EXERCISES 9: LECTURE FOUNDATIONS OF MATHEMATICS

Exercise 1. A total order on a set X is called well-ordered from below (above) if there exists a smallest (biggest) element for any non-empty subset of X with respect to the fixed total order. Such orders are called well-orders.

(a) Let X be a finite set. Find a well-order on X.

(b) Find a well-order on $X = \mathbb{N}_0$.

(c) Find a well-order on $X = \mathbb{Z}$.

(d) Find a well-order on $X = \mathbb{Z}^n$ for $n \geq 1$.

(e) Let $X = \{1, 2\}$. Find a well-order on $\mathfrak{P}(X)$.

In all the cases (a) to (d) you should find a well-ordering from below and above.

Exercise 2. Fix three integers $a, b, c \in \mathbb{Z}$. What condition needs to be satisfied by c such that there exist $x, y \in \mathbb{Z}$ with $ax + by = c$.

Exercise 3. Show:

(a) For all $n \in \mathbb{N}_0$, the set $n\mathbb{Z} = \{nz \mid z \in \mathbb{Z}\}$ is an ideal in $\mathbb{Z}$, i.e. a subset such that $x + y \in n\mathbb{Z}$ and $z_1 x z_2 \in n\mathbb{Z}$ hold for all $x, y \in n\mathbb{Z}$ and $z_1, z_2 \in \mathbb{Z}$.

(b) $n \in \mathbb{N}_0$, $n \geq 2$ is prime if and only if there do not exist $z_1, z_2 \in \mathbb{Z}$, $z_1, z_2 \notin n\mathbb{Z}$ such that $z_1 z_2 \in n\mathbb{Z}$.

(c) $n \in \mathbb{N}_0$, $n \geq 2$ is prime if and only if for all $z_1 \in \mathbb{Z}$ with $z_1 \notin n\mathbb{Z}$ there exists an integer $z_2 \in \mathbb{Z}$ such that $(z_1 z_2 - 1) \in n\mathbb{Z}$.

Exercise 4. Define recursively a map $f \colon \mathbb{N} \to \mathbb{Z}$ via $f(1) = 1$, $f(2) = 1$ and $f(n+1) = f(n) + f(n-1)$ for $n > 2$. The numbers $f(n)$ are called Fibonacci numbers. Apply the euclidean algorithm on two consecutive Fibonacci numbers. What kind of pattern occurs? (Explain the pattern, and prove your claim.)

Submission of the exercise sheet: 18.Nov.2019 during the exercise sessions. **Return of the exercise sheet**: 21.Nov.2019 during the exercise sessions.