Cyclotomic quiver Hecke algebras III

The Ariki-Brundan-Kleshchev categorification theorem

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Outline of lectures

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   - Categorification of highest weight modules
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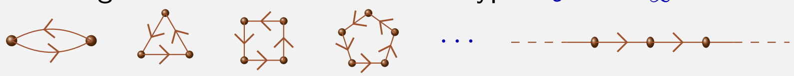
Ariki-Brundan-Kleshchev categorification theorem

Let C be a generalised Cartan matrix of type $A_e^{(1)}$ or A_∞:



The aim for this lecture is to explain and understand:

Theorem (Ariki, Brundan-Kleshchev, Brundan-Stroppel, Rouquier)

Let C be a Cartan matrix of type $A_e^{(1)}$ or A_∞ and let $\Bbbk$ be a field. Then

$$L_{\mathbb{A}}(\Lambda) \cong \bigoplus_{n\ge 0} Proj(\mathscr{R}_n^\Lambda) \quad \text{and} \quad L_{\mathbb{A}}(\Lambda)^\vee \cong \bigoplus_{n\ge 0} Rep(\mathscr{R}_n^\Lambda)$$

Moreover, if $\Bbbk = \mathbb{C}$ then

- *The canonical basis of $L_{\mathbb{A}}(\Lambda)$ coincides with* $\{\, [P] \mid \text{self dual projective indecomposable } \mathscr{R}_n^\Lambda\text{-module}, n \ge 0 \,\}$
- *The dual canonical basis of $L_{\mathbb{A}}(\Lambda)$ coincides with* $\{\, [D] \mid \text{self dual irreducible } \mathscr{R}_n^\Lambda\text{-modules}, n \ge 0 \,\}$

Ariki proved the ungraded analogue of this result in 1996, establishing and generalising the LLT conjecture. This result motivated Chuang-Rouquier's $\mathfrak{sl}_2$-categorification paper and the introduction of quiver Hecke algebras

Multipartitions and dominance

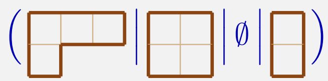
A multipartition, or ℓ-partition, of n is an ordered ℓ-tuple of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}|\lambda^{(2)}|\dots|\lambda^{(\ell)})$ such that $|\boldsymbol{\lambda}| = |\lambda^{(1)}| + \dots + |\lambda^{(\ell)}| = n$

Let $\mathcal{P}_n^\Lambda$ be the set of ℓ-partitions of n

We identify ℓ-partitions and their diagrams:

$$[\boldsymbol{\lambda}] = \{\, (l, r, c) \mid 1 \le l \le \ell, 1 \le c \le \lambda_r^{(l)} \,\}$$

For example, if $\boldsymbol{\lambda} = (3,1|2,2|\emptyset|1^2)$ then



A node is any triple (l, r, c) in a diagram. The set $\{1,\dots,\ell\} \times \mathbb{N}^2$ of nodes is totally ordered by the lexicographic order

The set $\mathcal{P}_n^\Lambda$ is a post under dominance, where if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\Lambda$ then

$$\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu} \text{ if } \sum_{k=1}^{l-1} |\lambda^{(k)}| + \sum_{j=1}^{i} \lambda_j^{(l)} \ge \sum_{k=1}^{l-1} |\mu^{(k)}| + \sum_{j=1}^{i} \mu_j^{(l)}$$

Dominance corresponds to moving nodes in the diagrams up and to the left

Addable and removable nodes

Recall from last lecture that we fixed integers $\kappa_1, \dots, \kappa_\ell$ such that

$$\#\{\, 1 \le l \le \ell \mid \kappa_l \equiv i \pmod{e} \,\} = (h_i, \Lambda), \qquad \text{for } i \in I$$

An addable node of $\boldsymbol{\lambda}$ is a node $B \notin \boldsymbol{\lambda}$ such that $\boldsymbol{\lambda} + A := \boldsymbol{\lambda} \cup \{B\} \in \mathcal{P}^\Lambda_{n+1}$

A removable node of $\boldsymbol{\lambda}$ is a node $B \in \boldsymbol{\lambda}$ with $\boldsymbol{\lambda} - A := \boldsymbol{\lambda} \setminus \{B\} \in \mathcal{P}^\Lambda_{n-1}$

A node $(l, r, c) \in \{1, 2, \dots, \ell\} \times \mathbb{N}^2$ is an i-node if it has residue

$$i = \kappa_l + c - r + e\mathbb{Z} \in I = \mathbb{Z}/e\mathbb{Z}$$

Let $\mathrm{Add}_i(\boldsymbol{\lambda})$ and $\mathrm{Rem}_i(\boldsymbol{\lambda})$ be the sets of addable and removable i-nodes

Definition (Brundan-Kleshchev-Wang)

If A is an addable or removable i-node of $\boldsymbol{\mu}$ define:

$$d^A(\boldsymbol{\mu}) = \#\{\, B \in \mathrm{Add}_i(\boldsymbol{\mu}) \mid A > B \,\} - \#\{\, B \in \mathrm{Rem}_i(\boldsymbol{\mu}) \mid A > B \,\}$$
$$d_A(\boldsymbol{\mu}) = \#\{\, B \in \mathrm{Add}_i(\boldsymbol{\mu}) \mid A < B \,\} - \#\{\, B \in \mathrm{Rem}_i(\boldsymbol{\mu}) \mid A < B \,\}$$
$$d_i(\boldsymbol{\mu}) = \#\,\mathrm{Add}_i(\boldsymbol{\mu}) - \#\,\mathrm{Rem}_i(\boldsymbol{\mu})$$

