A Diagrammatic Categorification of the Boson-Fermion Correpondence

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HIM 2017 Winter School:
Categorification, Representation Theory, and Symplectic Geometry

The Decategorified Story

Heisenberg Algebra $\mathfrak{h}$

Generated by: $\langle \mathrm{p}^{(m)}, \mathrm{q}^{(n)} \rangle$ modulo:

- $[\mathrm{p}^{(m)}, \mathrm{p}^{(n)}] = 0$ all $m, n \in \mathbb{N}$
- $[\mathrm{q}^{(m)}, \mathrm{q}^{(n)}] = 0$
- $[\mathrm{q}^{(n)}, \mathrm{p}^{(m)}] = \sum_{k>0} \mathrm{p}^{(m-k)}\mathrm{q}^{(n-k)}$

Also gen. $\langle \mathrm{p}^{(1^m)}, \mathrm{q}^{(1^n)} \rangle_{m,n\in\mathbb{N}}$ mod. $\sim$

Clifford Algebra $\mathfrak{Cl}$

Generated by: $\langle \Psi_i, \Psi_j^* \rangle$ modulo:

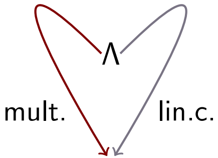
- $\{\Psi_i, \Psi_j\} = 0$ for all $i, j \in \mathbb{Z}$
- $\{\Psi_i^*, \Psi_j^*\} = 0$
- $\{\Psi_i, \Psi_j^*\} = \delta_{i,j}$

$\mathrm{p}^{(m)} \longrightarrow h_m$

$\mathrm{q}^{(n)} \longrightarrow h_n^\perp$

$\mathrm{p}^{(1^m)} \longrightarrow e_m$

$\mathrm{q}^{(1^n)} \longrightarrow e_n^\perp$



Bosonic Fock Space

$\Lambda =$ ring of symmetric functions

$\cong \mathbb{C}[x_1, x_2, \ldots]^{\mathrm{Sym}}$

The Boson-Fermion Correspondence

We can express the action of $\mathfrak{Cl}$ on Λ in terms of the $\mathfrak{h}$ action and vice-versa.

The Boson-Fermion Correspondence:

The fermionic operators on $\mathbb{Z} \otimes \Lambda$ given by:

$$\psi_i(n \otimes v) := n + 1 \otimes C_{i+n}(v)$$

$$\psi_i^*(n + 1 \otimes v) := n \otimes C_{i+n}^*(v)$$

where

$$C_i := \begin{cases} \sum_{k\geq 0} p^{(k)} q^{(1^{i+k})}, & \text{for } i \geq 0, \\ \sum_{k\geq 0} p^{(-i+k)} q^{(1^k)}, & \text{for } i \leq 0. \end{cases} \quad C_i^* := \begin{cases} \sum_{k\geq 0} p^{(1^{i+k})} q^{(k)}, & \text{for } i \geq 0, \\ \sum_{k\geq 0} p^{(1^k)} q^{(-i+k)}, & \text{for } i \leq 0. \end{cases}$$

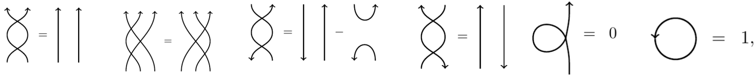
satisfy the Clifford algebra relations:

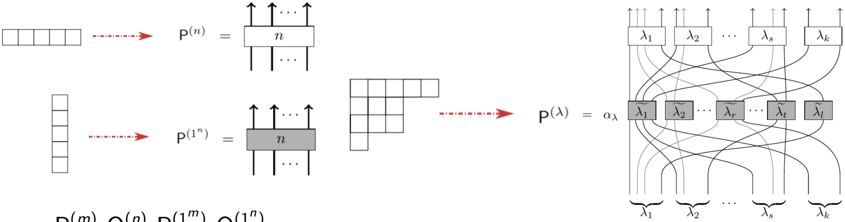
$$\bullet\{\psi_i, \psi_j\} = 0 \qquad \bullet\{\psi_i^*, \psi_j^*\} = 0 \qquad \bullet\{\psi_i, \psi_j^*\} = \delta_{i,j} \qquad \forall i, j \in \mathbb{Z}$$

