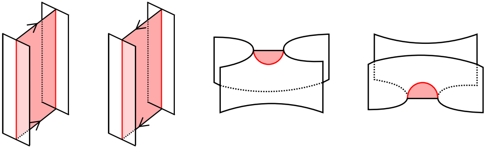
with labels $k_i \in \{0, 1, 2\}$. We only picture edges labeled 1 in black and edges labeled 2 as a dotted leashes. Moreover, we picture a "left-plus-ladder" with an arrow to the left and vice versa for a "right-plus-ladder".

The advantage of this was that it was "easy" to generalize to $n > 2$ and we were able to see an $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-action on the $\mathbf{U}_q(\mathfrak{sl}_2)$-webs!

Rigid $\mathfrak{sl}_2$-foams: Sloppy version

Instead of giving the formal definition of the rigid $\mathfrak{sl}_2$-foam category $\mathbf{Foam}_2$ let me just give some examples.

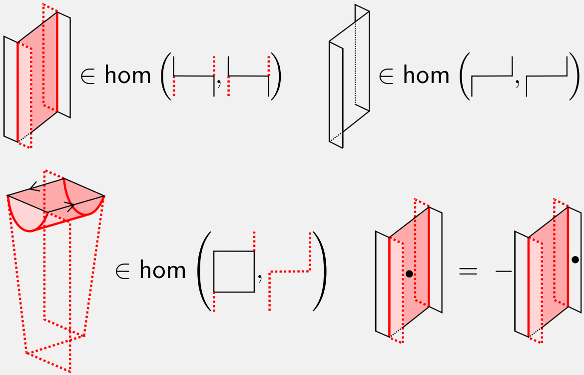
- The rigid versions of the $\mathfrak{sl}_2$-foams are locally generated by



  where facet get the numbers of their incident edges. Facets labeled 0 are removed, facets labeled 1 really exists and facet labeled 2 are pictured using leashes as boundary (but they exist). Thus, these will be singular surfaces!
- The singular surfaces above are called identities and singular saddles.
- Facets with label 1 are allowed to carry dots. Dots move freely on a facet but are not allowed to cross singular lines.
- There are some relations and the 2-category is graded by a slight rearrangement of the geometrical Euler characteristic.

Exempli gratia

Rigid examples



Think: Leash-faces take care of sign-issues coming from the fact that $\Lambda^0\bar{\mathbb{Q}}^2$ and its dual $\Lambda^2\bar{\mathbb{Q}}^2$ are only isomorphic. Moreover: "Easy" to generalize, since one needs singular surfaces already for non-rigid $\mathfrak{sl}_3$-foams.

The overview

$$\begin{tikzcd}
\mathcal{U}(\mathfrak{sl}_d) \arrow[rr, squiggly, "\text{``Higher'' } q\text{-skew Howe}", "\mathcal{U}(\mathfrak{sl}_d)\text{ acts}"'] \arrow[d, "K_0^{\oplus}"'] & \text{How it should be!} & H_n(\vec{k})\text{-}\,(\mathrm{p})\mathbf{Mod}_{gr} \arrow[d, "K_0^{\oplus}"] \\
\dot{\mathbf{U}}_q(\mathfrak{sl}_d) \arrow[rr, squiggly, "q\text{-skew Howe}", "\dot{\mathbf{U}}_q(\mathfrak{sl}_d)\text{ acts}"'] & & W_n(\vec{k})
\end{tikzcd}$$

This is how it should be: There is an $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-action on the $\mathfrak{sl}_n$-web spaces (for us it was mostly the case $n = 2$). Moreover, suitable module categories over the $\mathfrak{sl}_n$-web algebras $H_n(\vec{k})$ categorify these spaces.

On the left side: There is Khovanov-Lauda's categorification of $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$ denoted by $\mathcal{U}(\mathfrak{sl}_d)$ (which I briefly recall on the next slides).

Conclusion: There should be a 2-action of $\mathcal{U}(\mathfrak{sl}_d)$ on the top right!

Khovanov-Lauda's 2-category $\mathcal{U}(\mathfrak{sl}_d)$

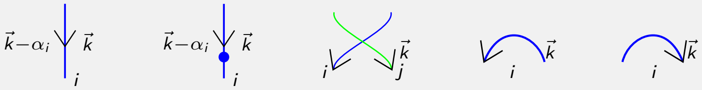
Idea(Khovanov-Lauda)

The algebra $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$ has a basis with surprisingly nice behaviour, e.g. positive structure coefficients. Thus, there should be a categorification of $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$ pulling the strings from the background!