Standard tableaux

A $\boldsymbol{\lambda}$-tableau is a map $\mathtt{t} : [\boldsymbol{\lambda}] \longrightarrow \{1, 2, \dots, n\}$, which we identify with a labelling of $[\boldsymbol{\lambda}]$. A tableau $\mathtt{t}$ is standard if its entries increase along rows and down columns in each component

Let $\mathrm{Std}(\boldsymbol{\lambda})$ be the set of standard $\boldsymbol{\lambda}$-tableaux

Example Let $\boldsymbol{\lambda} = (3, 2|2^2|(2, 1)$. Then two standard $\boldsymbol{\lambda}$-tableau are:

$$\mathtt{t}^{\boldsymbol{\lambda}} = \left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \end{array} \middle| \begin{array}{cc} 6 & 7 \\ 8 & 9 \end{array} \middle| \begin{array}{cc} 10 & 11 \\ 12 & \end{array} \right) \qquad \mathtt{t}_{\boldsymbol{\lambda}} = \left(\begin{array}{ccc} 8 & 10 & 12 \\ 9 & 11 & \end{array} \middle| \begin{array}{cc} 4 & 6 \\ 5 & 7 \end{array} \middle| \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \right)$$

These are the initial and final $\boldsymbol{\lambda}$-tableau, respectively

If $\mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ define permutations $d(\mathtt{t})$ and $d'(\mathtt{t}) \in \mathfrak{S}_n$ by

$$\mathtt{t}^{\boldsymbol{\lambda}} d(\mathtt{t}) = \mathtt{t} = \mathtt{t}_{\boldsymbol{\lambda}} d'(\mathtt{t})$$

The residue sequence of $\mathtt{t} \in \mathrm{Std}(\mathcal{P}^\Lambda_n)$ is $\mathbf{i}^{\mathtt{t}} \in I^n$ where $\mathtt{t}^{-1}(m)$ is an $i^{\mathtt{t}}_m$-node

Let $A = \mathtt{t}^{-1}(n)$. Define the degree and codegree, respectively, of $\mathtt{t}$ by:

$$\deg \mathtt{t} = \deg \mathtt{t}_{\downarrow(n-1)} + d^A(\boldsymbol{\mu}) \quad \text{and} \quad \deg' \mathtt{t} = \deg' \mathtt{t}_{\downarrow(n-1)} + d_A(\boldsymbol{\mu})$$

such that "empty $(0|\dots|0)$-tableau" has degree and codegree 0

By definition, $\deg \mathtt{t}, \deg' \mathtt{t} \in \mathbb{Z}$. They can be positive, negative or zero

Cellular bases

The algebra $\mathscr{R}^\Lambda_n$ has two natural "dual" graded cellular bases.

For $\boldsymbol{\lambda} \in \mathcal{P}^\Lambda_n$ define polynomials $y^{\boldsymbol{\lambda}} = y(\mathtt{t}^{\boldsymbol{\lambda}})$ and $y_{\boldsymbol{\lambda}} = y(\mathtt{t}_{\boldsymbol{\lambda}})$ inductively by

$$y^{\boldsymbol{\lambda}} = y(\mathtt{t}^{\boldsymbol{\lambda}}_{\downarrow(n-1)}) y_n^{d^A(\boldsymbol{\lambda})} \quad \text{and} \quad y_{\boldsymbol{\lambda}} = y(\mathtt{t}_{\boldsymbol{\lambda}\downarrow(n-1)}) y_n^{d_A(\boldsymbol{\lambda})}$$

Then these two cellular bases have the following properties

Poset	$(\mathcal{P}^\Lambda_n, \trianglerighteq)$	$(\mathcal{P}^\Lambda_n, \trianglelefteq)$
Basis	$\{\, \psi_{\mathtt{st}} \mid (\mathtt{s}, \mathtt{t}) \in \mathrm{Std}^2(\mathcal{P}^\Lambda_n) \,\}$	$\{\, \psi'_{\mathtt{st}} \mid (\mathtt{s}, \mathtt{t}) \in \mathrm{Std}^2(\mathcal{P}^\Lambda_n) \,\}$
Definition	$\psi_{\mathtt{st}} = \psi^*_{d(\mathtt{s})} i^{\boldsymbol{\lambda}} y^{\boldsymbol{\lambda}} \psi_{d(\mathtt{t})}$	$\psi_{\mathtt{st}} = \psi^*_{d'(\mathtt{s})} i_{\boldsymbol{\lambda}} y_{\boldsymbol{\lambda}} \psi_{d'(\mathtt{t})}$
Degree	$\deg \psi_{\mathtt{st}} = \deg \mathtt{s} + \deg \mathtt{t}$	$\deg' \psi_{\mathtt{st}} = \deg' \mathtt{s} + \deg' \mathtt{t}$
Residues	$\mathbf{i}^{\mathtt{s}}$ and $\mathbf{i}^{\mathtt{t}}$	$\mathbf{i}^{\mathtt{s}}$ and $\mathbf{i}^{\mathtt{t}}$
Cell modules	$S^{\boldsymbol{\lambda}}$	$S_{\boldsymbol{\lambda}}$
Simple modules	D^μ	D_μ

