Categorification in topology

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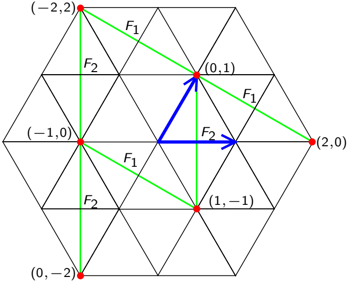
Fun with highest weight modules!



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The famous Jones polynomial

Theorem (Jones 1984)

There is a polynomial $J(\cdot)$ from the set of oriented link diagrams which is invariant under the three Reidemeister moves. Thus, it gives rise to a map from the set of all oriented links in S^3 to $\mathbb{Z}[q, q^{-1}]$: The Jones polynomial.

- It was also extended to other set-ups.
- Nowadays the Jones polynomial is known to be related to different fields of modern mathematics and physics, e.g. the Witten-Reshetikhin-Turaev invariants of 3-manifolds originated from the Jones polynomial.
- The Jones revolution: Before Jones there was only one link polynomial. After Jones there were whole families of link polynomials.
- Thus, the question has changed: Instead of getting new link polynomials, we have to order them!
- Reshetikhin-Turaev: Representation theory of $\mathbf{U}_q(\mathfrak{sl}_2)$ does the trick!

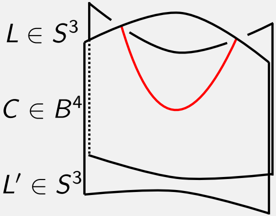
Its categorification

Theorem(Khovanov 1999)

There is a chain complex $\mathbf{Kh}(\cdot)$ of graded vector spaces whose homotopy type is a link invariant. Its graded Euler characteristic gives the Jones polynomial.

Theorem(Khovanov, Bar-Natan, Clark-Morrison-Walker,...)

The $\mathbf{Kh}(\cdot)$ can be extended to a functor from the category of links in S^3 to the category chain complexes of graded vector spaces.



$\xmapsto{\mathbf{Kh}}$

$$\begin{array}{c} \text{gr. chain complex} \\ \mathbf{Kh}(L) \\ \uparrow\downarrow \quad \begin{array}{c}\text{gr. chain maps} \\ \mathbf{Kh}(C)\end{array} \\ \text{gr. chain complex} \\ \mathbf{Kh}(L') \end{array}$$

History repeats itself

- Khovanov's construction can be extended to different set-ups.
- Rasmussen obtained from the homology an invariant that "knows" the slice genus and used it to give a combinatorial proof of the Milnor conjecture.
- Rasmussen also gives a way to combinatorial construct exotic $\mathbb{R}^4$.
- Kronheimer and Mrowka showed that Khovanov homology detects the unknot. This is still an open question for the Jones polynomial.
- Even better: Hedden-Ni and Batson-Seed proved that it detects unlinks. This is known to be false for the Jones polynomial.
- Before I forget: It is a strictly stronger invariant.

After Khovanov lots of other homologies of "Khovanov-type" were discovered. So we need to understand this better.

Since I have all the time in the world, I go into all gory details today.

Categorified symmetries

Let A be some algebra, M be a A-module and $\mathcal{C}$ be a suitable category.

$$\text{“Usual”} \rightsquigarrow \text{“Higher”}$$

$$a \mapsto f_a \in \mathrm{End}(M) \rightsquigarrow a \mapsto \mathcal{F}_a \in \mathbf{End}(\mathcal{C})$$

$$(f_{a_1} \cdot f_{a_2})(m) = f_{a_1 a_2}(m) \rightsquigarrow (\mathcal{F}_{a_1} \circ \mathcal{F}_{a_2})\binom{X}{\varphi} \cong \mathcal{F}_{a_1 a_2}\binom{X}{\varphi}$$

A (weak) categorification of the A-module M should be though of a categorical action of A on a suitable category $\mathcal{C}$ with an isomorphism ψ such that

$$\begin{tikzcd} K_0(\mathcal{C}) \otimes A \arrow[r, "{[\mathcal{F}_a]}"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowleft"] & K_0(\mathcal{C}) \otimes A \arrow[d, "\psi"] \\ M \arrow[r, "\cdot a"] & M. \end{tikzcd}$$

Highest weight modules are (very) unique

$$\begin{tikzcd}
\mathcal{U}(\mathfrak{sl}_d) \arrow[rr, squiggly, "\mathcal{U}(\mathfrak{sl}_d)\text{ acts}"] \arrow[d, "K_0^{\oplus}"'] \arrow[drr, phantom, "\text{Highest weight categorification}"] & & \mathcal{V}_\Lambda \arrow[d, "K_0^{\oplus}"] \\
\dot{\mathbf{U}}_q(\mathfrak{sl}_d) \arrow[rr, squiggly, "\dot{\mathbf{U}}_q(\mathfrak{sl}_d)\text{ acts}"'] & & V_\Lambda
\end{tikzcd}$$

Theorem(Rouquier 2008, Cautis-Lauda 2011, Cautis 2014)

Up to small preconditions: There is a unique category $\mathcal{V}_\Lambda$ that categorifies the $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-module of highest weight Λ determined on the level of K_0.

Conclusion(Morally: Khovanov homology is the unique link homology)

We get Khovanov homology using $\mathcal{V}_\Lambda$. Moreover, any other link homology that on the level of K_0 (plus ε) agrees with Khovanov homology give Khovanov homology.

There is still much to do...

Thanks for your attention!