EXERCISES 3: LECTURE FOUNDATIONS OF MATHEMATICS

Exercise 1. Let X, Y, Z be sets. Moreover, let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps. Show:

(a) If f and g are injective, then $g \circ f$ injective.

(b) If f and g are surjective, then $g \circ f$ surjective.

(c) f is injective if and only if there exists $h\colon Y \to X$ such that $h \circ f = \mathrm{id}_X$.

(d) f is surjective if and only if there exists $h\colon Y \to X$ such that $f \circ h = \mathrm{id}_Y$.

Above id_X resp. id_Y denote the identity maps on X resp. Y.

Exercise 2. Let X, Y be sets. Further, let $f\colon X \to Y$ be a map whose preimage is denoted by f^{-1}. Show that the following are equivalent:

(i) f is injective.

(ii) $f^{-1}(f(A)) = A$ for all $A \subset X$.

(iii) $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subset X$.

(iv) For all $A, B \subset X$ with $A \cap B = \emptyset$ one has $f(A) \cap f(B) = \emptyset$.

(v) For all $A, B \subset X$ with $B \subset A$ one has $f(A \setminus B) = f(A) \setminus f(B)$.

Exercise 3. Let W, X, Y, Z be sets, and $f\colon W \to X$, $g\colon X \to Y$ and $h\colon Y \to Z$ be maps. Show that f, g, h are bijective in case $g \circ f$ and $h \circ g$ are.

Exercise 4. Let X, Y be sets, and let $f\colon X \to Y$ be a map whose preimage is denoted by f^{-1}. Let A, B be subsets of X and C, D be subsets of Y.

Decide which of the following statements are true and which are false.

(a) If $A \neq \emptyset$, then $f(A) \neq \emptyset$.

(b) If $C \neq \emptyset$, then $f^{-1}(C) \neq \emptyset$.

(c) If $A \subset B$, then $f(A) \subset f(B)$.

(d) If $C \subset D$, then $f^{-1}(C) \subset f^{-1}(D)$.

(e) $f(A \cap B) = f(A) \cap f(B)$.

(f) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

(g) $f(A \cup B) = f(A) \cup f(B)$.

(h) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

(i) If $B \subset A$, then $f(A \setminus B) = f(A) \setminus f(B)$.

(j) If $D \subset C$, then $f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$.

Justify your answer with a proof or a counterexample.

Submission of the exercise sheet: 07.Oct.2019 before the lecture. **Return of the exercise sheet**: 10.Oct.2019 during the exercise classes.