Let $\mathcal{K}^\Lambda_n = \{\, \boldsymbol{\mu} \in \mathcal{P}^\Lambda_n \mid D^{\boldsymbol{\mu}} \neq 0 \,\}$ and $\mathcal{K}^n_\Lambda = \{\, \boldsymbol{\mu} \in \mathcal{P}^\Lambda_n \mid D_{\boldsymbol{\mu}} \neq 0 \,\}$. Then

$$\{\, D^{\boldsymbol{\mu}}\langle k \rangle \mid \boldsymbol{\mu} \in \mathcal{K}^\Lambda_n, k \in \mathbb{Z} \,\} \quad \text{and} \quad \{\, D_{\boldsymbol{\mu}}\langle k \rangle \mid \boldsymbol{\mu} \in \mathcal{K}^n_\Lambda, k \in \mathbb{Z} \,\}$$

are both complete sets of pairwise non-isomorphic irreducible $\mathscr{R}^\Lambda_n$-modules

For symmetric groups, the Specht modules and simple modules are interchanged by tensoring with the sign representation

Graded decomposition matrices

For $\boldsymbol{\lambda} \in \mathcal{P}^\Lambda_n$ and $\boldsymbol{\mu} \in \mathcal{K}^\Lambda_n$ define graded decomposition numbers

$$d_{\boldsymbol{\lambda\mu}}(q) = [S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]_q = \textstyle\sum_{k \in \mathbb{Z}} [S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}\langle k \rangle]\, q^k \quad \in \mathbb{N}[q, q^{-1}]$$

Let $\mathbf{d}_q = (d_{\boldsymbol{\lambda\mu}}(q))$ be the graded decomposition matrix

Let $Y^{\boldsymbol{\mu}}$ be the (graded) projective cover of $D^{\boldsymbol{\mu}}$

Let $\mathbf{c}_q = ([Y^{\boldsymbol{\mu}} : D^{\boldsymbol{\nu}}]_q)_{\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{K}^\Lambda_n}$ be the graded Cartan matrix

Theorem

Suppose that $\boldsymbol{\lambda} \in \mathcal{P}^\Lambda_n$ and $\boldsymbol{\mu} \in \mathcal{K}^\Lambda_n$. Then

$$d_{\boldsymbol{\mu\mu}}(q) = 1 \text{ and } d_{\boldsymbol{\lambda\mu}}(q) \neq 0 \text{ only if } \boldsymbol{\lambda} \trianglerighteq \mu$$

Moreover, $Y^{\boldsymbol{\mu}}$ has a filtration by graded Specht modules in which $S^{\boldsymbol{\lambda}}$ appears with multiplicity $d_{\boldsymbol{\lambda\mu}}(q)$

$$\implies \quad \mathbf{c}_q = \mathbf{d}_q^T \mathbf{d}_q$$

Proof This follows from the general theory of (graded) cellular algebras

Remark Specht filtration multiplicities are not well-defined, but the import of the theorem is that $\quad [Y^{\boldsymbol{\mu}} : S^{\boldsymbol{\lambda}}]_q = [S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]_q$

Induction and restriction

For $i \in I$ define $1_{n,i} = \sum_{\mathbf{j} \in I^n} 1_{\mathbf{j}i} \in \mathscr{R}^\Lambda_{n+1}$

Lemma

Let $i \in I$. There is an embedding of graded algebras $\mathscr{R}^\Lambda_n \hookrightarrow \mathscr{R}^\Lambda_{n+1}$ given by

$$1_{\mathbf{j}} \mapsto 1_{\mathbf{j}i}, \quad y_r \mapsto y_r 1_{n,i} \quad \text{and} \quad \psi_s \mapsto \psi_s 1_{n,i}$$

This induces an exact functor

$$i\text{-Ind} : \mathscr{R}^\Lambda_n\text{-Mod} \longrightarrow \mathscr{R}^\Lambda_{n+1}\text{-Mod};\ M \mapsto M \otimes_{\mathscr{R}^\Lambda_n} 1_{n,i}\mathscr{R}^\Lambda_{n+1}$$

Moreover, $\mathrm{Ind} = \bigoplus_{i\in I} i\text{-Ind}$

Proof Check the KLR relations and use the KLR basis theorems

The functor i-Ind has a natural left adjoint:

$$i\text{-Res}\, M = Me_{1,i} \cong \mathrm{Hom}_{\mathscr{R}^\Lambda_n}(1_{n,i}\mathscr{R}^\Lambda_n, M)$$

Theorem (Kashiwara)

Suppose $i \in I$. Then $(i\text{-Res}, i\text{-Ind})$ is a biadjoint pair.

Induction and restriction of Specht modules

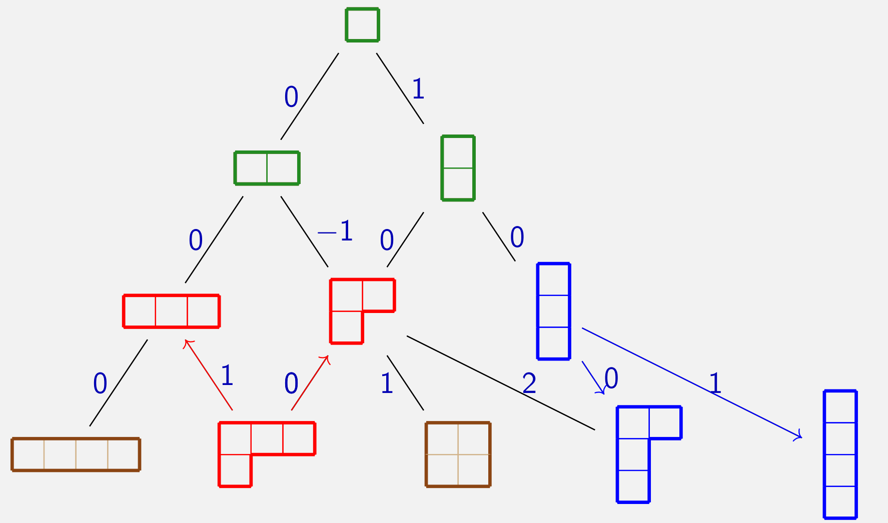
Theorem (Brundan-Kleshchev-Wang, Hu-Mathas)

Suppose that $\Bbbk$ is an integral domain and $\boldsymbol{\lambda} \in \mathcal{P}^\Lambda_n$.