Definition(Khovanov-Lauda 2008)

The 2-category $\mathcal{U}(\mathfrak{sl}_d)$ is defined by (everything suitably $\mathbb{Z}$-graded and $\bar{\mathbb{Q}}$-linear):

- The objects in $\mathcal{U}(\mathfrak{sl}_d)$ are the weights $\vec{k} \in \mathbb{Z}^{d-1}$.
- The 1-morphisms are finite formal sums of the form $\mathcal{E}_{\underline{i}}\mathbf{1}_{\vec{k}}\{t\}$ and $\mathcal{F}_{\underline{i}}\mathbf{1}_{\vec{k}}\{t\}$.
- 2-cells are graded, $\bar{\mathbb{Q}}$-vector spaces generated by compositions of diagrams (additional ones with reversed arrows) as illustrated below plus relations.



$\mathfrak{sl}_2$-foamation (works for all $n > 1$!)

We define a 2-functor

$$\Psi \colon \mathcal{U}(\mathfrak{sl}_d) \to \bigoplus_{\vec{k}\in\Lambda(2\ell,2\ell)_2} H_2(\vec{k})\text{-}\,(\mathrm{p})\mathbf{Mod}_{gr} = \mathcal{W}^{(p)}_{(2^\ell)}$$

called $\mathfrak{sl}_2$-foamation, in the following way.

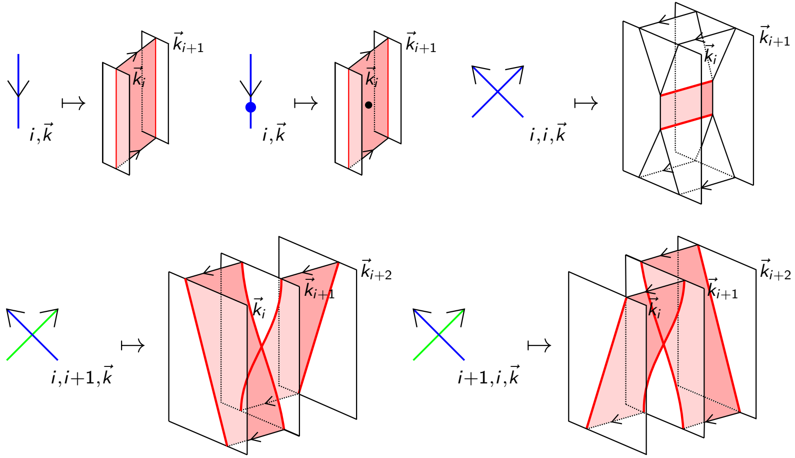
On objects: The functor is defined by sending an $\mathfrak{sl}_d$-weight $\vec{k} = (\vec{k}_1, \dots, \vec{k}_{d-1})$ to an object $\Psi(\lambda)$ of $\mathcal{W}^{(p)}_{(2^\ell)}$ by

$$\Psi(\lambda) = S,\ S = (a_1, \dots, a_\ell),\ a_i \in \{0,1,2\},\ \lambda_i = a_{i+1} - a_i,\ \sum_{i=1}^{\ell} a_i = 2\ell.$$

On morphisms: The functor on morphisms is by glueing the ladder webs from before on top of the $\mathfrak{sl}_2$-webs in $W_{(2^\ell)} = \bigoplus_{\vec{k}\in\Lambda(2\ell,2\ell)_2} W_2(\vec{k})$.

$\mathfrak{sl}_2$-foamation (Part 2)

On 2-**cells:** We define



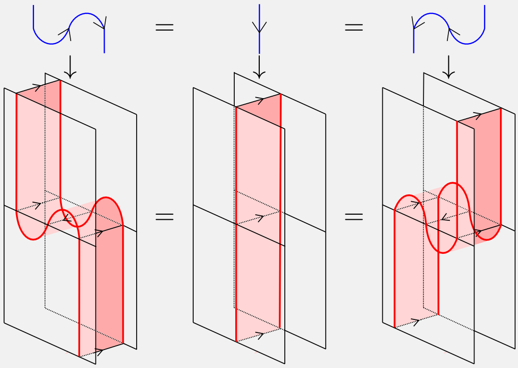
And some others (that are not important today).

Everything fits

Theorem

The 2-functor $\Psi\colon \mathcal{U}(\mathfrak{sl}_d) \to \mathcal{W}^{(p)}_{(2^\ell)}$ categorifies q-skew Howe duality.

Example without labels (One has to check well-definedness!)