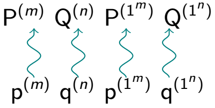
The Categorified Story

In 2010 Khovanov defined the **Heisenberg Category** $\mathcal{H}$: a monoidal, idempotent complete category whose object are generated by P, Q, and morphisms by:

subject to the relations:







Categorical Fock Space

$$V_{Fock} = \bigoplus_{n\in\mathbb{N}} \mathbb{C}[S_n] - mod$$

Induction and Restriction

The Categorified Story

Define the **categorical matrix algebra** $\mathrm{Mat}_{\mathbb{Z}\times\mathbb{Z}}(\mathrm{Kom}(\mathcal{H}))$ whose objects are infinite dimensional matrices with finitely many nonzero entries in each row and column and whose entries are of the form (n, A, m) with $n, m \in \mathbb{Z}$ and $\mathrm{A} \in \mathrm{Kom}(\mathcal{H})$, the homotopy category of $\mathcal{H}$, and whose morphisms are given by matrices in $\mathrm{Mor}(\mathrm{Kom}(\mathcal{H}))$ acting component-wise.

On each entry, the monoidal structure is given by:

$$(n, \mathrm{A}, m) \otimes (n', \mathrm{B}, m') = (n, \delta_{m,n'} \mathrm{A} \otimes \mathrm{B}, m')$$

In 2015, Cautis and Sussan defined functors in terms of the complexes below and conjectured they satisfied categorical Clifford-like relations.

$$\mathrm{C}_i := \begin{cases} \left(\cdots \to \mathrm{P}^{(k)} Q^{(1^{i+k})} \to \cdots \to \mathrm{P}\mathrm{Q}^{(1^{i+1})} \to \mathrm{Q}^{(1^{i})}\right), & i \geq 0, \\ \left(\cdots \to \mathrm{P}^{(-i+k)}\mathrm{Q}^{(1^k)} \to \cdots \to \mathrm{P}^{(-i+1)}\mathrm{Q} \to \mathrm{P}^{(-i)}\right)[-i], & i \leq 0. \end{cases}$$

$$\mathrm{C}_i^* := \begin{cases} \left(\mathrm{P}^{(1^i)} \to \mathrm{P}^{(1^{i+1})}\mathrm{Q} \to \cdots \to \mathrm{P}^{(1^{i+k})}\mathrm{Q}^{(k)} \to \dots\right), & i \geq 0, \\ \left(\mathrm{Q}^{(-i)} \to \mathrm{P}\mathrm{Q}^{(-i+1)} \to \cdots \to \mathrm{P}^{(1^k)}\mathrm{Q}^{(-i+k)} \to \dots\right)[i], & i \leq 0. \end{cases}$$

Categorical Boson-Fermion Correspondence

Theorem (G)

The categorical operators in $\mathrm{Mat}_{\mathbb{Z}\times\mathbb{Z}}(\mathrm{Kom}(\mathcal{H}))$:

$$\Psi_i := [(n+1, \mathrm{C}_{i+n}, n)]_{n\in\mathbb{Z}}$$

$$\Psi_i^* := [(n, \mathrm{C}^*_{i+n}, n+1)]_{n\in\mathbb{Z}}$$

act on categorical Fock space, $\mathbb{Z} \times \mathrm{V}_{Fock}$, and satisfy the following **categorical Clifford relations**.

- $(\Psi_i)^2 \cong 0$
- $\Psi_i \Psi_j \cong \begin{cases} \Psi_j \Psi_i[-1] & \text{if } i < j \\ \Psi_j \Psi_i[1] & \text{if } i > j \end{cases}$
- $(\Psi_i^*)^2 \cong 0$
- $\Psi_i^* \Psi_j^* \cong \begin{cases} \Psi_j^* \Psi_i^*[-1] & \text{if } i < j \\ \Psi_j^* \Psi_i^*[1] & \text{if } i > j \end{cases}$

Categorical Boson-Fermion Correspondence

Conjectured: (by Cautis-Sussan 2015)

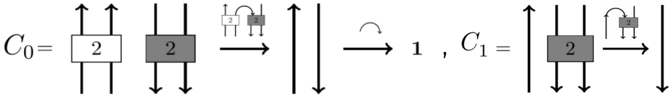
Moreover, they also satisfy:

- $\Psi_i \Psi_j^* \cong \begin{cases} \Psi_j^* \Psi_i[1] & \text{if } i < j \\ \Psi_j^* \Psi_i[-1] & \text{if } i > j \end{cases}$