1. *Let $B_1 > B_2 > \cdots > B_y$ be the removable i-nodes of $\boldsymbol{\lambda}$. Then i-Res $S^{\boldsymbol{\lambda}}$ and i-Res $S_{\boldsymbol{\lambda}}$ have graded Specht filtrations*
$$0 = R_0 \subset R_1 \subset \cdots \subset R_y = i\text{-Res}\, S^{\boldsymbol{\lambda}}$$
$$0 = R_{y+1} \subset R_y \subset \cdots \subset R_1 = i\text{-Res}\, S_{\boldsymbol{\lambda}}$$
such that $R_j/R_{j-1} \cong q^{d_{B_j}(\boldsymbol{\lambda})} S^{\boldsymbol{\lambda}-B_j}$ and $R_j/R_{j+1} \cong q^{d^{B_j}(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}-B_j}$
2. *Let $A_1 > A_2 \cdots > A_z$ be the addable i-nodes of $\boldsymbol{\lambda}$. Then i-Ind $S^{\boldsymbol{\lambda}}$ and i-Ind $S_{\boldsymbol{\lambda}}$ have graded Specht filtrations*
$$0 = I_{z+1} \subset I_z \subset \cdots \subset I_1 = i\text{-Ind}\, S^{\boldsymbol{\lambda}}$$
$$0 = I_0 \subset I_1 \subset \cdots \subset I_z = i\text{-Ind}\, S_{\boldsymbol{\lambda}}$$
such that $I_j/I_{j+1} \cong q^{d_{A_j}(\boldsymbol{\lambda})} S^{\boldsymbol{\lambda}+A_j}$ and $I_j/I_{j-1} \cong q^{d^{A_j}(\boldsymbol{\lambda})} S_{\boldsymbol{\lambda}+A_j}$

Proof Reduce to the semisimple case and then use the seminormal form

Graded branching rules and tableaux degrees



$\Longrightarrow$ $[\mathrm{Res}\, S^{(3,1)}] = q[S^{(3)}] + [S^{(2,1)}]$ (Brundan, Kleshchev and Wang) and $[\mathrm{Ind}\, S_{(1^3)}] = [S_{(2,1^2)}] + q[S_{(1^4)}]$ (Hu and M.)

Paths still index a basis $\Longrightarrow$ $\dim_q S_{(3,1)} = q + q^{-1} + q$

$\Longrightarrow$ $\dim_q D_{(3,1)} = q + q^{-1}$

Defect and duality

Let $*$ be the unique (homogeneous) anti-isomorphism of $\mathscr{R}^\Lambda_n$ that fixes each of the KLR generators

$\Longrightarrow$ $(\psi_{\mathtt{st}})^* = \psi_{\mathtt{ts}}$ and $(\psi'_{\mathtt{st}})^* = \psi'_{\mathtt{ts}}$, so $*$ is the cellular basis involution for both the ψ and ψ'-bases

If M is an $\mathscr{R}^\Lambda_n$-module then $M^\circledast = \mathrm{Hom}_\Bbbk(M, \Bbbk)$ is an $\mathscr{R}^\Lambda_n$-module with action: $(h \cdot f)(m) = f(h^* m)$, for $h \in \mathscr{R}^\Lambda_n$, $f \in M^\circledast$ and $m \in M$

$\Longrightarrow$ $\dim_q M^\circledast = \overline{\dim_q M}$

where $\overline{f(q)} = f(q^{-1})$ is the $\mathbb{Z}$-linear bar involution on $\mathbb{Z}[q, q^{-1}]$

Previously, we noted that $(D^\mu)^\circledast \cong D^\mu$ and $(D_\nu)^\circledast \cong D_\nu$

To describe duality on the Specht modules define the defect of $\beta \in Q^+$

$$\mathrm{def}\,\beta = (\Lambda, \beta) - \tfrac{1}{2}(\beta, \beta) = \tfrac{1}{2}\Big((\Lambda, \Lambda) - (\Lambda - \beta, \Lambda - \beta)\Big) \in \mathbb{N}$$

For $\boldsymbol{\lambda} \in \mathcal{P}^\Lambda_n$ set $\beta_{\boldsymbol{\lambda}} = \sum_{k=1}^n \alpha_{i^{\mathtt{t}}_k} \in Q^+$, for any $\mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda})$.

The defect of $\boldsymbol{\lambda}$ is $\mathrm{def}\,\boldsymbol{\lambda} = \mathrm{def}\,\beta_{\boldsymbol{\lambda}}$

Symmetrizing form

A graded $\Bbbk$-algebra A is a graded symmetric algebra if there exists a homogeneous non-degenerate trace form $\tau : A \longrightarrow \Bbbk$, where $\Bbbk$ is in degree zero. That is, $\tau(ab) = \tau(ba)$ and if $0 \neq a \in A$ then there exists $b \in A$ such that $\tau(ab) \neq 0$.

Theorem (Hu-M., Kang-Kasiwara, Webster)

Suppose that $\beta \in Q_n^+$. Then $\mathscr{R}_\beta^\Lambda$ a graded symmetric algebra with homogeneous trace form τ_β of degree $-2\operatorname{def}\beta$.