Khovanov's categorification of the Jones polynomial

Recall the rules for the Jones polynomial.

- $\langle \emptyset \rangle = 1$ (normalization).
- $\langle \diagup\!\!\!\!\diagdown \rangle = \langle)(\rangle - q\langle \asymp \rangle$ (recursion step 1).
- $\langle \bigcirc \amalg L_D \rangle = [2] \cdot \langle L_D \rangle$ (recursion step 2).
- $[2]J(L_D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L_D \rangle$ (Re-normalization).

Definition/Theorem(Khovanov 1999)

Let L_D be a diagram of an oriented link. Denote by $A = \bar{\mathbb{Q}}[X]/X^2$ the dual numbers with $\mathrm{qdeg}(1) = 1$ and $\mathrm{qdeg}(X) = -1$ - this is a Frobenius algebra with a given comultiplication Δ. We assign to it a chain complex $[\![L_D]\!]$ of $\mathbb{Z}$-graded $\bar{\mathbb{Q}}$-vector spaces using the categorified rules:

- $[\![\emptyset]\!] = 0 \to \bar{\mathbb{Q}} \to 0$ (normalization).
- $[\![\diagup\!\!\!\!\diagdown]\!] = \Gamma\left(0 \to [\![ )( ]\!] \xrightarrow{d} [\![\asymp]\!] \to 0\right)$ with $d = m, \Delta$ (recursion step 1).
- $[\![\bigcirc \amalg L_D]\!] = A \otimes_{\bar{\mathbb{Q}}} [\![L_D]\!]$ (recursion step 2).
- $\mathbf{Kh}(L_D) = [\![L_D]\!][-n_-]\{n_+ - 2n_-\}$ (Re-normalization).

Then $\mathbf{Kh}(\cdot)$ is an invariant of oriented links whose graded Euler characteristic gives $\chi_q(\mathbf{Kh}(L_D)) = [2]J(L_D)$.

This is better than the Jones polynomial

- Khovanov's construction can be extended to a categorification of the HOMFLY-PT polynomial.
- It is functorial (in this formulation only up to a sign).
- Kronheimer and Mrowka showed that Khovanov homology detects the unknot. This is still an open question for the Jones polynomial.
- Rasmussen obtained from the homology an invariant that "knows" the slice genus and used it to give a combinatorial proof of the Milnor conjecture.
- Rasmussen also gives a way to combinatorial construct exotic $\mathbb{R}^4$.
- The categorification is not unique, e.g. the so-called "odd Khovanov homology" differs over $\bar{\mathbb{Q}}$.
- Before I forget: It is a strictly stronger invariant.

History repeats itself: After Khovanov lots of other homologies of "Khovanov-type" were discovered. So we need to understand this better, e.g. how to extend this to tangles?

Resolutions are $H_2(m) - H_2(n)$-bimodules

Let $T_D^{m,n}$ be a oriented diagram of a tangle with numbered crossings $c_1, \dots, c_r$ and $2m$ bottom and $2n$ top boundary points. A resolution $R(T_D^{m,n})_k$ of $T_D^{m,n}$ is a local replacement of the c_i by either) (or $\asymp$.

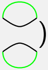
Definition

Define a $H_2(m) - H_2(n)$-bimodule for $a = R(T_D^{m,n})_k$ by

$$\mathcal{F}(a) = \bigoplus_{u \in \mathrm{Arc}(n), v \in \mathrm{Arc}(m)} \mathcal{F}(u^* a v),$$

that is all $\mathfrak{sl}_2$-foams $\emptyset \to u^* a v$ for all suitable u, v. This is an $H_2(m) - H_2(n)$-bimodule where the elements of $H_2(m)$ act by stacking from the bottom and the elements of $H_2(n)$ act by stacking from the top.

Example

$H_2(1) - H_2(1)$-bimodules: $\mathcal{F}() \, ()= \mathrm{hom}(\emptyset,$  $)$ and $\mathcal{F}(\asymp) = \mathrm{hom}(\emptyset,$  $)$.