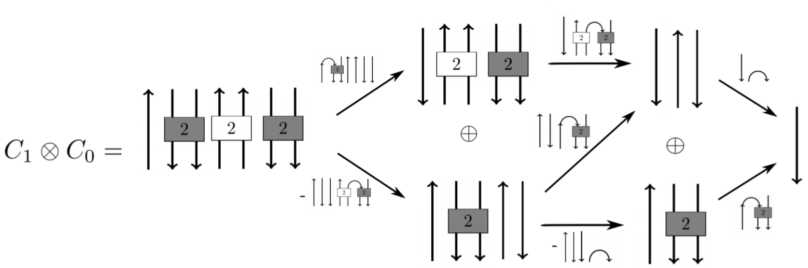
- there exists a distinguished triangle $\Psi_i \Psi_i^* \to \mathbf{1} \to \Psi_i^* \Psi_i$.

Example

Suppose $i = 0$ and $Q^m = 0$ for $m \geq 3$. Then $\Psi_0 \otimes \Psi_0 \cong 0$
$\Leftrightarrow C_{n+1} \otimes C_n \cong 0$ for all n.

Let $n = 0$ then 

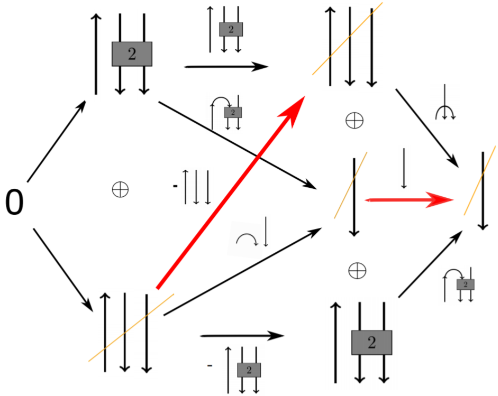
so:



Example

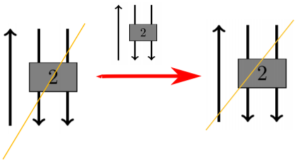
Apply isomorphism QPQ $\cong$ PQQ $\oplus$ Q then:

$C_1 \otimes C_0 \cong$



Example

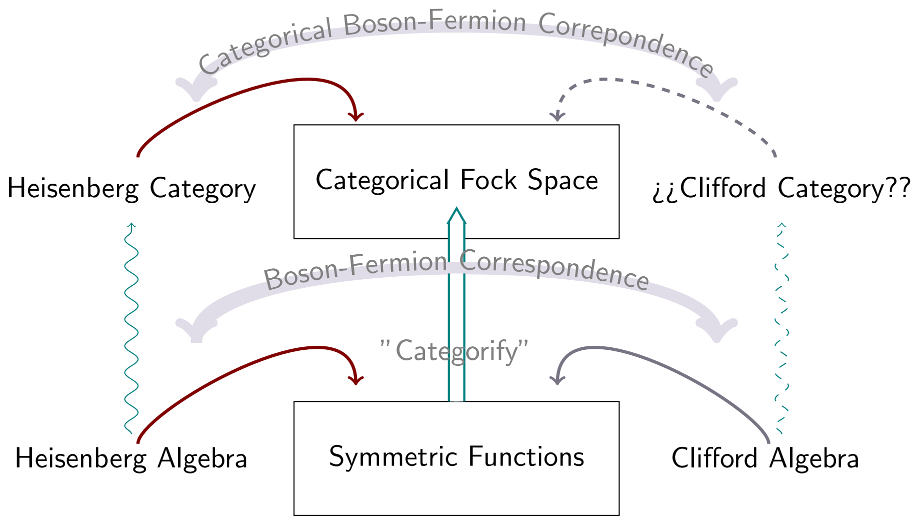
$C_1 \otimes C_0 \cong \quad 0 \longrightarrow$



$\longrightarrow 0$

$\cong \quad 0$

The Big Picture



Thank you!

Thank you for listening!

References:

- Cautis, S., and Sussan, J. On a categorical Boson-Fermion Correspondance, Communications in Mathematical Physics,(2015) 336, 649
- Khovanov, M. Heisenberg algebra and a graphical calculus, Fundamenta Mathematicae 225, (2014), 169-210