Proof Our proof reduces to the trace-form on $\mathscr{H}_n^\Lambda$. A key part of the argument is the observation that

$$\tau_\beta(\psi_{\mathtt{st}}\psi'_{\mathtt{uv}}) \neq 0 \text{ only if } \mathtt{u} \trianglerighteq \mathtt{t} \text{ and that } \tau_\beta(\psi_{\mathtt{st}}\psi'_{\mathtt{ts}}) \neq 0$$

Corollary (Hu-M.)

Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Then $S^{\boldsymbol{\lambda}} \cong q^{\operatorname{def}\boldsymbol{\lambda}} S_{\boldsymbol{\lambda}}^{\circledast}$ and $S_{\boldsymbol{\lambda}} = q^{\operatorname{def}\boldsymbol{\lambda}} (S^{\boldsymbol{\lambda}})^{\circledast}$

Proof By the remarks above, an isomorphism is given by sending $\psi_{\mathtt{t}} \in S^{\boldsymbol{\lambda}}$ to the map $\theta_{\mathtt{t}} \in q^{\operatorname{def}\boldsymbol{\lambda}} S_{\boldsymbol{\lambda}}^{\circledast}$ that is given by $\theta_{\mathtt{t}}(\psi'_{\mathtt{u}}) = \tau_\beta(\psi_{\mathtt{t}^{\boldsymbol{\lambda}}\mathtt{t}}\psi'_{\mathtt{u}\mathtt{t}^{\boldsymbol{\lambda}}})$

The Hom-dual

Define $\#$ to be the graded endofunctor of $\operatorname{Rep}(\mathscr{R}_n^\Lambda)$ and $\operatorname{Proj}(\mathscr{R}_n^\Lambda)$ given by

$$M^\# = \operatorname{Hom}_{\mathscr{R}_n^\Lambda}(M, \mathscr{R}_n^\Lambda)$$

In particular, note that $(Y^{\boldsymbol{\mu}})^\# \cong Y^{\boldsymbol{\mu}}$ since $Y^{\boldsymbol{\mu}}$ is a summand of $\mathscr{R}_n^\Lambda$

A straightforward argument using the adjointness of $\otimes$ and Hom gives:

Lemma

Let $\beta \in Q^+$. As endofunctors of $\operatorname{Rep}(\mathscr{R}_\beta^\Lambda)$, there is an isomorphism of functors $\# \cong q^{2\operatorname{def}\beta} \circ \circledast$.

We use will $\circledast$ as the duality for the dual canonical bases and $\#$ for the canonical basis

The quantum group $U_q(\widehat{\mathfrak{sl}}_e)$

Given our choice of Cartan matrix, we need to work with $U_q(\widehat{\mathfrak{sl}}_e)$

The quantum group $U_q(\widehat{\mathfrak{sl}}_e)$ associated with the Cartan matrix C is the $\mathbb{Q}(q)$-algebra generated by $\{\, E_i, F_i, K_i^\pm \mid i \in I \,\}$, subject to the relations:

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = 1, \quad K_iE_jK_i^{-1} = q^{c_{ij}}E_j$$
$$K_iF_jK_i^{-1} = q^{-c_{ij}}F_j, \quad [E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}},$$
$$\textstyle\sum_{0\le c\le 1-c_{ij}} (-1)^c \left[\!\left[{1-c_{ij} \atop c}\right]\!\right]_i E_i^{1-c_{ij}-c}E_jE_i^c = 0$$
$$\textstyle\sum_{0\le c\le 1-c_{ij}} (-1)^c \left[\!\left[{1-c_{ij} \atop c}\right]\!\right]_i F_i^{1-c_{ij}-c}F_jF_i^c = 0$$

where $\left[\!\left[{d \atop c}\right]\!\right]_i = \frac{[\![d]\!]_i!}{[\![c]\!]_i![\![d-c]\!]_i!}$ and $[\![m]\!]_i! = \prod_{k=1}^m \frac{q^k - q^{-k}}{q - q^{-1}}$

Recall that $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$

Let $U_{\mathbb{A}}(\widehat{\mathfrak{sl}}_e)$ be Lusztig's $\mathbb{A}$-form of $U_q(\widehat{\mathfrak{sl}}_e)$, which is the $\mathbb{A}$-subalgebra of $U_q(\widehat{\mathfrak{sl}}_e)$ generated by the quantised divided powers

$$E_i^{(k)} = E_i^k/[\![k]\!]_i! \quad \text{and} \quad F_i^{(k)} = F_i^k/[\![k]\!]_i!$$

For each $\Lambda \in P^+$ there is a irreducible integrable highest weight module $L(\Lambda)$ of highest weight Λ.

The combinatorial Fock space

The **combinatorial Fock space** $\mathscr{F}_{\mathbb{A}}^\Lambda$ is the free $\mathbb{A}$-module with basis the set of symbols $\{\, |\boldsymbol{\lambda}\rangle \mid \boldsymbol{\lambda} \in \mathcal{P}^\Lambda \,\}$, where $\mathcal{P}^\Lambda = \bigcup_{n\ge 0} \mathcal{P}_n^\Lambda$. For future use, let $\mathcal{K}^\Lambda = \bigcup_{n\ge 0} \mathcal{K}_n^\Lambda$. Set $\mathscr{F}_{\mathbb{Q}(q)}^\Lambda = \mathscr{F}_{\mathbb{A}}^\Lambda \otimes_{\mathbb{A}} \mathbb{Q}(q)$. Then, $\mathscr{F}_{\mathbb{Q}(q)}^\Lambda$ is an infinite dimensional $\mathbb{Q}(q)$-vector space. We consider $\{\, |\boldsymbol{\lambda}\rangle \mid \boldsymbol{\lambda} \in \mathcal{P}^\Lambda \,\}$ as a basis of $\mathscr{F}_{\mathbb{Q}(q)}^\Lambda$ by identifying $|\boldsymbol{\lambda}\rangle$ and $|\boldsymbol{\lambda}\rangle \otimes 1_{\mathbb{Q}(q)}$.