How to build a chain complex $\mathbf{Kh}(T_D^{m,n})$

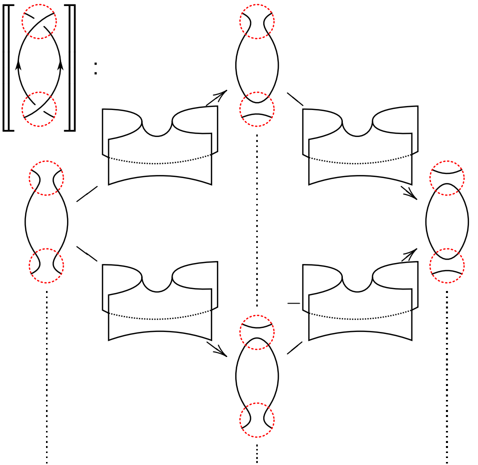
For an oriented diagram T_D with $r = n_+ + n_-$ crossings and resolutions $R(T_D^{m,n})_k$ ordered into $r+1$-columns in a suitable way - for two consecutive columns the local difference is $)\,(\to \overset{\smile}{\frown}$ or $\overset{\smile}{\frown} \to)\,($.

- For $i = 0, \dots, n$ the $i - n_-$ chain module is the formal direct sum of all $H_2(m) - H_2(n)$-bimodules for the resolutions of column i.
- Between resolutions of column i and $i+1$ the morphisms should be saddles between the resolutions. These are $H_2(m) - H_2(n)$-bimodule homomorphisms.
- Extra formal signs to make everything well-defined - skipped today.
- Shift everything suitable and obtain $\mathbf{Kh}(T_D^{m,n})$ - a complex of $H_2(m) - H_2(n)$-bimodules.

Theorem(Khovanov 2002)

The complex $\mathbf{Kh}(T_D^{m,n})$ is a functorial invariant of oriented tangles.

Exempli gratia - Khovanov homology using $\mathfrak{sl}_2$-foams



$$\mathcal{F}(\ ) \xrightarrow{d_1} \mathcal{F}(\ ) \oplus \mathcal{F}(\ ) \xrightarrow{d_2} \mathcal{F}(\ )$$

This is a $H_2(1) - H_2(1)$-bimodule!

Shouldn't $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_d)$ be sufficient?

We gave a method to obtain the $\mathbf{U}_q(\mathfrak{sl}_n)$-link polynomials using $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-highest weight representation theory because we could restrict to F's: $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_d)$ suffices.

Moreover, the $\mathfrak{sl}_n$-foamation connects the $\mathfrak{sl}_n$-web algebras $H_n(\Lambda)$ with Khovanov-Lauda's categorification $\mathcal{U}(\mathfrak{sl}_d)$.

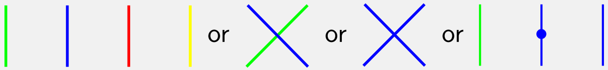
Moreover, which I explain in a second, there is a (easier to work with) "version" of $\mathcal{U}(\mathfrak{sl}_d)$, called the Khovanov-Lauda and Rouquier (KL-R) algebra R_d, and a cyclotomic quotient R_Λ ,called cyclotomic KL-R algebra, which categorify $\mathcal{U}_q^-(\mathfrak{sl}_d)$ and its highest weight representation V_Λ respectively.
This gives two natural questions:

- On the level of $\mathfrak{sl}_n$-link polynomials only F's suffice. Shouldn't the "same" hold for the $\mathfrak{sl}_n$-link homologies?
- If so, how can we use the cyclotomic KL-R algebra to "explain" the $\mathfrak{sl}_n$-link homologies as instances of $\mathcal{U}_q^-(\mathfrak{sl}_d)$-highest weight representation theory.

The KL-R algebra

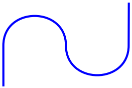
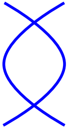
Definition/Theorem(Khovanov-Lauda, Rouquier 2008/2009)

Let R_d be a certain direct sum of subalgebras of $\mathrm{hom}_{\mathcal{U}(\mathfrak{sl}_d)}(\mathcal{F}_{\underline{i}}\mathbf{1}_{\vec{k}}\{t\}, \mathcal{F}_{\underline{j}}\mathbf{1}_{\vec{k}'}\{t\})$. Thus only downwards pointing arrows - aka only F's. That is, working with R_d enables us to ignore orientations and consider only diagrams of the form



The KL-R algebra has the structure of a $\mathbb{Z}$-graded, $\bar{\mathbb{Q}}$-algebra. We have (note that this works for more general $\mathfrak{g}$)

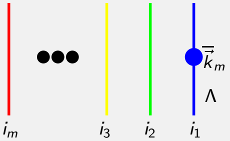
$$\dot{\mathbf{U}}_q^-(\mathfrak{sl}_d) \cong K_0^{\oplus}(R_d) \otimes_{\mathbb{Z}[q,q^{-1}]} \bar{\mathbb{Q}}(q).$$