Theorem (Hayashi, Misra-Miwa)

Suppose that $\Lambda \in P^+$. Then $\mathscr{F}_{\mathbb{Q}(q)}^\Lambda$ is an integrable $U_q(\widehat{\mathfrak{sl}}_e)$-module with $U_q(\widehat{\mathfrak{sl}}_e)$-action determined by

$$E_i|\boldsymbol{\lambda}\rangle = \sum_{B \in \mathrm{Rem}_i(\boldsymbol{\lambda})} q^{d_B(\boldsymbol{\lambda})}|\boldsymbol{\lambda} - B\rangle, \qquad F_i|\boldsymbol{\lambda}\rangle = \sum_{A \in \mathrm{Add}_i(\boldsymbol{\lambda})} q^{-d^A(\boldsymbol{\lambda})}|\boldsymbol{\lambda} + A\rangle$$

and $K_i|\boldsymbol{\lambda}\rangle = q^{d_i(\boldsymbol{\lambda})}|\boldsymbol{\lambda}\rangle$, for $i \in I$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$.

Proof A tedious check of the relations

It follows from the theorem that $L(\Lambda) \cong U_q(\widehat{\mathfrak{sl}}_e)|\mathbf{0}_\ell\rangle$, where $\mathbf{0}_\ell$ is the zero ℓ-partition. Define $L_{\mathbb{A}}(\Lambda) = U_{\mathbb{A}}(\widehat{\mathfrak{sl}}_e)|\mathbf{0}_\ell\rangle$

The CDE triangle in the Fock space

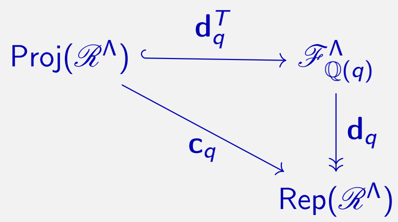
Recall that $\mathrm{Rep}(\mathscr{R}^\Lambda)=\bigoplus_{n\ge0}\mathrm{Rep}(\mathscr{R}^\Lambda_n)$ and $\mathrm{Proj}(\mathscr{R}^\Lambda)=\bigoplus_{n\ge0}\mathrm{Proj}(\mathscr{R}^\Lambda_n)$

Proposition

Suppose that $\Lambda\in P^+$. Then the i-induction and i-restriction functors induce a $U_q(\widehat{\mathfrak{sl}}_e)$-module structure on $\mathrm{Proj}(\mathscr{R}^\Lambda)\otimes_{\mathbb{A}}\mathbb{Q}(q)$ and $\mathrm{Rep}(\mathscr{R}^\Lambda)\otimes_{\mathbb{A}}\mathbb{Q}(q)$ so that, as $U_q(\widehat{\mathfrak{sl}}_e)$-modules,

$$\mathrm{Proj}(\mathscr{R}^\Lambda)\otimes_{\mathbb{A}}\mathbb{Q}(q)\cong L(\Lambda)\cong \mathrm{Rep}(\mathscr{R}^\Lambda)\otimes_{\mathbb{A}}\mathbb{Q}(q)$$

Proof The decomposition matrix defines the linear maps shown. As vector space homomorphisms, $\mathbf{d}_q^T$ is injective and $\mathbf{d}_q$ is surjective. Using the graded induction and restriction formulas it remains to observe that E_i coincides with i-Res and that $q^{-1}F_iK_i$ coincides with i-Ind.



The result then follows since $L(\Lambda)=U_q(\widehat{\mathfrak{sl}}_e)v_\Lambda\subseteq \operatorname{im}\mathbf{d}_q^T$

Cartan pairing

A semilinear map of $\mathbb{A}$-modules is a $\mathbb{Z}$-linear map $\theta: M\longrightarrow N$ such that $\theta(f(q)m)=\overline{f(q)}\theta(m)$, for all $f(q)\in\mathbb{A}$ and $m\in M$.

A sesquilinear map $f: M\times N\longrightarrow\mathbb{A}$, where M and N are $\mathbb{A}$-modules, is a function that is semilinear in the first variable and linear in the second.

Define the Cartan pairing $\langle[P],[M]\rangle=\dim_q\mathrm{Hom}_{\mathscr{R}^\Lambda_n}(P,M)$, for $P\in\mathrm{Proj}(\mathscr{R}^\Lambda_n)$ and $M\in\mathrm{Rep}(\mathscr{R}^\Lambda_n)$. This is a sesquilinear form because

$$\mathrm{Hom}_{\mathscr{R}^\Lambda_n}(P\langle k\rangle,M)\cong\mathrm{Hom}_{\mathscr{R}^\Lambda_n}(P,M\langle -k\rangle)$$

$\implies\ \langle[Y^\mu],[D^\nu]\rangle=\delta_{\mu\nu}$

The biadjointness of (E_i,F_i) implies that

$$\langle i\text{-Ind}\,x,y\rangle=\langle x,i\text{-Res}\,y\rangle\ \text{and}\ \langle i\text{-Res}\,x,y\rangle=\langle x,i\text{-Ind}\,y\rangle$$

Using the uniqueness of the Shapovalov form, we obtain:

$\implies$ If $x\in\mathrm{Proj}(\mathscr{R}^\Lambda)$ and $y\in\mathrm{Rep}(\mathscr{R}^\Lambda)$ then $\langle\mathbf{d}_q^T(x),y\rangle=\langle x,\mathbf{d}_q(y)\rangle$

Corollary

As $U_{\mathbb{A}}(\widehat{\mathfrak{sl}}_e)$-modules, $\mathrm{Proj}(\mathscr{R}^\Lambda)=L_{\mathbb{A}}(\Lambda)$ and
$\mathrm{Rep}(\mathscr{R}^\Lambda)=L_{\mathbb{A}}(\Lambda)^\vee=\{\,x\in L_{\mathbb{Q}(q)}(\Lambda)\mid\langle x,y\rangle\in\mathbb{A}$ for all $y\in\mathbb{L}_{\mathbb{A}}(\Lambda)\,\}$

Dualities on Fock space

The dualities $\circledast$ and $\#$ on $\mathrm{Rep}(\mathscr{R}^\Lambda)$ induce semilinear endomorphisms on $\mathrm{Rep}(\mathscr{R}^\Lambda)$ and $\mathrm{Proj}(\mathscr{R}^\Lambda)$ by

$$[M]^\circledast=[M^\circledast]\ \text{and}\ [M]^\#=[M^\#]$$

We concentrate on $\circledast$. Write $\mathbf{d}_q^{-1}=(e_{\mu\nu}(-q))$

Lemma

Let $\boldsymbol{\lambda}\in\mathcal{K}^\Lambda_n$. Then $[S^\mu]^\circledast=[S^\mu]+\sum_{\mu\rhd\tau\in\mathcal{K}^\Lambda_n}a_{\mu\tau}(q)[S^\tau]$

Proof We just compute using the decomposition matrix:

$$[S^\mu]^\circledast=\Big(\sum_{\mu\rhd\nu\in\mathcal{K}^\Lambda_n}d_{\mu\nu}(q)[D^\nu]\Big)^\circledast=\sum_{\mu\trianglerighteq\nu}\overline{d_{\mu\nu}(q)}\,[D^\nu]$$
$$=[S^\mu]+\sum_{\substack{\tau\in\mathcal{K}^\Lambda_n\\ \mu\rhd\tau}}\Big(\sum_{\substack{\nu\in\mathcal{K}^\Lambda_n\\ \mu\trianglerighteq\nu\trianglerighteq\tau}}\overline{d_{\mu\nu}(q)}\,e_{\nu\tau}(-q)\Big)[S^\tau]$$

Lusztig's Lemma

Proposition (Lusztig's lemma)

There exists a unique basis $\{\,B^\mu\mid\mu\in\mathcal{K}^\Lambda\,\}$ of $\mathrm{Rep}(\mathscr{R}^\Lambda)$ such that

$$(B^\mu)^\circledast=B^\mu\quad\text{and}\quad B^\mu=[S^\mu]+\sum_{\mu\rhd\tau\in\mathcal{K}^\Lambda_n}b^{\mu\tau}(q)[S^\tau]$$

where $b^{\mu\tau}(q)\in\delta_{\mu\tau}+q\mathbb{Z}[q]$

Proof

Uniqueness If B^μ and $\dot B^\mu$ are two such elements then

$$B^\mu-\dot B^\mu=\textstyle\sum_{\mu\rhd\tau}c^{\mu\tau}(q)[S^\tau],\qquad\text{for } c^{\mu\tau}(q)\in q\mathbb{Z}[q].$$

The left-hand side is $\circledast$-invariant and $\overline{c^{\mu\tau}(q)}\in q^{-1}\mathbb{Z}[q^{-1}]$. If $\tau\ne\mu$ is maximal such that $c^{\mu\tau}(q)\neq0$ then the last lemma forces

$$c^{\mu\tau}(q)\in q\mathbb{Z}[q]\cap q^{-1}\mathbb{Z}[q^{-1}]=\{0\},$$

a contradiction! Hence, $B^\mu=\dot B^\mu$

Lusztig's lemma – existence

Existence: argue by induction on dominance

If $\mu \in \mathcal{K}_n^\Lambda$ is minimal in $\mathcal{K}_n^\Lambda$ then $B^\mu = [S^\mu] = [D^\mu] = (B^\mu)^\circledast$.

If $\mu \in \mathcal{K}_n^\Lambda$ is not minimal set $C^\mu = [D^\mu]$

$\implies$ $(C^\mu)^\circledast = C^\mu$ and $C^\mu = [S^\mu] + \sum_{\mu \rhd \tau} c^{\mu\tau}(q)[S^\tau]$, for $c^{\mu\tau}(q) \in \mathbb{A}$

If $c^{\mu\tau}(q) \in q\mathbb{Z}[q]$ for all τ, set $B^\mu = C^\mu$ – we're done

If not, let ν be maximal such that $c^{\mu\nu}(q) \notin q\mathbb{Z}[q]$

Replace C^μ with the element $C^\mu - a^{\mu\nu}(q)B^\nu$, where $a^{\mu\nu}(q)$ is the unique Laurent polynomial such that $\overline{a^{\mu\nu}(q)} = a^{\mu\nu}(q)$ and $c^{\mu\nu}(q) - a^{\mu\nu}(q) \in q\mathbb{Z}[q]$.

$\implies$ $(C^\mu)^\circledast = C^\mu$ and the coefficient of $[S^\nu]$ in C^μ belongs to $q\mathbb{Z}[q]$.