NOT allowed:  But  $= 0$ is the Nil-Hecke relation

The cyclotomic quotient

Definition(Khovanov-Lauda, Rouquier 2008/2009)

Fix a dominant $\mathfrak{sl}_d$-weight Λ. The cyclotomic KL-R algebra R_Λ is the subquotient of $\mathcal{U}(\mathfrak{sl}_d)$ defined by the subalgebra of only downward (only F's!) pointing arrows and rightmost region labeled Λ modulo the so-called cyclotomic relation



Theorem(Brundan-Kleshchev, Lauda-Vazirani, Webster, Kang-Kashiwara,...>2008)

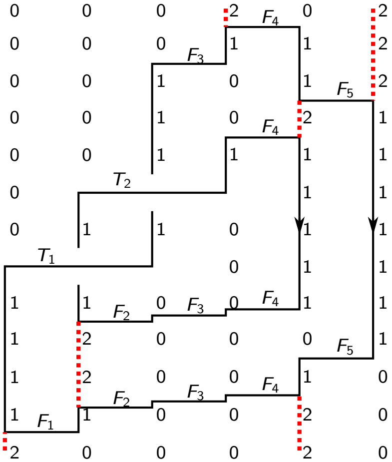
Let V_Λ be the $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-module of highest weight Λ. We have

$$V_\Lambda \cong K_0^{\oplus}(R_d) \otimes_{\mathbb{Z}[q,q^{-1}]} \bar{\mathbb{Q}}(q)$$

as $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-modules (note that this works for more general $\mathfrak{g}$).

Recall: Only F's suffices!

Hopf :



$$F_4^{(2)} F_4 F_3 F_5 F_4 T_2 T_1 F_4 F_3 F_2 F_5 F_4 F_3 F_2 F_1 F_4^{(2)} F_3^{(2)} F_2^{(2)} v_{220000} = F_t T_2 T_1 F_b v_{220000}$$

Exempli gratia (The Hopf link - part two)

The Hopf link example from before will give a complex

$$\begin{tikzcd}
 & F_t F_4 F_3 F_2 F_3 F_b v_h\{5\} \arrow[dr, "\tilde{\Psi}(\times)\colon F_2F_3 \to F_3F_2"] & \\
F_t F_3 F_4 F_2 F_3 F_b v_h\{4\} \arrow[ur, "\tilde{\Psi}(\times)\colon F_3F_4 \to F_4F_3"] \arrow[dr, "\tilde{\Psi}(\times)\colon F_2F_3 \to F_2F_3"'] & \oplus & F_t F_3 F_4 F_2 F_3 F_b v_h\{6\} \\
 & F_t F_4 F_3 F_2 F_3 F_b v_h\{5\} \arrow[ur, "-\tilde{\Psi}(\times)\colon F_3F_4 \to F_4F_3"'] &
\end{tikzcd}$$

that, up to some degree conventions, agrees with the $\mathfrak{sl}_2$-link homology of **Hopf**, because the $\times$ "are" the saddles.

Observation - a more "down to earth" point of view

One can use the Hu-Mathas basis for the cyclotomic KL-R algebra to write down a basis for each of the $\mathfrak{sl}_2$-web algebra modules. The $\times$ are homomorphisms: Calculating the homology reduces to linear algebra because we only need to track the image of the basis elements!

The $\mathfrak{sl}_n$-homologies using $\mathfrak{sl}_d$-symmetries

Let us summarize the connection between $\mathfrak{sl}_n$-homologies and the higher q-skew Howe duality.

- Khovanov, Khovanov-Rozansky and others: The $\mathfrak{sl}_n$-link homology can be obtained using the $\mathfrak{sl}_n$-web algebras.
- Only "F's": The $\mathfrak{sl}_n$-foams are part of the (Karoubian) of some KL-R algebra.
- Conclusion: The $\mathfrak{sl}_n$-homologies are instances of highest $\mathcal{U}_q(\mathfrak{sl}_d)$-weight representation theory!
- If L_D is a link diagram, then its homology is obtained by "jumping via higher F's" from a highest $\mathcal{U}_q(\mathfrak{sl}_d)$-object v_h to a lowest $\mathcal{U}_q(\mathfrak{sl}_d)$-object v_l!
- Missing: Connection to Webster's categorification of the RT-polynomials!
- Missing: Is the module category of the cyclotomic KL-R algebra braided?
- Missing: Details about colored $\mathfrak{sl}_n$-homologies have to be worked out!

There is still much to do...

Thanks for your attention!