Repeating this process, after finitely many steps we construct an element B^μ with the required properties. □

Canonical basis

Using an almost identical argument starting with $X_\mu = \sum_{\lambda \in \mathcal{K}_n^\Lambda} e_{\lambda\mu}(-q)[Y^\lambda] \in \mathrm{Proj}(\mathscr{R}^\Lambda)$ we obtain:

Proposition (Lusztig's lemma)

There exists a unique basis $\{ B_\mu \mid \mu \in \mathcal{K}^\Lambda \}$ of $\mathrm{Proj}(\mathscr{R}^\Lambda)$ such that

$$(B^\mu)^\# = B^\mu \quad \text{and} \quad B^\mu = [S_\mu] + \sum_{\tau \rhd \mu \in \mathcal{K}_n^\Lambda} b^{\tau\mu}(q)[X_\tau]$$

where $b^{\tau\mu}(q) \in \delta_{\tau\mu} + q\mathbb{Z}[q]$

The basis $\{B^\mu\}$ is the dual canonical basis of $L_{\mathbb{A}}(\lambda)^\vee \cong \mathrm{Rep}(\mathscr{R}^\Lambda)$ and $\{ D_\mu \mid \mu \in \mathcal{K}^\Lambda \}$ is the canonical basis of $L_{\mathbb{A}}(\Lambda) \cong \mathrm{Proj}(\mathscr{R}_n^\Lambda)$

As their names suggest, these two bases are dual under the Cartan pairing:

Corollary

Suppose that $\lambda, \mu \in \mathcal{K}^\Lambda$. Then $\langle B_\mu, B^\lambda \rangle = \delta_{\lambda\mu}$

Ariki's categorification theorem

Let $\mathrm{Proj}(\mathscr{H}^\Lambda) = \bigoplus_{n \ge 0} \mathrm{Proj}(\mathscr{H}_n^\Lambda)$ be the Grothendieck group of the *ungraded* algebras $\mathscr{H}_n^\Lambda$, for $n \ge 0$.

$\implies$ $\mathrm{Proj}(\mathscr{H}^\Lambda)$ is the free $\mathbb{Z}$-module with basis $\{ \underline{Y}^\mu \mid \mu \in \mathcal{K}^\Lambda \}$, where $M \mapsto \underline{M}$ is the forgetful functor that forgets the grading

Let $L_1(\Lambda)$ be the irreducible integrable highest weight module with highest weight Λ when $q = 1$

Theorem (Ariki's Categorification Theorem)

Suppose that $\Bbbk$ is a field of characteristic zero. Then the canonical basis of $L_1(\Lambda)$ coincides with the basis of (ungraded) projective indecomposable $\mathscr{H}_n^\Lambda$-modules $\{ [\underline{Y}^\mu] \mid \mu \in \mathcal{K}^\Lambda \}$ of $\mathrm{Proj}(\mathscr{H}_n^\Lambda)$.

Corollary

Suppose that $\Bbbk$ is a field of characteristic zero. Then $\{ [D^\mu] \mid \mu \in \mathcal{K}^\Lambda \}$ is the dual canonical basis of $L_{\mathbb{A}}(\Lambda)$

$\implies$ $d_{\lambda\mu}(q) \in \delta_{\lambda\mu} + q\mathbb{N}[q]$

Further reading I

- S. Ariki, *On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$*, J. Math. Kyoto Univ., **36** (1996), 789–808.
- ——, *Representations of quantum algebras and combinatorics of Young tableaux*, University Lecture Series, **26**, American Mathematical Society, Providence, RI, 2002. Translated from the 2000 Japanese edition and revised by the author.
- S. Ariki and A. Mathas, *The number of simple modules of the Hecke algebras of type $G(r, 1, n)$*, Math. Z., **233** (2000), 601–623.
- J. Brundan and A. Kleshchev, *Graded decomposition numbers for cyclotomic Hecke algebras*, Adv. Math., **222** (2009), 1883–1942.
- ——, *The degenerate analogue of Ariki's categorification theorem*, Math. Z., **266** (2010), 877–919. arXiv:0901.0057.
- J. Brundan, A. Kleshchev, and W. Wang, *Graded Specht modules*, J. Reine Angew. Math., **655** (2011), 61–87. arXiv:0901.0218.
- M. Geck, *Representations of Hecke algebras at roots of unity*, Astérisque, 1998, Exp. No. 836, 3, 33–55. Séminaire Bourbaki. Vol. 1997/98.

Further reading II

- J. Hu and A. Mathas, *Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type A*, Adv. Math., **225** (2010), 598–642. arXiv:0907.2985.
- ———, *Graded induction for Specht modules*, Int. Math. Res. Not. IMRN, **2012** (2012), 1230–1263. arXiv:1008.1462.
- A. Kleshchev, *Linear and projective representations of symmetric groups*, Cambridge Tracts in Mathematics, **163**, Cambridge University Press, Cambridge, 2005.
- A. Lascoux, B. Leclerc, and J.-Y. Thibon, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Comm. Math. Phys., **181** (1996), 205–263.
- A. Mathas, *Cyclotomic quiver Hecke algebras of type A*, in Modular representation theory of finite and p-adic groups, G. W. Teck and K. M. Tan, eds., National University of Singapore Lecture Notes Series, **30**, World Scientific, 2015, ch. 5, 165–266. arXiv:1310.2142.

Further reading III

- ———, *Restricting Specht modules of cyclotomic Hecke algebras*, Science China Mathematics, 2017, 1–12. Special Issue on Representation Theory, arXiv:1610.09729.
- R. Rouquier, *Quiver Hecke algebras and 2-Lie algebras*, Algebra Colloq., **19** (2012), 359–410.
- M. Varagnolo and E. Vasserot, *Canonical bases and KLR-algebras*, J. Reine Angew. Math., **659** (2011), 67–100.
- B. Webster, *Canonical bases and higher representation theory*, Compos. Math., **151** (2015), 121–166. arXiv:1209.0051.